On the finite-time consensus problem of multiple Euler–Lagrange systems: An Energy Shaping Approach ⋆

Emmanuel Cruz-Zavala *, **Emmanuel Nuño** * and **Jaime A. Moreno** **

* *Departamento de Ciencias Computacionales, CUCEI, Universidad de Guadalajara. Guadalajara, Jalisco, Mexico.*
** *Instituto de Ingeniería, Universidad Nacional Autónoma de México (UNAM), Ciudad de Mexico, Mexico.*

Abstract: This paper deals with the consensus problem of multiple Euler-Lagrange (EL) systems using the energy shaping plus damping injection principles of passivity-based control. We propose a family of novel decentralized controllers that are capable of solving the leaderless and the leader-follower consensus problems in networks of fully-actuated EL-systems in finite-time. Simulations, using a ten robot manipulator network, are presented to show the performance of the proposed approach.

Keywords: Energy Shaping, Finite-time consensus, Robot Manipulators.

1. INTRODUCTION

According to the existence of a leader or not, consensus is classified into leaderless and leader-follower. When multiple leaders appear, then consensus disappears and a containment control scenario arises. In the leaderless case, all agents agree at a common coordinate value; while in the leader-follower control, the network has to be regulated at the given leader coordinate. The practical applications of consensus are diverse and can be found in different fields, such as biology, physics, control systems and robotics (Cao and Ren, 2011; Cao et al., 2013). Solutions to first and second order linear agents are extensively reported in the literature to solve the leaderless and leader-follower consensus problem under different scenarios (Ren, 2008; Meng et al., 2014).

A wide number of physical systems—including mechanical, electrical and electromechanical systems (Ortega et al., 1998)–can be described by the Euler–Lagrange (EL) equations. Since the seminal works (Chopra and Spong, 2005) and (Nuño et al., 2011) a plethora of different controllers have been proposed to solve both consensus problems, from simple Proportional plus damping (P+d) schemes (Ren, 2009; Nuño et al., 2013; Mei et al., 2011; Ren, 2010; Chen et al., 2013) to more elaborated adaptive controllers (Nuño et al., 2011). However, in all these previous results only asymptotic stability has been ensured. In practical applications, it is often desirable to achieve consensus in Finite-Time (FT). Compared with the asymptotic control approach, FT control is an effective approach with high performance and good robustness to uncertainty and disturbance rejection. Finite-time consensus for multi-agent systems with first-order and second-order integrator dynamics has been also well studied in (Wang and Hong, 2008; Wang and Feng, 2010; Cao et al., 2011; Li et al., 2011, 2012; Zhao et al., 2013). Recently, the FT consensus problems have been investigated for multiple EL-dynamics in (Mei et al., 2011; Ren, 2010; Chen et al., 2013), respectively. These last works have employed sliding-mode control techniques.

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In this paper, inspired on the energy shaping plus damping injection methodology (Ortega et al., 1998), we propose a framework to design consensus (continuous) controllers for EL-systems in finite-time. The energies of the system and the controller are added to make the resulting total energy a suitable Lyapunov function and damping is added to achieve asymptotic stability. The controller's potential and dissipative-like energies are imposed to satisfy some properties of (weighted) homogeneous functions in order to assure the finite-time convergence of the closed-loop system trajectories. The main purpose of this work is a first step to extend (and unify) some existing results on consensus algorithms applied to solve the distributed finite-time control problem with multiple EL dynamics (Mei et al., 2011; Ren, 2010; Chen et al., 2013).

1.1 Background

Throughout the paper, the following **notation** is employed. $\mathbb{R} := (-\infty,\infty)$, $\mathbb{R}_+ := (0,\infty)$, $\mathbb{R}_{\geq 0} := [0,\infty)$, $\mathbb{N}_+ := \{1,2,3,\dots\}$ and $\bar{m} := \{1,2,...,m\}$ for $m \in \mathbb{N}$. For any $m \in \mathbb{N}_+$, $B_\delta = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| < \delta\}$ is an open ball, centered at the origin with radius $\delta \in \mathbb{R}_+$; and $S_c^{m-1} := \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| = c\}$ is a $m-1$ sphere with radius $c \in \mathbb{R}_+$. A function $\mathbf{f}(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^m$ is said to be of class $\mathscr{C}^k$, for $k \in \mathbb{N}_+$, if its derivatives $\dot{\mathbf{f}}, \ddot{\mathbf{f}}, ..., \mathbf{f}^{(k)}$ exist and are continuous. For any $\mathbf{x} \in \mathbb{R}^m$, $\nabla_{\mathbf{x}} := [\partial_{x_1}, \dots, \partial_{x_m}]^\top$ stands for the gradient operator of a scalar function and $\nabla^2_{\mathbf{x}} := \left[\partial_{x_i}\partial_{x_j}\right]$ is the Hessian operator where $\partial_{x_i} := \frac{\partial}{\partial x_i}$ and $i, j \in \bar{m}$.

Definition 1. Consider a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$. Assume that $\mathbf{x} = \mathbf{0}$ is an equilibrium point, i.e., $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. The point $\mathbf{x} = \mathbf{0}$ is said to be Finite-Time Stable (FTS) if it is Lyapunov stable and there exists a locally bounded function $T : B_R \mapsto \mathbb{R}_{\geq 0}$ such that for each $\mathbf{x}_0 \in B_R \setminus \{\mathbf{0}\}$, any solution $\mathbf{x}(t, \mathbf{x}_0)$ of (1) is defined on $t \in [0, T(\mathbf{x}_0))$ and $\mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}$ for all $t \geq T(\mathbf{x}_0)$. T is the settling-time function. In addition, if $B_R = \mathbb{R}^n$, $\mathbf{x} = \mathbf{0}$ is globally finite-time stable (GFTS) (Bhat and Bernstein, 2000, 2005).

The finite-time convergence analysis usually relies on the properties of homogeneous systems. We recall important results of the theory of such systems. Let $r_i > 0$, $i \in \bar{m}$, be the weights of the elements x_i of $\mathbf{x} \in \mathbb{R}^m$ and define the vector of weights as $\mathbf{r} := [r_1, ..., r_m]^\top \in \mathbb{R}^m$. Let $\Delta_\varepsilon^{\mathbf{r}}$ be the dilation operator such that $\Delta_\varepsilon^{\mathbf{r}}\mathbf{x} := [\varepsilon^{r_1}x_1, ..., \varepsilon^{r_m}x_m]^\top$. A function $V : \mathbb{R}^m \mapsto \mathbb{R}$ (resp. a vector field $\mathbf{f} : \mathbb{R}^m \mapsto \mathbb{R}^m$) is said to be $\mathbf{r}$-homogeneous of degree $l \in \mathbb{R}$, or $(\mathbf{r}, l)$-homogeneous for short, if for all $\varepsilon \in \mathbb{R}_+$ and for all $\mathbf{x} \in \mathbb{R}^m$ the equality $V(\Delta_\varepsilon^{\mathbf{r}}\mathbf{x}) = \varepsilon^l V(\mathbf{x})$ (resp., $\mathbf{f}(\Delta_\varepsilon^{\mathbf{r}}\mathbf{x}) = \varepsilon^l \Delta_\varepsilon^{\mathbf{r}}\mathbf{f}(\mathbf{x})$) holds. System (1) is $\mathbf{r}$-homogeneous if the vector field $\mathbf{f}$ is $\mathbf{r}$-homogeneous.

