# Saturated linear PI control of a class of exothermic tubular reactors

**Hugo Franco** * **Jesús Alvarez** **

* *Universidad Nacional Autónoma de México, 04510 Mexico City, Mexico (e-mail: HFrancoR@iingen.unam.mx).*
** *Universidad Autónoma Metropolitana, 09340 Mexico City, Mexico (e-mail: jac@xanum.uam.mx).*

**Abstract:** The problem of controlling and open-loop unstable spatially distributed exothermic tubular reactor by means of a saturated linear proportional-integral (PI) control with anti-windup (AW) protection is addressed. The aim is to upgrade the existing industrial designs, in terms of: (i) *a priori* guarantee of robust nonlocal closed-loop stability, and (ii) systematicity of control gain and limit choices. The problem is solved combining concepts from nonlinear dynamics and control as well as industrial practice insight. The proposed control design approach is applied to a representative example through numerical simulations, and put in perspective with related industrial techniques.

*Keywords:* Chemical tubular distributed reactor, PI control, anti-windup protection, robust stability.

## 1. INTRODUCTION

Spatially distributed jacketed tubular reactors are important units in the chemical process industry, where one or several reactants are converted into a product by means of an exothermic reaction. Since the reactors are described by a set of nonlinear partial differential equations, their complex (with multiplicity and bifurcation global phenomena [Jørgensen, 1986]) dynamics are described with numerical PDE solvers.

In industry [Del Vecchio and Petit, 2005], these [possibly open-loop (OL) unstable] reactors are controlled by adjusting with a proportional-integral (PI) controller the heat extraction on the basis of a temperature measurement at the most sensitive (with the largest slope change before the hot spot) axial location [Singh et al., 2008]. Due to their simplicity, reliability and low investment-maintenance cost, the PI regulators are by far the most widely employed controllers in the chemical process industry. Since exothermic reactors exhibit strong parametric sensitivity, the handling of control saturation through anti-windup protection (AW) is a key issue that must be carefully considered for safety and operability purposes. The PI and AW gains are chosen with procedures that have as point of departure, for the PI part, the well-known Ziegler Nichols (Åström and Hägglund [2006]) or internal model control (IMC) [Rivera et al., 1986] techniques; and for the AW *ad hoc* recommendations [Åström and Hägglund, 2006, Visioli, 2006]. However, the implementation and maintenance of the PI with AW (PI+AW) controllers for exothermic reactors requires a large dosage of per-reactor knowledge, insight and experience. In fact, a bad choice of control limits can induce a catastrophic (explosive) straneous stable steady-state (SS) attractor [Alvarez et al., 1991, Chen and Chang, 1985].

Given that the nonlinear estimation and control theory for PDE systems is mathematically complex, and lags far behind the ones for finite-dimensional (ODE) systems, the advanced model-based studies for distributed chemical reactors are based on finite-dimensional systems models and lead to output-feedback (OF) controllers [Christofides, 2012] that are quite complex for industrial standards, and it is not clear that those controllers outperform the PI+AW ones employed in industry.

The preceding considerations motivate the scope of the present study: the upgrade of the existing PI+AW control scheme for exothermic tubular distributed reactors, in the sense of a more efficient and reliable design with: (i) *a priori* guarantee of robust nonlocal closed-loop (CL) stability, and (ii) systematicity of the control gain and limit choices.

The paper is organized as follows. In section 2, the tubular reactor is introduced and the control problem is stated. In sections 3 and 4, the proposed approach is developed. In section 5, a comparison study between our design some of the most common AW techniques is done. Finally, conclusions are drawn in section 6.

## 2. SATURATED CONTROL PROBLEM

### *2.1 The tubular reactor*

Consider the spatially distributed jacketed tubular reactor depicted in Fig. 1, where a reactant is converted into product by means of an exothermic reaction. From mass-heat conservation and thermodynamics principles [Varma, 1980], the reactor dynamics in dimensionless form are given by the two PDEs:

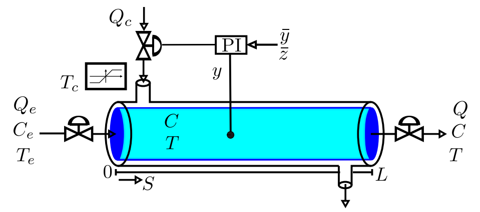


Fig. 1. Tubular reactor with PI+AW control.