Definition 2. For any $p \geq 1$ and each $\mathbf{x} \in \mathbb{R}^m$, let $\|\mathbf{x}\|_{\mathbf{r},p} := \left(\sum_{i=1}^{m} |x_i|^{p/r_i}\right)^{1/p}$ be the $\mathbf{r}$-*homogeneous norm*. The origin of (1) is h-Exponentially Stable (h-ES) if there exist $\alpha, \beta, \delta > 0$ such that for any $\mathbf{x}_0 \in B_\delta$, all $t \geq 0$ and $p \geq 1$, the solutions $\mathbf{x}(t, \mathbf{x}_0)$ of (1) satisfy $\|\mathbf{x}(t, \mathbf{x}_0)\|_{\mathbf{r},p} \leq \alpha e^{-\beta t}\|\mathbf{x}_0\|_{\mathbf{r},p}$. △

Plenty nonlinear systems are non-homogeneous. However, as it occurs in the linearization approach, homogeneous approximations are used to study the stability of their equilibria (Hong et al., 2002; Bacciotti and Rosier, 2005; Orlov, 2009; Zavala-Río and Fantoni, 2014; Zavala-Río and Zamora-Gómez, 2016).

Lemma 1. (Bacciotti and Rosier, 2005) Consider system (1) with $\mathbf{f}(\mathbf{x}) = \mathbf{f}_H(\mathbf{x}) + \mathbf{f}_{NH}(\mathbf{x})$. Suppose that [1] $\mathbf{f}_H(\mathbf{x})$ is an $(\mathbf{r}, l)$-homogeneous continuous vector field such that $\mathbf{f}_H(\mathbf{0}) = \mathbf{0}$ is a locally Asymptotically Stable (AS) equilibrium point. Assume that there exists $c \in \mathbb{R}_+$ and that $\mathbf{f}_{NH}(\mathbf{x})$ is a continuous vector field such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-(l+r_i)} f_{NH_i}(\Delta_\varepsilon^{\mathbf{r}}\mathbf{x}) = 0, \quad \forall i \in \bar{m}, \tag{2}$$

uniformly with respect to (w.r.t.) $\mathbf{x} \in S_c^{m-1}$. Then, the origin is locally AS. Furthermore, (i) if $l = 0$, the origin is locally h-ES; and if $l < 0$, the origin is locally FTS. △

Direct consequences of Lemma 1 are: (a) if condition (2) is fulfilled for all $\varepsilon > 0$ and $\mathbf{x} = \mathbf{0}$ is globally AS (GAS), the statements (i) and (ii) become global; (b) if condition (2) is fulfilled for some $\varepsilon > 0$ arbitrarily small, $\mathbf{x} = \mathbf{0}$ is GAS and $l < 0$, the origin becomes GFTS, (Hong et al., 2002; Bacciotti and Rosier, 2005).

2. DYNAMIC MODEL AND CONTROL OBJECTIVE

Consider a network of N, *fully-actuated* n–DoF, EL-systems of the form $\frac{d}{dt}\left(\nabla_{\dot{\mathbf{q}}_i}\mathfrak{L}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\right) - \nabla_{\mathbf{q}_i}\mathfrak{L}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) = \tau_i$, where $\mathfrak{L}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$ is the Lagrangian that is defined as $\mathfrak{L}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) = \mathscr{K}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) - \mathscr{U}_i(\mathbf{q}_i)$, with ${}^s\mathscr{K}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) := \frac{1}{2}\dot{\mathbf{q}}_i^\top \mathbf{M}_i(\mathbf{q}_i)\dot{\mathbf{q}}_i$ the kinetic energy and ${}^s\mathscr{U}_i(\mathbf{q}_i)$ the potential energy. $\mathbf{q}_i, \dot{\mathbf{q}}_i \in \mathbb{R}^n$ are the generalized position and velocity, respectively, $\mathbf{M}_i(\mathbf{q}_i) \in \mathbb{R}^{n\times n}$ is the generalized inertia matrix, which is positive definite and bounded, and $\tau_i \in \mathbb{R}^n$ is the vector of external forces.

The EL-equations of motion of each agent can be written as

$$\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i + \nabla_{\mathbf{q}_i}{}^s\mathscr{U}_i(\mathbf{q}_i) = \tau_i \tag{3}$$

where $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal forces matrix, defined via the Christoffel symbols of the first kind.

[1] The vector field $\mathbf{f}_H(\mathbf{x})$ is known as the $\mathbf{r}$-homogeneous approximation of $\mathbf{f}(\mathbf{x})$. Similarly, an $\mathbf{r}$-homogeneous function $V_H : \mathbb{R}^m \mapsto \mathbb{R}$ is said to be $\mathbf{r}$-homogeneous approximation of $V : \mathbb{R}^m \mapsto \mathbb{R}$ if there exists $V_{NH} : \mathbb{R}^m \mapsto \mathbb{R}$ such that $V = V_H + V_{NH}$ and $\lim_{\varepsilon \mapsto 0} \varepsilon^{-l} V_{NH}(\Delta_\varepsilon^{\mathbf{r}}\mathbf{x}) = 0$ uniformly w.r.t. $\mathbf{x} \in S_c^{m-1}$ (Sepulchre and Aeyels, 1996; Andrieu et al., 2008).

We focus on (3) satisfying the assumptions for each i:

A1. There exist strictly positive constants m_1 and m_2, such that $m_1\mathbf{I}_n \leq \mathbf{M}_i(\mathbf{q}_i) \leq m_2\mathbf{I}_n$. Furthermore, there exists $L_m > 0$ such that $\|\mathbf{M}_i^{-1}(\mathbf{q}_1) - \mathbf{M}_i^{-1}(\mathbf{q}_2)\| \leq L_m\|\mathbf{q}_1 - \mathbf{q}_2\|$, $\forall \mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^n$. ◁

A2. The potential energy $\mathscr{U}_i(\mathbf{q}) \in \mathscr{C}^2$ is bounded from below. Furthermore, for all $\mathbf{q} \in \mathbb{R}^n$, there exists $k_g, L_g > 0$, such that $\|\nabla_{\mathbf{q}_i}\mathscr{U}_i(\mathbf{q}_i)\| \leq k_g$ and $\left\|\nabla_{\mathbf{q}_i}^2\mathscr{U}_i(\mathbf{q}_i)\right\| \leq L_g$. ◁

Assumption **A2** implies that there exist constants $k_{gi} > 0$ such that $|\partial_{q_i}\mathscr{U}_i(\mathbf{q}_i)| \leq k_{gi}$, for all $i \in \bar{n}$, where $\partial_{q_i}\mathscr{U}_i(\mathbf{q}_i)$ is the ith-element of $\nabla_{\mathbf{q}_i}\mathscr{U}_i(\mathbf{q}_i)$.