$$s \in S = (0,1), \quad t \in \mathbb{R}^+ :$$

$$\partial_t c = d_m \partial_{ss} c - q \partial_s c - r(c,\tau),\ u \in U = [u^-, u^+] \quad \text{(1a)}$$

$$\partial_t \tau = d_h \partial_{ss} \tau - q \partial_s \tau + r(c,\tau) - \upsilon(\tau - u),\ y = \tau(s_y, t), \quad \text{(1b)}$$

$$s = 0, t > 0 :\ d_m \partial_s c = q(c - c_e),\ d_h \partial_s \tau = q(\tau - \tau_e) \quad \text{(1c)}$$

$$s = 1, t > 0 : \partial_s c = 0, \quad \partial_s \tau = 0, \quad z = c(1,\tau) \quad \text{(1d)}$$

$$t = 0, 0 \le s \le 1 :\ c(s,0) = c_0(s),\ \tau(s,0) = \tau_0(s) \quad \text{(1e)}$$

where

$$0 \le c(s,t) \le c_e^+,\ \tau^- \le \tau(s,\tau) \le \tau^+ \quad \text{(2a)}$$

$$\tau^- = \min(\tau_e^-, u^-),\ \tau^+ = \tau_a \max(\tau_e^+, u^+),\ \tau_a = 1 \quad \text{(2b)}$$

$$\Xi = \left\{ [c(s), \tau(s)] \mid 0 \le c(s) \le c_e^+,\ \tau^- \le \tau(s) \le \tau^+ \right\} \quad \text{(2c)}$$

$t$ is the time, $s$ the axial position, $c(s,t)$ [or $\tau(s,t)$] is the concentration (or temperature) state profile at time $t$, $c_e$ and $c_0$ (or $\tau_e$ and $\tau_0$) are the feed and initial concentration (or temperature), $q$ is the dilution (volumetric flow-to-volume) rate, $u = \tau_c$ is the coolant temperature control input, $u^-$ (or $u^+$) is the lower (or upper) control limit, $y$ is the measured temperature output at length $s_y \in S$, $z$ is the exit concentration (indirectly) regulated output, $d_m$ (or $d_h$) is the mass (or heat) dispersion number, $\upsilon$ is the heat transfer parameter, $r$ is the reaction rate, and the bounded set $\Xi$ (2) reflects mass-heat conservation and heat direction flow, $t_a$ is the adiabatic temperature increment, and $c_e$ the maximum value of the inlet concentration.

In compact form, the PDE model (1) is written as

$$\partial_t \boldsymbol{\chi} = \boldsymbol{\mathcal{F}}(\boldsymbol{\chi}, \boldsymbol{d}, u),\ \boldsymbol{\chi}(0) = \boldsymbol{\chi}_0,\ \boldsymbol{\chi} \in \Xi \quad \text{(3a)}$$

$$y = \mathcal{H}_y(\boldsymbol{\chi}),\ z = \mathcal{H}_z(\boldsymbol{\chi}),\ \boldsymbol{\chi} = [c(s,t), \tau(s,t)]^\top, \quad \text{(3b)}$$

with statics

$$\boldsymbol{\mathcal{F}}(\bar{\boldsymbol{\chi}}_j, \bar{\boldsymbol{d}}, u) = 0, \quad j = 1, \ldots, n_s \ge 1 \quad \text{(3c)}$$

where $\boldsymbol{\chi}$ is the state profile, $\boldsymbol{d} = [c_e, \tau_e, q]^\top$ is the exogenous input, $\bar{\boldsymbol{\chi}}_j$ is the $j$-th steady-state (SS), $n_s$ is the number of SSs, and $\bar{\boldsymbol{\chi}}$ is the (possibly OL unstable) nominal target SS.

The dynamics (3) are robust if they are structurally stable [Hubbard and West, 1992], meaning that small parameter changes do not yield a qualitatively change in the global dynamics. Otherwise, the dynamics (3) are fragile. Nonlocal practical stability means for admissible delimited input-initial state and state excursion sizes [La Salle and Lefschetz, 2012].

As representative case example, with OL dynamics extensively studied over its parameter space [Varma, 1980], let us consider an OL bistable reactor with irreversible first-order reaction rate $r$ and with Arrhenius temperature dependency:

$$r(c,\tau) = c\alpha(\tau),\ \alpha(\tau) = \mathrm{e}^{\phi - \frac{\gamma}{\tau}},\ (\phi, \gamma) = (23.719, 50) \quad \text{(4a)}$$

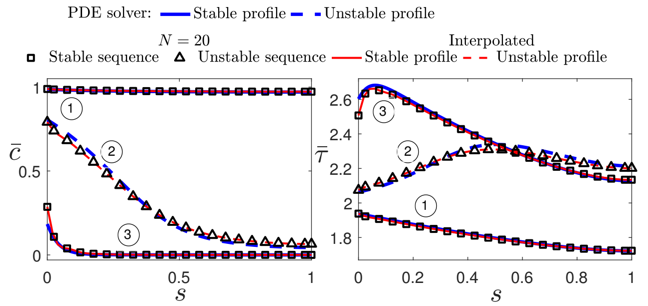


Fig. 2. Global statics of the bistable reactor drawn with: (i) a $N_{PDE}(= 200)$-stage FD PDE numerical solver (blue plots), and (ii) with $N(= 20)$-stage model (7) (sequence and interpolated profile in red). ①: stable ignition SS $(\bar{\boldsymbol{\chi}}_I, \boldsymbol{x}_I)$, ②: unstable saddle SS $(\bar{\boldsymbol{\chi}}_U, \boldsymbol{x}_U)$, and ③: stable extinction SS $(\bar{\boldsymbol{\chi}}_E, \boldsymbol{x}_E)$.