Also, (3) has the following fundamental properties (Kelly et al., 2005):

P1. Matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric and there exists $L_c > 0$ such that $\|\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i\| \leq L_c\|\dot{\mathbf{q}}_i\|^2$. ◁

Define the vectors $\mathbf{q} := \mathrm{col}(\mathbf{q}_i)$ and $\tau := \mathrm{col}(\tau_i)$, $i \in \bar{N} := \{1, \ldots, N\}$. Then, the Hamiltonian (total energy) function of the complete N EL-systems is given by ${}^s\mathscr{H}(\mathbf{q}, \dot{\mathbf{q}}) = {}^s\mathscr{K}(\mathbf{q}, \dot{\mathbf{q}}) + {}^s\mathscr{U}(\mathbf{q})$, where ${}^s\mathscr{K}(\mathbf{q}, \dot{\mathbf{q}}) := \sum_{i\in\bar{N}} \mathscr{K}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$, ${}^s\mathscr{U}(\mathbf{q}) := \sum_{i\in\bar{N}} \mathscr{U}_i(\mathbf{q}_i)$, are the total kinetic and potential energies, respectively. The dynamics of the overall system can be compactly written as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \nabla_{\mathbf{q}}{}^s\mathscr{U}(\mathbf{q}) = \tau. \tag{4}$$

where we defined the overall inertia and Coriolis and centrifugal forces matrices as $\mathbf{M}(\mathbf{q}) := \mathrm{blockdiag}\{\mathbf{M}_i(\mathbf{q}_i)\}$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) := \mathrm{blockdiag}\{\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\}$.

System (4) verifies the following input–output property (Ortega et al., 1998).

Fact 1. System (4) defines a passive operator $\Sigma_s : \tau \to \dot{\mathbf{q}}$ with storage function ${}^s\mathscr{H}(\mathbf{q}, \dot{\mathbf{q}})$. More precisely, ${}^s\dot{\mathscr{H}}(\mathbf{q}, \dot{\mathbf{q}}) = \tau^\top\dot{\mathbf{q}}$. ◇

We use graphs to represent the communication topology among agents. A weighted graph consists of a set of nodes $\mathscr{V} = \{1, ..., N\}$, an edge set $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $\mathbf{A}_N = [a_{ij}] \in \mathbb{R}^{N\times N}$. A weighted undirected graph is defined such that $(j, i) \in \mathscr{E}$ implies $(i, j) \in \mathscr{E}$, clearly the edge (i, j) denotes that agents i and j can obtain information from one another. An undirected graph is connected if there is an undirected path between every pair of distinct nodes. The neighbors of node i are defined as the set $\mathscr{N}_i := \{i | (i, j) \in \mathscr{E}\}$, that is, each set $\mathscr{N}_i \subset \bar{N}$ contains the set of agents transmitting information to the ith agent. The weighted adjacency matrix $\mathbf{A}_N$ of a weighted undirected graph satisfies $a_{ij} = a_{ji} > 0$ and $a_{ii} = 0$ if $(i, j) \in \mathscr{E}$, for all $i, j \in \bar{N}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathbf{L} := \{L_{ij}\} \in \mathbb{R}^{N\times N}$ associated with $\mathbf{A}_N$ is defined as $L_{ii} = \sum_{j\in\mathscr{N}_i} a_{ij}$ and $L_{ij} = -a_{ij}$ for $i \neq j$. By construction, $\mathbf{L}$ has a zero row sum. For an undirected and connected graph, $\mathbf{L}$ is symmetric positive semidefinite with a single zero-eigenvalue (with the associated eigenvector [2] $\mathbf{1}_N$) and all of the other eigenvalues are strictly positive.

Throughout the paper, we assume that there are not time-delays in the interconnection of the EL-agents. Also, these agents exchange information according to the following assumption:

[2] $\mathbf{1}_N$ is a column vector with N components equal to one. Thus, $\mathrm{rank}(\mathbf{L}) = N - 1$. Therefore, exists $\alpha \in \mathbb{R}$ such that $\ker(\mathbf{L}) = \alpha\mathbf{1}_N$.

A3. The EL-agents interconnection graph is *undirected and connected*.

Our control objective is as follows:

Consider a network of N EL-systems of the form (3). Assume that velocities are *available* for measurement and suppose that the interconnection graph fulfills Assumption **A3**. Design a *decentralized* controller to solve the following two consensus problems:

(FTLC) Finite-Time Leaderless Consensus Problem. The network has to reach a consensus position in finite-time. That is, *there exists* a constant $\mathbf{q}_c \in \mathbb{R}^n$ such that, for all $i \in \bar{N}$,

$$\lim_{t \to T(\mathbf{q}(0),\dot{\mathbf{q}}(0))} |\dot{\mathbf{q}}_i(t)| = 0, \qquad \lim_{t \to T(\mathbf{q}(0),\dot{\mathbf{q}}(0))} \mathbf{q}_i(t) = \mathbf{q}_c, \quad (5)$$

(FTLFC) Finite-Time Leader-Follower Consensus Problem. Given an *extra leader* node, labeled ℓ, the network has to be regulated at the leader's constant position $\mathbf{q}_\ell \in \mathbb{R}^n$, whose value is *not available* to all agents, but only to a nonempty subset of them. That is, for all $i \in \bar{N}$, [3]

$$\lim_{t \to T(\mathbf{q}(0),\dot{\mathbf{q}}(0))} |\dot{\mathbf{q}}_i(t)| = 0, \qquad \lim_{t \to T(\mathbf{q}(0),\dot{\mathbf{q}}(0))} \mathbf{q}_i(t) = \mathbf{q}_\ell, \quad (6)$$

regardless of the initial conditions $\mathbf{q}_0 = \mathbf{q}(0), \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}(0) \in \mathbb{R}^n$. It means that the joint position $\mathbf{q}(t)$ tends to the consensus point at some finite-time moment T.

3. CONTROL DESIGN FOR THE **FTLC** PROBLEM

Let $\delta\mathbf{q}$ be the a column stack vector of all $\mathbf{q}_i - \mathbf{q}_j$ and let η be its corresponding dimension. Let $\mathscr{U}_a(\mathbf{q}) := \mathscr{U}_c(\delta\mathbf{q}) - {}^s\mathscr{U}(\mathbf{q})$ be the *artificial* potential energy induced by the controller, then we define the *desired* (total) potential energy as

$$\mathscr{U}_d(\mathbf{q}) := \mathscr{U}_a(\mathbf{q}) + {}^s\mathscr{U}(\mathbf{q}) = \mathscr{U}_c(\delta\mathbf{q}). \quad (7)$$

Function $\mathscr{U}_c(\delta\mathbf{q})$ can be understood as the (elastic) potential energy imposed to the interconnection between the agents i and j. In fact, it is given by $\mathscr{U}_c(\delta\mathbf{q}) := \sum_{i=1}^N \sum_{j=1}^N a_{ij}\mathscr{U}_{ij}(\delta\mathbf{q}_{ij})$, where $\delta\mathbf{q}_{ij} := \mathbf{q}_i - \mathbf{q}_j$ and a_{ij} is the (i,j)th entry of the adjacency matrix associated with the undirected graph.

Now, let us define the *desired total* energy as $\mathscr{H}_d(\mathbf{q},\dot{\mathbf{q}}) := {}^s\mathscr{H}(\mathbf{q},\dot{\mathbf{q}}) + \mathscr{U}_a(\mathbf{q})$. From Fact 1, we obtain that $\dot{\mathscr{H}}_d(\mathbf{q},\dot{\mathbf{q}}) = \tau^\top\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top\nabla_{\mathbf{q}}\mathscr{U}_a(\mathbf{q})$. Hence, designing the controller to have the following form

$$\tau = -\nabla_{\delta\mathbf{q}}\mathscr{U}_a(\mathbf{q}) - \nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}}), \quad (8)$$

ensures that $\dot{\mathscr{H}}_d(\mathbf{q},\dot{\mathbf{q}}) = -\dot{\mathbf{q}}^\top\nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}})$, where $\mathscr{F}_a : \mathbb{R}^{nN} \mapsto \mathbb{R}$ represents the *energy dissipation* function (which gives rise to the damping injection term) (Ortega et al., 1998).

Let $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{1}_n \otimes \mathbf{q}_c$ be the consensus position error vector. Although $\mathbf{q}_c$ is an unknown point it is constant, that is, $\dot{\mathbf{q}}_c = 0$. Note that the fact $\tilde{\mathbf{q}} = \mathbf{0} \Leftrightarrow \mathbf{q}_i = \mathbf{q}_j$ (or $\delta\mathbf{q} = \mathbf{0}$) comes true for all i and j. Since $\tilde{\mathbf{q}}_{ij} := \tilde{\mathbf{q}}_i - \tilde{\mathbf{q}}_j = \mathbf{q}_i - \mathbf{q}_j$, the CL system (4) and (8) can be rewritten as

$$\begin{aligned}\dot{\tilde{\mathbf{q}}} =& \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} =& -\mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{1}_n \otimes \mathbf{q}_c)\mathbf{C}(\tilde{\mathbf{q}} + \mathbf{1}_n \otimes \mathbf{q}_c, \dot{\mathbf{q}})\dot{\mathbf{q}} \\ & -\mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{1}_n \otimes \mathbf{q}_c)\left[\nabla_{\tilde{\mathbf{q}}}\mathscr{U}_c(\tilde{\mathbf{q}}) + \nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}})\right].\end{aligned} \quad (9)$$

where $\nabla_{\tilde{\mathbf{q}}}\mathscr{U}_c(\tilde{\mathbf{q}}) := col\left(\sum_{j=1}^N a_{ij}\nabla_{\tilde{\mathbf{q}}_{ij}}\mathscr{U}_{ij}(\tilde{\mathbf{q}}_{ij})\right)$, $i \in \bar{N}$.

[3] We stress the fact that the leader *is not* a moving agent of the network, but a virtual *constant* position. To the best of our knowledge, the former is a much harder problem that stills open.

The equilibrium point $(\tilde{\mathbf{q}},\dot{\mathbf{q}}) = (\mathbf{0},\mathbf{0})$ of (9) will be GAS by satisfying two conditions: i) the desired energy $\mathscr{H}_d(\mathbf{q},\dot{\mathbf{q}})$ has a global minimum at $(\mathbf{q},\dot{\mathbf{q}}) = (\mathbf{1}_n \otimes \mathbf{q}_c, \mathbf{0}_{nN})$ (which implies $(\tilde{\mathbf{q}},\dot{\mathbf{q}}) = (\mathbf{0}_{nN},\mathbf{0}_{nN})$); and ii) the dissipation function entails the following properties $\nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}}) = \mathbf{0} \Leftrightarrow \dot{\mathbf{q}} = \mathbf{0}$ and $\dot{\mathbf{q}}^\top\nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}}) > 0$ for all $\dot{\mathbf{q}} \neq \mathbf{0}$.

Let $\mathbf{r}_{\tilde{\mathbf{q}}} = r_1\mathbf{1}_n \otimes \mathbf{1}_N$ and $\mathbf{r}_{\dot{\mathbf{q}}} = r_2\mathbf{1}_n \otimes \mathbf{1}_N$ ($r_1, r_2 \in \mathbb{R}_+$) be the homogeneity weights of the vectors $\tilde{\mathbf{q}}$ and $\dot{\mathbf{q}}$, respectively. [4] Also, let $\delta\tilde{\mathbf{q}}$ be the a column stack vector of all $\tilde{\mathbf{q}}_i - \tilde{\mathbf{q}}_j$.

A4. For each $r_1, r_2 \in \mathbb{R}_+$ such that $2r_2 > r_1 > r_2 > 0$
(a) The potential energy $\mathscr{U}_c(\delta\mathbf{q}) = \mathscr{U}_c(\delta\tilde{\mathbf{q}})$ satisfies

(i) ${}^s\mathscr{U}_c(\mathbf{0}) = 0$ and $\mathscr{U}_c(\delta\tilde{\mathbf{q}}) \geq \begin{cases} \beta_1\|\delta\tilde{\mathbf{q}}\|^{\frac{2r_2}{r_1}} & \text{if } \|\delta\tilde{\mathbf{q}}\| < \delta_{\mathscr{U}}, \\ \beta_2\|\delta\tilde{\mathbf{q}}\| & \text{if } \|\delta\tilde{\mathbf{q}}\| \geq \delta_{\mathscr{U}}, \end{cases}$

(ii) $\delta\tilde{\mathbf{q}}^\top\nabla_{\delta\tilde{\mathbf{q}}}\mathscr{U}_c(\delta\tilde{\mathbf{q}}) \geq \begin{cases} \beta_1'\|\delta\tilde{\mathbf{q}}\|^{\frac{2r_2}{r_1}} & \text{if } \|\delta\tilde{\mathbf{q}}\| < \delta_{\mathscr{U}}', \\ \beta_2'\|\delta\tilde{\mathbf{q}}\| & \text{if } \|\delta\tilde{\mathbf{q}}\| \geq \delta_{\mathscr{U}}', \end{cases}$

(iii) $\mathscr{U}_c(\delta\tilde{\mathbf{q}})$ splits as $\mathscr{U}_c(\delta\tilde{\mathbf{q}}) = \mathscr{U}_c^H(\tilde{\mathbf{q}}) + \mathscr{U}_c^{NH}(\delta\tilde{\mathbf{q}})$, where $\mathscr{U}_c^H(\tilde{\mathbf{q}})$ is $(\mathbf{r}, 2r_2)$-homogeneous, $\mathscr{U}_c^H(\mathbf{0}) = \mathscr{U}_c^{NH}(\mathbf{0}) = 0$ and

$$\lim_{\varepsilon \to 0} \varepsilon^{-2r_2}\left\|\mathscr{U}_c^{NH}(\varepsilon^{r_1}\delta\tilde{\mathbf{q}})\right\| = 0, \quad \forall \delta\tilde{\mathbf{q}} \in S_\delta^{\eta-1} \quad (10)$$

for some constants $\beta_1, \beta_2, \beta_1', \beta_2', \delta_{\mathscr{U}}, \delta_{\mathscr{U}}' \in \mathbb{R}_+$ and for any $\delta < \delta_{\mathscr{U}}$.
(b) The energy dissipation function $\mathscr{F} : \mathbb{R}^{nN} \mapsto \mathbb{R}$ satisfies