$$d_m = d_h = 1/5,\ \bar{c}_e = \bar{q} = \upsilon = 1,\ \bar{\tau}_e = 2, \bar{\tau}_c = 3/4 \quad \text{(4b)}$$

the corresponding three SS profiles pairs $[\bar{c}(s), \bar{\tau}(s)]_{i=1,\ldots,3}$, computed with a finite difference (FD) numerical PDE solver ($N_{PDE} = 200$ internal nodes) are shown (blue plots) in Fig. 2: extinction (③: $\bar{\boldsymbol{\chi}}_E$) and ignition (①: $\bar{\boldsymbol{\chi}}_I$) stable SSs, and an unstable saddle (②: $\bar{\boldsymbol{\chi}}_U$) in between.

### 2.2 Control problem

As a worst-case situation to subject the control to a severe test, the CL reactor must operate about the OL unstable SS $\bar{\boldsymbol{\chi}}_U = \bar{\boldsymbol{\chi}}$. Given that back-calculation (BC) is the most effective and common AW scheme [Åström and Hägglund, 2006, Visioli, 2006], let us consider the associated PI+AW controller

$$u = \bar{u} + \pi(\psi) + \alpha_{AW}(u - u_s),\ \psi = y - \bar{y},\ u_s = \mathrm{sat}(u) \quad \text{(5a)}$$

where

$$\pi(\cdot) = -k_p \left[ (\cdot) + t_I^{-1} \int_0^t (\cdot) \mathtt{d}\mathtt{t} \right] \quad \text{(5b)}$$

$$\alpha_{AW}(\cdot) = -t_a^{-1} \int_0^t (\cdot) \mathtt{d}\mathtt{t} \quad \text{(5c)}$$

$$\mathrm{sat}(u) = \begin{cases} u^+ & \text{if } u > u^+ \\ u & \text{if } u_- \le u \le u^+ \\ u^- & \text{if } u < u_- \end{cases} \quad \text{(5d)}$$

$$(k_p, t_I, t_a, u_-, u^+)^\top := \boldsymbol{k}_{PI} \quad \text{(5e)}$$

$\pi$ is the linear PI controller with proportional gain $k_p$ and reset time $t_I$, $\alpha_{AW}$ is the BC-based anti-windup protector [Åström and Hägglund, 2006] with reset time $t_a$, $\psi$ is the deviation measurement value, $u_s$ is the saturation, and $k_{PI}$ is the five-parameter adjustable vector. Typically, the PI pair $(k_p, t_I)$ is tuned with standard (Ziegler-Nichols, IMC, etc.) rules. For the AW reset time $t_a$ there are a diversity of recommendations [Visioli, 2006].

Our problem consists in designing the five-parameter set $k_{PI}$ (5e) of the saturated PI+AW temperature controller (5) so that the (possibly unstable or fragilely stable) nominal steady-state is the unique robust nonlocal attractor of the associated closed loop dynamics (3), (5) with emphasis on: (i) *a priori* guarantee of reliability: the assurance of CL robust nonlocal practical stability conditions, and (ii) derivation of tuning procedure and a gain limit choice criterion that is more systematic and simple that the ones employed in industrial practice.

## 3. PROPORTIONAL CONTROL

As a methodological step of the saturated PI control design, in this section the saturated P control design problem is addressed on the basis of a finite-(sufficiently large)-dimensional approximation of the PDE tubular reactor model (1).

### 3.1 *Staged model*

The application to the PDE model (1) of N-internal node FDs [LeVeque, 2007]

$$[\partial_s(\cdot)]_{s=s_i} \approx \Delta(\cdot)/N, \qquad \Delta(\cdot) = (\cdot)_i - (\cdot)_{i-1}$$
$$[\partial_{ss}(\cdot)]_{s=s_i} \approx \Delta^2(\cdot)/N^2, \qquad \Delta(\cdot) = (\cdot)_{i+1} - 2(\cdot)_i - (\cdot)_{i-1}$$

yields the finite-dimensional model:

$$1 \le i \le N,\; N^- < N < N^+ < N_{PDE},\; t > 0$$

$$\dot{c}_i = \theta_m \Delta^2 c_i - \theta \Delta c_i - r(c_i, \tau_i), \quad c_i(0) = c_{0i} \tag{6a}$$
$$\dot{\tau}_i = \theta_h \Delta^2 \tau_i - \theta \Delta \tau_i + r(c_i, \tau_i) - \upsilon(\tau_i - u),\; \tau(0) = \tau_{i0} \tag{6b}$$
$$\theta_m(c_1 - c_0) = \theta(c_1 - c_e), \quad \theta_h(\tau_1 - \tau_0) = \theta(\tau_1 - \tau_e) \tag{6c}$$
$$c_{N+1} = c_N, \qquad \tau_{N+1} = \tau_N \tag{6d}$$