(iv) $\mathscr{F}_a(\mathbf{0}) = 0$ and $\mathscr{F}_a(\dot{\mathbf{q}}) \geq \begin{cases} \kappa_1\|\dot{\mathbf{q}}\|^{\frac{3r_2 - r_1}{r_2}} & \text{if } \|\dot{\mathbf{q}}\| < \delta_{\mathscr{F}}, \\ \kappa_2\|\dot{\mathbf{q}}\| & \text{if } \|\dot{\mathbf{q}}\| \geq \delta_{\mathscr{F}}, \end{cases}$

(v) $\dot{\mathbf{q}}^\top\nabla_{\dot{\mathbf{q}}}\mathscr{F}_a(\dot{\mathbf{q}}) \geq \begin{cases} \kappa_1'\|\dot{\mathbf{q}}\|^{\frac{3r_2 - r_1}{r_2}} & \text{if } \|\mathbf{q}\| < \delta_{\mathscr{F}}', \\ \kappa_2'\|\dot{\mathbf{q}}\| & \text{if } \|\dot{\mathbf{q}}\| \geq \delta_{\mathscr{F}}', \end{cases}$

(vi) $\mathscr{F}_a(\dot{\mathbf{q}})$ splits as $\mathscr{F}_a(\dot{\mathbf{q}}) := \mathscr{F}_a^H(\dot{\mathbf{q}}) + \mathscr{F}_a^{NH}(\dot{\mathbf{q}})$, where $\mathscr{F}_a^H(\dot{\mathbf{q}})$ is $(\mathbf{r}, 3r_2 - r_1)$-homogeneous, $\mathscr{F}_a^H(\mathbf{0}) = \mathscr{F}_a^{NH}(\mathbf{0}) = 0$ and

$$\lim_{\varepsilon \to 0} \varepsilon^{-(3r_2 - r_1)}\left\|\mathscr{F}_a^{NH}(\varepsilon^{r_2}\dot{\mathbf{q}})\right\| = 0, \quad \forall \dot{\mathbf{q}} \in S_\delta^{nN-1} \quad (11)$$

for some constants $\kappa_1, \kappa_2, \kappa_1', \kappa_2', \delta_{\mathscr{F}}, \delta_{\mathscr{F}}' \in \mathbb{R}_+$ and any $\delta < \delta_{\mathscr{F}}$.

Some remarks about **A4** are in order:

R1. For any $2r_2 > r_1 > r_2$, conditions above on $\mathscr{U}_d(\tilde{\mathbf{q}})$ and $\mathscr{F}_a(\dot{\mathbf{q}})$ characterize the class of controllers (8) derived from energy-like functions and which solve the consensus problems in finite-time. The case $r_2 = r_1$ gives rise to the conditions for global asymptotic stability but with local exponential stability behaviour.

R2. Radially unboundedness and positive-definiteness of the term $\mathscr{U}_d(\tilde{\mathbf{q}})$ w.r.t. $\tilde{\mathbf{q}}$ is supported by ***(i)***. While ***(ii)*** ensures that $\tilde{\mathbf{q}} = \mathbf{0}$ is a unique and global minimum. Analogue conclusions can be drawn for the energy dissipation function.

R3. **A4** includes total potential and dissipation energy-like functions having bounded gradients for all robot positions and velocities. It implies that for some $k_u, k_f \in \mathbb{R}_+$

$$\sup_{\tilde{\mathbf{q}}_{ij} \in \mathbb{R}^n} \left\|\nabla_{\tilde{\mathbf{q}}_{ij}}\mathscr{U}_c(\tilde{\mathbf{q}}_{ij})\right\| < k_u, \ \sup_{\dot{\mathbf{q}}_i \in \mathbb{R}^n} \left\|\nabla_{\dot{\mathbf{q}}_i}\mathscr{F}_a(\dot{\mathbf{q}}_i)\right\| < k_f, \quad (12)$$

This class induces saturated position and velocity error feedback providing a wide range of FT controllers that globally

[4] It clearly implies that $\mathbf{r}_{\tilde{\mathbf{q}}_i} = r_1\mathbf{1}_n$ and $\mathbf{r}_{\dot{\mathbf{q}}_i} = r_2\mathbf{1}_n$ are the weights of $\mathbf{q}_i$ and $\dot{\mathbf{q}}_i$, respectively.

solves the consensus problems with bounded torque actions. Recall that in this case, each joint actuator is only able to supply a known maximum torque $\bar{\tau}_{ij} \in \mathbb{R}_+$ such that

$$|\tau_{ij}| \leq \bar{\tau}_{ij}, \quad j \in \{1, ..., n\}, i = \bar{N}, \tag{13}$$

where τ_{ij} stands for the ijth-element of vector τ.

At this point we are ready to state our first result.

Proposition 1. Consider the network of EL-agents (4) with the interconnection graph verifying Assumptions **A1–A4**. Controller (8) solves the desired control objective **FTLC** problem if r_1, r_2 are set as $r_1 > r_2 > 0$. Moreover, if $r_1 = r_2 > 0$, then the origin of (9) is GAS and locally h-ES. $\diamond$

Proof. (**Sketch**) Assumption **A4.b** guarantees that $\dot{\mathbf{q}}^\top \nabla_{\dot{\mathbf{q}}} \mathscr{F}_a(\dot{\mathbf{q}})$ is strictly positive. Hence $\dot{\mathscr{H}}_d = -\dot{\mathbf{q}}^\top \nabla_{\dot{\mathbf{q}}} \mathscr{F}_a(\dot{\mathbf{q}}) \leq 0$. It is not difficult to see that $\tilde{\mathbf{q}} = 0$ implies $-\nabla_{\tilde{\mathbf{q}}} \mathscr{U}_c(\tilde{\mathbf{q}}) = \mathbf{0}$. Then, by Krasovskii-LaSalle's Invariance principle, it follows that $(\tilde{\mathbf{q}}, \dot{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is a GAS equilibrium of (9). Global FT stability of such equilibrium point is concluded by showing that the CL system (9) admits a homogeneous approximation of negative degree. For sake of space such part of the proof is omitted here. $\triangleleft$

Corollary 1. Proposition 1 holds with bounded artificial potential energy and bounded dissipation functions if, additional to Assumptions **A1**–**A4**, condition (12) is ensured and the maximum torques $\overline{\tau}_i$ satisfy the condition $\overline{\tau}_i \geq k_{gi} + k_{ui} + k_{fi}$, where k_{gi} is given below **A2**. $\triangle$

4. CONTROL DESIGN FOR THE **FTLFC** PROBLEM

Given a desired constant position $\mathbf{q}_\ell$ of a stationary leader, the control goal is to guarantee the followers positions to track those of the leader in finite-time. We make the following assumption for the leader-follower interconnection.

A5. There is a non-empty set of follower agents that has direct access to the leader's desired position $\mathbf{q}_\ell$, *i.e.*, in the graph of $N+1$ nodes, being node 0 the leader node, there exists at least one path from the leader to any of the N followers.

Following similar steps as above, we obtain the control law

$$\tau = -\nabla_{\tilde{\mathbf{q}}} \mathscr{U}_c(\tilde{\mathbf{q}}) - \nabla_{\dot{\mathbf{q}}} \mathscr{F}_a(\dot{\mathbf{q}}), \tag{14}$$

where $\nabla_{\tilde{\mathbf{q}}} \mathscr{U}_c(\tilde{\mathbf{q}}) := col\left(\sum_{j=\ell}^{N} a_{ij} \nabla_{\tilde{\mathbf{q}}_{ij}} \mathscr{U}_{ij}(\tilde{\mathbf{q}}_{ij})\right)$, $\tilde{\mathbf{q}}_{ij} := \tilde{\mathbf{q}}_i - \tilde{\mathbf{q}}_j = \mathbf{q}_i - \mathbf{q}_j$, $i \in \{\ell, \bar{N}\}$.

In this case, $\tilde{\mathbf{q}}$ is a column stack vector of $\mathbf{q}_i - \mathbf{q}_\ell$, $i \in N$, or equivalently $\mathbf{q} = \tilde{\mathbf{q}} + \mathbf{1}_n \otimes \mathbf{q}_\ell$. Assumptions **A3** and **A5** ensure that the leader is *globally reachable* from any of the N follower nodes. The proof of the next result follows *verbatim* the proof of Proposition 1 and thus is omitted for sake of space.

Proposition 2. Consider the network of EL-agents (4) with the interconnection graph verifying Assumptions **A1–A5**. The controller (14) solves the **FTLFC** problem. $\diamond$

5. CONTROL EXAMPLES

Here, we derive some controller examples. For that, we make use of the next functions.

Definition 3. A power-sign function $\lceil x \rfloor^p : \mathbb{R} \mapsto \mathbb{R}$ is a strictly increasing odd function given by $\lceil x \rfloor^p := |x|^p \mathrm{sign}(x)$, for any $x \in \mathbb{R}$ and any $p \in \mathbb{R}_+$, where $\mathrm{sign}(x)$ is the standard *sign* function. $\triangle$

Function $\lceil x \rfloor^p : \mathbb{R} \mapsto \mathbb{R}$ is continuous everywhere. Further, for any $\mathbf{z} \in \mathbb{R}^n$ we define $\lceil \mathbf{z} \rfloor^p := [\lceil z_1 \rfloor^p, ..., \lceil z_n \rfloor^p]^\top$ and $sat_\varepsilon(\lceil \mathbf{z} \rfloor^p) := [sat_\varepsilon(\lceil z_1 \rfloor^p), ..., sat_\varepsilon(\lceil z_n \rfloor^p)]^\top$, where

$$sat_\varepsilon(\lceil z \rfloor^p) := \begin{cases} \lceil z \rfloor^p & \text{if } |z| < \varepsilon, \\ \varepsilon^p \,\mathrm{sign}(z) & \text{if } |z| \geq \varepsilon, \end{cases}$$

Notice that for any $m, n > 0$, we have

(1) $\lceil z \rfloor^m = z^m$ for any odd integer m and $|z|^m = z^m$ for any even integer m.
(2) $\lceil z \rfloor^m \lceil z \rfloor^n = |z|^{m+n}$.
(3) $\partial_z \lceil z \rfloor^m = m|z|^{m-1}$ and $\partial_z |z|^m = m\lceil z \rfloor^{m-1}$.
(4) $\int_0^x sat_\delta(\lceil z \rfloor^p) dz = s(x)$, for all $x \in \mathbb{R}$ and $p, \delta > 0$, where

$$s(x) := \begin{cases} \dfrac{1}{p+1} |x|^{p+1} & \text{if } |x| < \delta, \\ \delta^p |x| - \dfrac{p}{p+1} \delta^{p+1} & \text{if } |x| \geq \delta. \end{cases} \tag{15}$$

Note that $\delta^p |x| - \frac{p}{p+1} \delta^{p+1} \geq \frac{1}{p+1} \delta^p |x|$ for all $|x| \geq \delta$.

Lemma 2. (Cao and Ren, 2011) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous odd function satisfying $\varphi(x) > 0$ if $x > 0$. Define $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{P} \in \mathbb{R}^{m \times m}$, and $\mathbf{C} = [c_{ij}] \in \mathbb{R}^{m \times m}$. If $\mathbf{C}$ is symmetric, then

$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} (\mathbf{y}_i - \mathbf{y}_j)^\top \varphi(\mathbf{x}_i - \mathbf{x}_j) = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \mathbf{y}_i^\top \varphi(\mathbf{x}_i - \mathbf{x}_j).$$

For all the upcoming examples, we set $2r_2 > r_1 > r_2 > 0$ and $\mathbf{P} := \mathrm{blockdiag}\{\mathbf{P}_i\}$, $\mathbf{P}_\ell := \mathrm{blockdiag}\{\mathbf{P}_{\ell i}\}$, $\mathbf{D} := \mathrm{blockdiag}\{\mathbf{D}_i\}$, $\mathbf{D}_\ell := \mathrm{blockdiag}\{\mathbf{D}_{\ell i}\}$, with $\mathbf{P}_i$, $\mathbf{P}_{\ell i}$, $\mathbf{D}_i$ and $\mathbf{D}_{\ell i}$ being diagonal positive-definite matrices. Set the dissipation function as

$$\mathscr{F}(\dot{\mathbf{q}}) = \frac{r_2}{3r_2 - r_1} \dot{\mathbf{q}}^\top \mathbf{D} \lceil \dot{\mathbf{q}} \rfloor^{\frac{2r_2 - r_1}{r_2}} + \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{D}_l \dot{\mathbf{q}}, \tag{16}$$

which implies $\kappa = \min_i \{D_i\}$.

Finite-time PD control: For the **FTLC** problem, choose $\mathscr{U}_c(\tilde{\mathbf{q}})$ as

$$\mathscr{U}_c(\delta \mathbf{q}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=\mathscr{N}_i} a_{ij} \frac{r_1}{2r_2} \mathbf{1}_n^\top \mathbf{P}_i |\mathbf{q}_i - \mathbf{q}_j|^{\frac{2r_2}{r_1}} + \mathbf{q}^\top (\mathbf{P}_l \mathbf{L} \otimes \mathbf{I}) \mathbf{q},$$

thus ${}^s\mathscr{U}_d(\delta \mathbf{q}) = \mathscr{U}_c(\delta \mathbf{q})$, and (8) becomes the PD control

$$\begin{aligned} \tau_i = \quad & \nabla_{\mathbf{q}_i} {}^s\mathscr{U}_i(\mathbf{q}_i) - \sum_{j \in \mathscr{N}_i} a_{ij} \mathbf{P}_i \lceil \mathbf{q}_i - \mathbf{q}_j \rfloor^{\frac{2r_2 - r_1}{r_1}} \\ & - \sum_{j \in \mathscr{N}_i} a_{ij} \mathbf{P}_{li} (\mathbf{q}_i - \mathbf{q}_j) - \mathbf{D}_i \lceil \dot{\mathbf{q}} \rfloor^{\frac{2r_2 - r_1}{r_2}} - \mathbf{D}_{li} \dot{\mathbf{q}}_i \end{aligned}$$

The total potential energy satisfies conditions on **A4.(a)** with $\varepsilon = \varepsilon' > 0$, $\beta_1 = \beta_2 = \frac{r_1}{2r_2} \underline{\lambda}\{\mathbf{P}\}$, and $\beta_1' = \beta_2' = \underline{\lambda}\{\mathbf{P}\}$.