where

$$\theta = Nq, \quad \theta_m = N^2 d_m, \quad \theta_h = N^2 d_h \tag{6e}$$

In compact form, this *N-staged reactor model* (6) is written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) + \boldsymbol{g}(\boldsymbol{x})u, \quad \boldsymbol{x}_0 = \boldsymbol{x}_0, \quad \boldsymbol{x} \in \boldsymbol{X} \tag{7a}$$
$$y = h_y(\boldsymbol{x}), \quad z = h_z(\boldsymbol{x}), \quad \dim \boldsymbol{x} = n = 2N \tag{7b}$$

where

$$\boldsymbol{x} = (\boldsymbol{x}_1^\top, \ldots, \boldsymbol{x}_N^\top)^\top, \quad \boldsymbol{x}_i = (c_i, \tau_i)^\top$$
$$\boldsymbol{f} = (\boldsymbol{f}_1^\top, \ldots, \boldsymbol{f}_N^\top)^\top, \quad \boldsymbol{f}_i = (f_i^m, f_i^h)^\top$$
$$X = \left\{\boldsymbol{x} \in \mathbb{R}^n \,|\, 0 \le c_i \le c_e^+,\, \tau^- \le \tau \le \tau^+,\, i = 1, \ldots, N\right\} \tag{8}$$

and $X$ is a bounded invariant set determined by $\Xi$ (2c). The corresponding statics with $n_s$ SSs ($\bar{\boldsymbol{x}}_i$) are

$$\boldsymbol{f}(\bar{\boldsymbol{x}}_i, \bar{\boldsymbol{d}}) + \boldsymbol{x}(\bar{\boldsymbol{x}}_i) = 0,\; i = 1, \ldots, n_s \ge 1,\; \bar{\boldsymbol{x}}_i \in X \tag{9}$$

Following previous stage-based modelling studies on bistable reactors [Badillo-Hernandez et al., 2013, Najera et al., 2015], it was found that the PDE model (1) of our bistable reactor example (4) can be described, up to typical kinetics-transport parameter error, using model (7) with $N = 20$. The corresponding three SSs $\bar{\boldsymbol{x}}_{i=1,\ldots,3}$ are shown (symbol sequence and its interpolated profile in red) in Fig. 2: $\bar{\boldsymbol{x}}_E$ (③): stable extinction SS, $\bar{\boldsymbol{x}}_I$ (①): stable ignition SS, and $\bar{\boldsymbol{x}}_U := \bar{\boldsymbol{x}}$ (②): unstable saddle *target* SS.

### 3.2 *Passivity assessment*

As first steps in the controller design we choose the sensor location following industrial practice: sensitivity criterion established that the sensor must be located at the stage with maximum temperature gradient before the hot spot [Najera et al., 2015], this zone can be found computing minimum concavity temperature stage of model (7). The application of this criterion to Example (4) yields, according to Fig. 3a, that the sensor must be located at the stage: $m = 5$ (i.e., at axial dimensionless position $s_y \approx 0.25$). Once the output is fixed, we proceed

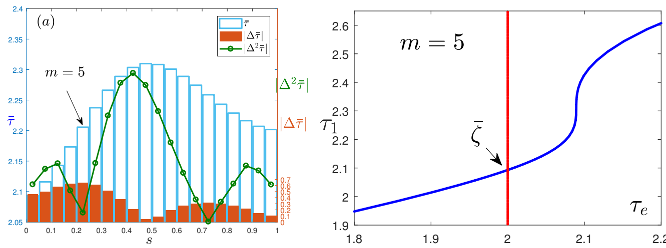


Fig. 3. Sensor location and the associated SS ZD: (a) Sensitivity criterion: $m = 5$: temperature, gradient, and concavity of the $N = 20$-stage model (9). (b) The ZD (11) (19) bifurcation: temperature ($\tau_1$)-feed temperature ($\tau_e$).

to characterize the associated passivity property. For the purpose at hand, let us rewrite the staged model (6) in the partitioned form

$$\dot{\tau}_m = f_m^h(\tau_m, \boldsymbol{\zeta}, \boldsymbol{d}) + g_m(\boldsymbol{x})u,\; y = \tau_m,\; \tau_m(0) = \tau_{m0} \tag{10a}$$
$$\dot{\boldsymbol{\zeta}} = \boldsymbol{f}_\zeta(\tau_m, \boldsymbol{\zeta}, \boldsymbol{d}) + \boldsymbol{g}_\zeta(\boldsymbol{x})u,\; \boldsymbol{z}(0) = \boldsymbol{\zeta}_0 \tag{10b}$$

where

$$\boldsymbol{x} = \boldsymbol{I}_\zeta(\tau_m, \boldsymbol{\zeta}^\top)^\top, \quad \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}, u) = \boldsymbol{I}_\zeta(f_m^h, \boldsymbol{f}_\zeta^\top)^\top$$

$\boldsymbol{I}_\zeta$ is a permutation matrix. From the enforcement of the restriction $\tau_m = \bar{y}$ upon model (10) the $(n-1)$-dimensional zero dynamics (ZD) [Isidori, 2013] follows:

$$\dot{\boldsymbol{\zeta}}^* = \boldsymbol{f}_\zeta^*(\boldsymbol{\zeta}^*, \boldsymbol{d}),\; \boldsymbol{\zeta}^*(0) = \boldsymbol{z}_0^*,\; \boldsymbol{f}_\zeta^*(\bar{\boldsymbol{\zeta}}^*, \bar{\boldsymbol{d}}) \tag{11}$$

where

$$\boldsymbol{f}_\zeta^*(\boldsymbol{\zeta}^*, \boldsymbol{d}) = \boldsymbol{f}_\zeta^*(\bar{y}, \boldsymbol{\zeta}^*, \boldsymbol{d}) + \boldsymbol{g}_\zeta(\boldsymbol{x})\mu_z(\bar{y}, \boldsymbol{\zeta}, \boldsymbol{d})$$
$$\mu_z(\bar{y}, \boldsymbol{\zeta}, \boldsymbol{d}) = -f_m^h(\bar{y}, \boldsymbol{\zeta}, \boldsymbol{d})/g_m(\boldsymbol{x})$$

In Fig. 3b is presented the $\tau_1^*$ versus $\tau_e$ bifurcation diagram, showing that the target restricted SS $\bar{\boldsymbol{x}}$: (i) is unique for the ZD (11), and (ii) is robustly stable (sufficiently away from mono-bistability bifurcation condition). It was verified with local Lyapunov first method that $\bar{\boldsymbol{\zeta}}$ is stable. The ZD conditions, together with the fact that the relative degree between the input $u$ and the output $y$ is always one since $\upsilon > 0$ implies that the staged model (7) is globally robust feedback passive. This in turn signifies that it can be robustly stabilized with feedback temperature control.

### 3.3 *Input-output bifurcation map*

From a previous study [Alvarez et al., 1991] we know that the derivation of analytic necessary control gains and limits conditions for the two-state continuous version of our tubular reactor (1) requires information contained in the static input-output bifurcation map (IOBM). Accordingly, in this subsection the state-control bifurcation pairs of our $2N$-dimensional $N$-staged tubular reactor model (6) are identified using the continuation package Matcont [Dhooge et al., 2003] to the model statics (9), with $N = 20$ of, the case example yielded the dependency of the SS set with the OL control input $u$. The corresponding IOBM is presented in Fig. 4, showing that for the staged reactor (6) of our case example (4), the control interval $U$ is partitioned according to behavior regions and bifurcation points as follows

$$U = U_m^E \cup u_* \cup U_b \cup u^* \cup U_m^I,\; n_b = 2,\; U_b = \{u_*, u^*\}$$

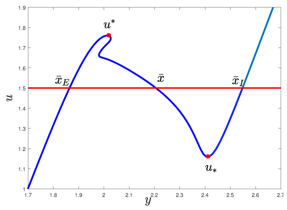

Fig. 4. Input-output bifurcation map: bifurcation values for the nominal value of the control input.

where: (i) $U_m^E$ (or $U_m^I$) is a control set with reactor extinction (or ignition)-type monostability, (ii) $U_b$ is a set with bistability (with an ignition-extinction pair of SSs and an unstable saddle in between), and (iii) $u_*$ (or $u^*$) is the bifurcation point where creation-destruction of ignition (or extinction) node-saddle pair occurs. Closeness to (or awayness from) the bifurcation values implies robustness (or fragility) of the reactor dynamics (3).

### 3.4 Closed-loop stability analysis

The application of the saturated proportional linear controller

$$u = \text{sat}\left[-k(y - \bar{y}) + \bar{u}\right],\ k > 0 \tag{12}$$

to the OL system (7) yields the CL dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) + \boldsymbol{g}(\boldsymbol{x})\text{sat}\left[-k(y - \bar{t}) + \bar{u}\right] := \boldsymbol{f}_c(\boldsymbol{x}, \boldsymbol{d}) \tag{13}$$

with statics

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) + \boldsymbol{g}(\boldsymbol{x})u = \boldsymbol{0},\ u = \text{sat}\left[-k(y - \bar{t}) + \bar{u}\right] \tag{14}$$

The solution $(\boldsymbol{x}, u)$ of the CL statics (14) is determined by the intersection $(y, u)$ of the OL-IOBM, and the saturated, at the control limits $u^+$ and $u_-$, P control curve: a straight line with slope $k$. These curves, with only one intersection at $(\bar{y}, \bar{u})$ are presented in Fig. 5a, showing that for $\bar{\boldsymbol{x}}$ (OL unstable) to be the unique CL SS necessary and sufficiently: (i) the gain $k$ must be chosen above the lower limit value $k_b$ and, (ii) the control limit $u^+$ (or $u_-$) must be chosen above (or below) of the bifurcation value $u^*$ (or $u_*$) determined by the OL-IOBM discussed in subsection 3.3. This result is stated next in proposition form.