Similarly, the PD controller

$$\begin{aligned} \tau_i = \nabla_{\mathbf{q}_i} {}^s\mathscr{U}_i(\mathbf{q}_i) - b_i \lceil \mathbf{q}_i - \mathbf{q}_\ell \rfloor^{\frac{2r_2 - r_1}{r_1}} - \sum_{j \in \mathscr{N}_i} a_{ij} \mathbf{P}_i \lceil \mathbf{q}_i - \mathbf{q}_j \rfloor^{\frac{2r_2 - r_1}{r_1}} \\ -\mathbf{D}_i \lceil \dot{\mathbf{q}}_i \rfloor^{\frac{2r_2 - r_1}{r_2}} - b_i (\mathbf{q}_i - \mathbf{q}_\ell) - \sum_{j \in \mathscr{N}_i} a_{ij} \mathbf{P}_{li} (\mathbf{q}_i - \mathbf{q}_j) - \mathbf{D}_{li} \dot{\mathbf{q}}_i, \end{aligned}$$

solves the **FTLFC** problem. The leader-follower interconnection is modeled by $b_i > 0$ if the leader position $\mathbf{q}_\ell$ is available to the ith-node and $b_i = 0$, otherwise. Clearly, both controllers are decentralized.

Saturated Finite-time PD control: Let us set the energy dissipation function as $\mathscr{F}(\dot{\mathbf{q}}) = \sum_{i=1}^{N} \sum_{j=1}^{n} h_{ij}^D (\dot{q}_j)$, where

$$h^D_{ij}(\dot{q}_j) := \begin{cases} \dfrac{r_2}{3r_2-r_1} D_{ij} |\dot{q}_j|^{\frac{3r_2-r_1}{r_2}} & \text{if } |\dot{q}_j| < \varepsilon, \\ D_{ij}\left(\varepsilon^{\frac{2r_2-r_1}{r_2}} |\dot{q}_j| - \dfrac{2r_2-r_1}{3r_2-r_1}\varepsilon^{\frac{3r_2-r_1}{r_2}}\right) & \text{if } |\dot{q}_j| \geq \varepsilon. \end{cases}$$

Choose $\mathscr{U}_c(\delta\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathscr{N}_i} a_{ij}\mathbf{1}_n^\top \mathbf{P}_i h_{ij}(\mathbf{q}_i - \mathbf{q}_j)$, $h_{ij}(\mathbf{q}_i - \mathbf{q}_j) := [h_{ij}(q_{i,1}-q_{j,1}), \ldots, h_{ij}(q_{i,n}-q_{j,n})]^\top$, where for all $k \in \bar{n}$

$$h_{ij}(q_{ijk}) := \begin{cases} \dfrac{r_1}{2r_2} |\delta q_{ijk}|^{\frac{2r_2}{r_1}} & \text{if } |\delta q_{ijk}| < \varepsilon, \\ \left(\varepsilon^{\frac{2r_2-r_1}{r_1}} |\delta q_{ijk}| - \dfrac{2r_2-r_1}{2r_2}\varepsilon^{\frac{2r_2}{r_1}}\right) & \text{if } |\delta q_{ijk}| \geq \varepsilon. \end{cases}$$

where $q_{ijk} := q_{i,k} - q_{j,k}$. This yields

$$\begin{aligned} \tau_i = {} & \nabla_{\mathbf{q}_i} {}^s\mathscr{U}_i(\mathbf{q}_i) - \sum_{j\in\mathscr{N}_i} a_{ij}\mathbf{P}_i sat_\varepsilon(\lceil \mathbf{q}_i - \mathbf{q}_j \rfloor^{\frac{2r_2-r_1}{r_1}}) \\ & - \mathbf{D}_i sat_\varepsilon(\lceil \dot{\mathbf{q}}_i \rfloor^{\frac{2r_2-r_1}{r_2}}) \end{aligned}$$

Similarly, the following PD controller solves the **FTLFC**

$$\begin{aligned} \tau_i = {} & \nabla_{\mathbf{q}_i} {}^s\mathscr{U}_i(\mathbf{q}_i) - b_i sat_\varepsilon(\lceil \mathbf{q}_i - \mathbf{q}_\ell \rfloor^{\frac{2r_2-r_1}{r_1}}) \\ & - \sum_{j\in\mathscr{N}_i} a_{ij}\mathbf{P}_i sat_\varepsilon(\lceil \mathbf{q}_i - \mathbf{q}_j \rfloor^{\frac{2r_2-r_1}{r_1}}) - \mathbf{D}_i sat_\varepsilon(\lceil \dot{\mathbf{q}}_i \rfloor^{\frac{2r_2-r_1}{r_2}}) \end{aligned}$$

6. SIMULATIONS

This section provides a numerical simulation study using a network of ten 2-DoF nonlinear manipulators with revolute joints. The simulations have been carried-out using the unsaturated PD controller

$$\tau_i = \nabla_{\mathbf{q}_i} {}^s\mathscr{U}_i(\mathbf{q}_i) - b_i \lceil \mathbf{q}_i - \mathbf{q}_\ell \rfloor^{p_1} - \sum_{j\in\mathscr{N}_i} a_{ij}\mathbf{P}_i \lceil \mathbf{q}_i - \mathbf{q}_j \rfloor^{p_1} - \mathbf{D}_i \lceil \dot{\mathbf{q}}_i \rfloor^{p_2}$$

where $p_1 = \frac{2r_2-r_1}{r_1}$ and $p_2 = \frac{2r_2-r_1}{r_2}$. It should be noted that when $2r_2 = r_1$ then $p_1 = p_2 = 0$ and the controller becomes discontinuous. However, when $r_2 = r_1$ then $p_1 = p_2 = 1$ and the controller becomes the linear P+d controller reported in (Nuño et al., 2013). The dynamics of each agent follow (3) with the inertia and Coriolis matrices given by

$$\mathbf{M}_i(\mathbf{q}_i) = \begin{bmatrix} \delta_{1i} + 2\delta_{2i}\mathrm{c}_{2_i} & \delta_{3i} + \delta_{2i}\mathrm{c}_{2_i} \\ \delta_{3i} + \delta_{2i}\mathrm{c}_{2_i} & \delta_{3i} \end{bmatrix},$$

$$\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) = \delta_{2i} \begin{bmatrix} -\mathrm{s}_{2_i}\dot{q}_{2_i} & -\mathrm{s}_{2_i}(\dot{q}_{1_i} + \dot{q}_{2_i}) \\ \mathrm{s}_{2_i}\dot{q}_{1_i} & 0 \end{bmatrix},$$