**Proposition 1**. The CL reactor dynamics (6) with the saturated P temperature control (12) has the prescribed (possibly OL fragilely stable or unstable) nominal SS $\bar{\boldsymbol{x}}$ (9) as unique robust steady-state if and only if the control gain $k$ and limit pair $(u^-, u^+)$ are chosen so that

$$k < k_b + \varepsilon_k, \qquad u^+ > u^* + \varepsilon^*, \qquad u_- < u_* + \varepsilon_* \tag{15}$$

where $\varepsilon_k$, $\varepsilon^*$ and $\varepsilon_*$ are margins to have a robust condition away from bifurcation.

For illustrative purposes, we can see in Fig 5b the CL statics when conditions (15) are not met: the OL-IOBM and saturated control curves have three intersections: at the nominal target output-input pair $(\bar{y}, \bar{u})$ (with locally stable target SS), ), and at two other output-input (with two undesired straneous SSs induced by the bad choice of the lower control $u_- > u_*$).

Knowing that the fulfillment of the control limits of Proposition 1 ensure that the target (OL unstable) SS $\bar{\boldsymbol{x}}$ is the unique SS of the CL system (13), let us complete the stability assessment by establishing non-local CL attractivity of $\bar{\boldsymbol{x}}$. For this aim we will use the notions of set invariance and stability, as well as Seibert's reduction

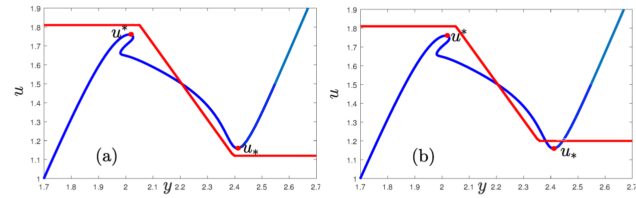


Fig. 5. Input-output bifurcation map for the CL system: (a) only one SS, condition (15) is met, (b) three SSs, condition (15) is not met

principle [Seibert, 1970]. By conservation arguments, the state set $X$ (2c) is an invariant in the sense that any trajectory born in $X$ remains on it for every future time. When the P controller (12) is not saturated: (i) the ZD set $Z$ is a local attractor for $X$, (ii) the nominal (OL unstable) target SS $\bar{\boldsymbol{x}}$ is the unique attractor in $Z$, (iii) $Z$ disconnects $X$, (iv) the P controller (12) without saturation steers the state motions $x(t)$ towards $Z$, and (v) consequently, the nominal SS $\bar{\boldsymbol{x}}$ is a global attractor for $X$. When the P controller ((12)) is saturated, the ZD set $Z$ shrinks, and the possibility of a limit cycle around $Z$exists. From a practical perspective, such mathematical possibility is a rare event, as an impractically large gain is required [Alvarez et al., 1991].

## 4. PROPORTIONAL-INTEGRAL CONTROL

In this section, the saturated linear PI+AW controller is designed, nonlocal CL robust stability conditions are derived, and a systematic-simple gain tuning procedure is obtained.

### 4.1 Model and controller design

Here, a suitable model for output feedback control design is obtained and then the PI+AW controller is designed, with emphasis in the tuning of the two control gains. On the basis of its passivity property, let us express the OL staged dynamics in the "parametric form" [Schaum et al., 2015]

$$\dot{\tau}_m = \lambda\tau_m + au + \iota,\ \tau_m(0) = \tau_{m0},\ a = \hat{\upsilon} > 0 \tag{16a}$$

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{f}_\zeta(\tau_m, \boldsymbol{\zeta}, \boldsymbol{d}, u) + \boldsymbol{g}_\zeta(\boldsymbol{x})u,\ \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \tag{16b}$$

where

$$\iota = \boldsymbol{f}_\iota(\tau_m, \boldsymbol{\zeta}, \boldsymbol{d}, u) := f_m^h(\tau_m, \boldsymbol{\zeta}, \boldsymbol{d}, u) - \lambda\tau_m - \tilde{a}u,\ \tilde{a} = u - \hat{u}$$

$\iota$ is an unknown-observable auxiliary input generated by the static-nonlinear map $f_\iota$; the input-output pairs $(u, y)$ and $(\iota, y)$ have relative degree equal to one, meaning that $u$ satisfies the matching condition for robust output-feedback control Krstic et al. [1995]. Since the input signal $\iota(t)$ is uniquely determined by the measured-known signal pair $(\dot{y}, u)(t)$, it can be quickly (with respect to reactor dynamics) estimated (up to measurement noise). Accordingly, let us to drop the nonlinear static $f_\iota$ and dynamic (16b) components of the OL dynamics (16) to obtain the linear-observable *model for OF control design*:

$$\dot{\tau}_m = au + \iota,\ \tau_m(0) = \tau_{m0}, \quad y = \tau_m \tag{17}$$

It must be pointed out that the robust stabilization through feedback control of this linear dynamical system (with unknown-observable input $\iota$) implies the same for the CL reactor, because its ZD (11) are robustly stable For this aim, let us recall the corresponding robustly convergent (reduced-order) input observer with gain $\omega$ [Schaum et al., 2015]

$$\dot{\chi} = -\omega\chi - \omega(\omega y + au),\ \chi(0) = \chi_0,\ \hat{\iota} = \chi + \omega y \quad (18)$$

the combination of the controller (12), with the load-observer (18) yields *the linear OF controller in IMC form*

$$\dot{\tilde{\chi}} = -\omega\tilde{\chi} - \omega(\omega\psi + av),\ \tilde{\chi}(0) = \tilde{\chi}_0,\ \psi = y - \bar{y} \quad (19a)$$

$$u = -\left[(k+\omega)\psi + \chi\right]/a + \bar{u},\ u_s = \mathrm{sat}(u),\ \tilde{\chi} = \chi - \bar{\chi} \quad (19b)$$

substituting (19a) on (19b), this controller is written in PI+AW form

$$u_s = \mathrm{sat}(u), \quad u = \bar{u} - \pi(\psi) + \alpha_{AW}(u_s - u) \quad (20a)$$

$$k_p = (k+\omega)/a, \quad t_I = k^{-1} + \omega^{-1}, \quad t_a = \omega^{-1} \quad (20b)$$

where

$$t_a = t_I\varsigma(k_p, t_I),\ 0 < \varsigma(k_p, t_I) < 1/2 \quad (20c)$$

$$\varsigma(k_p, t_I) = (1/2)\left[1 - \sqrt{1 - 4/(ak_pt_I)}\right] \quad (20d)$$

The adjustable parameters of this controller are: (i) the two gains ($k$ and $\omega$) of the PI part, and (ii) the two control limits ($u^+$ and $u_-$). The standard $[(k_p, t_I)]$ and IMC $[(k,\omega)]$ gain pairs are biunivocally related, and the IMC gain pair enables a rather systematic and efficient tuning procedure [González and Alvarez, 2005, Schaum et al., 2015]. Comparing with its industrial counterpart (5) (with three control gains), this controller has only two adjustable gains, because the AW time $t_a$ is determined by $(k,\omega)$. Eq. (20c), (20d) explains the heuristic rules to choose the AW time $t_a$.

### *4.2 Closed-loop stability*

The application of the PI+AW OF controller (19) to the OL reactor dynamics (6) yields the CL dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_s(\boldsymbol{x}, \boldsymbol{d}) + \upsilon\tilde{\iota} := \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{d}),\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ \boldsymbol{x} \in X \quad (21a)$$

$$\dot{\tilde{\iota}} = -\omega\tilde{\iota} - \vartheta(\boldsymbol{x}, \tilde{\iota}, \boldsymbol{d}, \dot{\boldsymbol{d}}, u, \dot{u}),\ \tilde{\iota}(0) = \tilde{\iota}_0,\ \tilde{\iota} = \hat{\iota} - \iota \in E \quad (21b)$$

where: (i) for (21a) with $\tilde{\iota} = 0$ [Input to State Stability (ISS) from P control design]

$$|\boldsymbol{x}_0| \le \delta_x \Rightarrow |\boldsymbol{x}(t)| \le a_x e^{-\lambda_x t}\delta_x \le a_x\delta_x = \varepsilon_x$$

and (ii) for the estimation error dynamics (21b) with $\vartheta = 0$ ($\tilde{\iota}$-observability with ISS)

$$|\tilde{\iota}| \le \delta_{\tilde{\iota}} \Rightarrow |\tilde{\iota}(t)| \le e^{-\lambda_\iota t} \le \delta_{\tilde{\iota}} = \varepsilon_\iota$$

From the application of the Small Gain Theorem (Schaum et al., 2015), the threshold stability condition follows

$$\omega = L_{\tilde{\iota}}^{\vartheta} + \upsilon L_x^{\vartheta} := \varpi(k, \omega, \varphi),\ \varphi = X \times E \quad (22)$$

where $L_x^{\vartheta}$ is the Lipschitz constant of $\vartheta$ with respect to $x$, over the domain state-error set $\varphi$, $\varpi$ grows linearly (or more than linearly) for small (or large) $\omega$, and the corresponding two solutions for $\omega$ of the threshold condition (22) are denoted by

$$\omega^- = \varpi^-(k,\varphi), \quad \omega^+ = \varpi^+(k,\varphi)$$

From these results, and the fact that the designed integral part does not modify the CL SS, the next proposition follows.

**Proposition 2**. Let the control gain and limits of the saturated PI+AW stabilizing controller (20) meet condition (15) of Proposition 1. Then, the CL reactor dynamics (21) are robustly monostable with unique SS $(\boldsymbol{x}, \tilde{\iota}) = (\bar{\boldsymbol{x}}, 0)$ if the observer gain $\omega$ is chosen as

$$\omega > k \quad \varpi^-(k,\varphi) < \omega < \varpi^+(k,\varphi) \quad \blacksquare \quad (23)$$