where $\delta_{1i} := l_{2_i}^2 m_{2_i} + l_{1_i}^2(m_{1_i} + m_{2_i})$, $\delta_{2i} := l_{1_i} l_{2_i} m_{2_i}$ and $\delta_{3i} := l_{2_i}^2 m_{2_i}$. c_{2_i} and s_{2_i} stand for the short notation of $\cos(q_{2_i})$ and $\sin(q_{2_i})$, respectively. q_{k_i} and $\dot{q}_{k_i}$ are the joint position and velocity, respectively, of link k of manipulator i, with $k \in \{1,2\}$. l_{k_i} and m_{k_i} are the respective lengths and masses of each link. The ten-agent network is composed of three different groups of robot manipulators, with equal members at each group. The physical parameters, for each group, are: $m_1 = 4$kg, $m_2 = 2$kg and $l_1 = l_2 = 0.4$m, for Agents 1, 2 and 3; $m_1 = 2.5$kg, $m_2 = 3$kg, $l_1 = 0.3$m and $l_2 = 0.5$m for Agents 4, 5 and 6; $m_1 = 3$kg, $m_2 = 2.5$kg, $l_1 = 0.5$m and $l_2 = 0.2$m for Agents 7, 8, 9 and 10. The Laplacian matrix of the network interconnection is

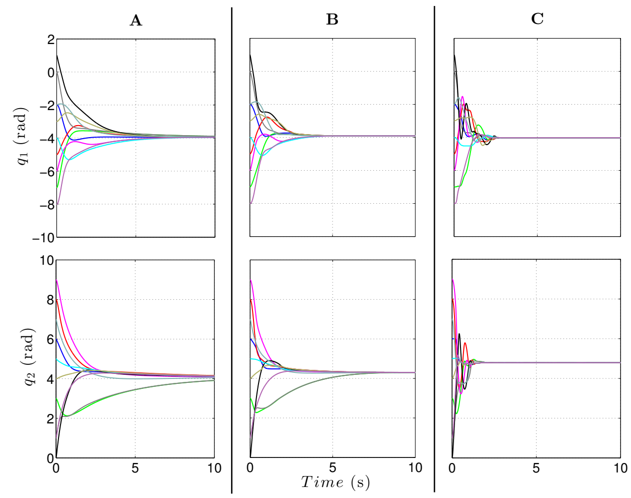

Fig. 1. Leaderless consensus for different values of r_2. In Columns **A**, **B** and **C**, r_2 has been set to 1.9, 1.5 and 1.1, respectively. In all cases $r_1 = 2$.

$$\mathbf{L} = \frac{1}{10}\begin{bmatrix} 14 & 0 & -3 & 0 & 0 & 0 & 0 & -4 & 0 & -7 \\ 0 & 9 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & -1 \\ -3 & 0 & 5 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -8 & 0 & 10 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -5 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 14 & 0 & 0 & -9 \\ -4 & 0 & -2 & 0 & 0 & -4 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & -2 & -3 & -2 & 0 & 0 & 7 & 0 \\ -7 & -1 & 0 & 0 & 0 & 0 & -9 & 0 & 0 & 17 \end{bmatrix}.$$

The proportional gains $\mathbf{P}_i$, for all the agents, have been set to $10\mathbf{I}$Nm. The damping gains are set to: $\mathbf{D}_1 = \mathbf{D}_7 = 9.8\mathbf{I}$, $\mathbf{D}_2 = 6.3\mathbf{I}$, $\mathbf{D}_3 = 3.5\mathbf{I}$, $\mathbf{D}_4 = \mathbf{D}_8 = 7\mathbf{I}$, $\mathbf{D}_5 = 5.6\mathbf{I}$, $\mathbf{D}_6 = 4.2\mathbf{I}$, $\mathbf{D}_9 = 4.9\mathbf{I}$, $\mathbf{D}_10 = 11.9\mathbf{I}$. The initial velocities have been set to zero and the initial positions are $\mathbf{q}^\top(0) = [-2,6,-7,3,-5,8,0,1,-6,9,1,0,-4,5,-3,4,-2,7,-8,1]$.

In all the simulation results $r_1 = 2$ and, to show how performance is increased when r_2 changes, we have set r_2 with different values.

Fig. 1 depicts the simulation results for the **FTLC**. Columns **A**, **B** and **C** plot the joint positions of the ten EL-agents for three different values of r_2, namely 1.9, 1.5 and 1.1, respectively. When $r_2 = 1.9$ then $p_1 = 0.9$ and $p_2 = 0.9474$, hence such exponents are *close* to the linear case $p_1 = p_2 = 1$ (when only GAS of the equilibrium can be established). Further, when $r_2 = 1.1$ then $p_1 = 0.1$ and $p_2 = 0.1818$, this case is *close* to the discontinuous scheme with $p_1 = p_2 = 0$. Fig. 1 shows that, when the exponents p_1 and p_2 are closer to zero, the convergence speed is increased. This can be corroborated from the plots in Columns **A**, **B** and **C**. As expected, in all cases, all the robots agree at a consensus position.

In the Leader-follower case, the leader desired position is only available to Agents 6 and 7. Therefore, the leader-follower interconnections for these agents have been set to $b_6 = b_7 = 20$. The rest of the gains b_i are set to zero. The leader position is $\mathbf{q}_\ell^\top = [-2,3]$rad. As in the leaderless case, Columns **A**, **B** and **C** of Fig. 2 show the different responses for the three different values of r_2, namely 1.9, 1.5 and 1.1, respectively. From these results, it can be concluded that all the robots reach the given leader position and that the convergence velocity is increased when the powers p_1 and p_2 are closer to zero.

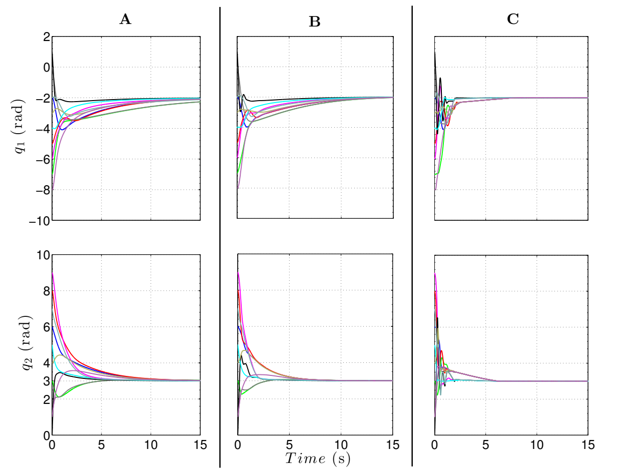

Fig. 2. Leader-follower consensus for $\mathbf{q}_l = [-2, 3]^\top$ and for different values of r_2. In Columns **A**, **B** and **C**, r_2 has been set to 1.9, 1.5 and 1.1, respectively. In all cases $r_1 = 2$.

7. CONCLUSIONS

Inspired on the energy shaping technique, this paper proposes a novel control method to solve the leaderless and the leader-follower consensus problems in finite-time for networks of multiple EL-agents. The main contribution is to show that FT control design can be done by modifying the total potential and dissipation-like energies of the EL-agent network in order to satisfy some homogeneity properties. The resulting controllers are preliminary extension of the results of (Nuño and Ortega, 2017) to the finite-time case.

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