This proposition ensures CL semiglobal robust practical stability with respect to bounded initial and excursion state sets [La Salle and Lefschetz, 2012]. According to (23), $\omega$ is typically between 10 and 30 times faster than the dominant CL dynamics [González and Alvarez, 2005, Schaum et al., 2015], i.e.,

$$\omega \approx n_k,\ n_\omega \in [10, 30], \quad k > k_b. \quad (24)$$

### *4.3 Summary*

The proposed design is done based on the fact that the controller can be implemented as a combination of a linear observer (18) and a linear controller (12) for a simplified model of the system (17) [González and Alvarez, 2005, Najera et al., 2015, Schaum et al., 2015] which retains only the dynamics of the measured output since the rest of the model corresponds with the stable ZD (11). In our design, we offer: (i) a criterion to choose control limits, (ii) a simpler and clear way to choose the control gains and anti-windup parameter, and (iii) assurance of stability of the CL system. The obtained formal reliability and control tuning results constitute an upgrade of the existing industrial PI+AW temperature controller for industrial tubular exothermic reactors.

## 5. CONTROL FUNCTIONING AND COMPARISON

In this section, the proposed controller (20) with BC AW scheme is compared with a PI with conditional integration and BC (CIBC) AW scheme, that according to [Visioli, 2006] is the one with the best functioning in a set of eight different AW schemes. For the proposed PI+AW design (20), the application of the control limit criteria (15) and gain choice (24) yielded: $(k, \omega, t_a) = (2, 30, 0.0333)$ [i.e., $(k_p, t_I) = (32, 0.5333)$] and $(u^-, u^+) = (1, 1.9)$. For the conventional designs, the application of Ziegler-Nichols rules yielded $(k_p, t_I) = (10, 3)$ and a reasonable approximation $(u^-, u^+) \approx (1.1, 1.87)$ of our *a priori* control limits was found after 5 trials, but to have an equal scenario for the test our limits are used also in the conventional controllers. For the AW schemes, we use the following recommendations [Visioli, 2006]: For the CIBC AW scheme, we use the following rules: the integral term is frozen when the control variable saturates and $u * \psi > 0$, with $t_a = 0.03t_I$. Given that, in practice, measurement noise limits control behavior, the proposed PI+AW and conventional controllers were tested with the temperature measurement noise

$$y = \tau_m + a_y \sin(\omega_y),\ a_y = 0.002,\ \omega_y = 30 \quad (25)$$

that corresponds to realistic fluctuations due to imperfect mixing. The behaviors of the proposed PI+AW controller (20) and the ones of its above mentioned three conventional counterparts are presented in Fig. 6, showing that in terms of measured temperature and unmeasured exit concentration transient and asymptotic offset, as well as control effort: the proposed controller (blue continuous plots) clearly outperforms the conventional ones. The proposed controller undergoes less saturation, has faster and smoother responses, and more efficient control action. Among the conventional controllers, the CI-BC has

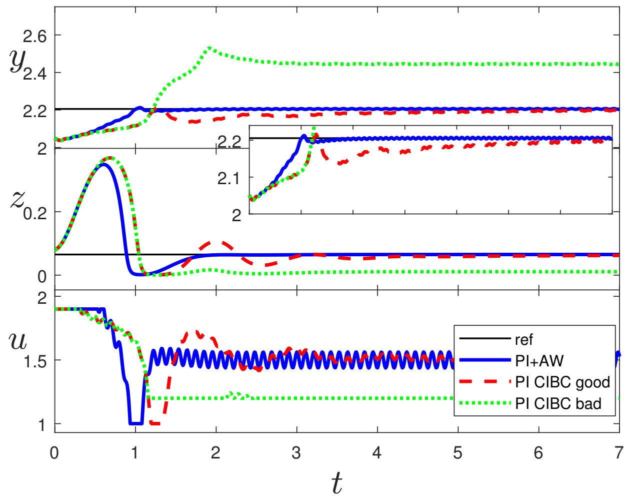


Fig. 6. CL behavior comparison with noisy measurement (25): proposed design, tuned with (24), and PI with CIBC AW scheme, both meeting control limit condition (15). Also, PI with CIBC AW scheme violating condition (15) (in cyan plots)

the best behavior, with less overshot but long settling time. Also in Fig.6 is shown PI+AW CIBC controller (cyan plots) when the lower control limit condition in (15) is violated. The latter situation reflects the realistic effect of an error in the determination of control limits through a rather cumbersome trial-and-error procedure. As expected from our theoretical developments, while the proposed controller manages to stabilize the reactor about the prescribed target SS, the conventional controller reaches a straneous ignition (i.e., dangerous) attractor SS with controller stuck at its lower saturation limit.

## 6. CONSLUSIONS

The industrial saturated PI temperature control design for spatially distributed exothermic tubular chemical reactors has been upgraded with: (i) a robust CL nonlocal (practical) mono-stability condition in terms of two control gains ($k_p$ and $t_I$) and two control limits ($u^-$ and $u^+$), and (ii) a rather simpler gain tuning procedure. The upgrade was attained by modelling the distributed reactor with a staged model, as well as combining notions and tools from global-nonlinear dynamics, and nonlinear control theory, and chemical reactor engineering insight. The present study is a point of departure to explore further improvement through the employment of nonlinear temperature feedback control.

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