Twisting based continuous controller for disturbed systems with uncertain control coefficient

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Abstract: In this paper a disturbed system with uncertain control coefficient and relative degree one is stabilized using a controller defined by the integral of a discontinuous law based on Twisting algorithm with extra terms that allows the rejection of the uncertainties produced by the disturbance and unknown control coefficient. Also the proposed control law ensures insensitivity to a kind of disturbances and finite time convergence to zero. The fact that the control is only integral could help to chattering reduction against FOSM control law.

Keywords: Sliding-mode control, Robust control, Lyapunov stability, Finite time, Homogeneity.

1. INTRODUCTION

In the last years, sliding mode control algorithms have become very important for control theory due to their properties of robustness and finite time convergence of the sliding variable.

Historically, the first sliding-mode controller was the sign signal (FOSM) (Utkin , 2016) that can stabilize a system with relative degree one sliding variable, nevertheless this kind of control (discontinuous) produce high frequency oscillation with bounded amplitude (called chattering). An intuitive way to deal with this problem is to propose continuous control laws, so thinking about that solution without losing some properties of robustness and finite time convergence, second order sliding-modes controllers (2-SMC) was proposed.

For systems with relative degree two the Twisting algorithm was introduced Emelyanov et. al. (1986) and this controller is able to drive the sliding variable and its derivative to zero in finite time (enforcing a second order sliding-mode), and for systems with relative degree one sliding variable the Super Twisting Algorithm (STA) (Levant , 1993) can enforce a second order sliding-mode too with a continuous control signal (reducing chattering), however STA has infinite gain a the origin that remains being bad for chattering avoidance. For that reason a control law defined by the integral of discontinuous Twisting controller was proposed (Efimov et. al., 2011), but this scheme only can stabilize the origin of the system locally, also the Twisting controller requires the derivative of the sliding variable (disturbance estimation), so if the control coefficient is known it's more practical to use a control law like STA or using the estimation in order to compensate the disturbance on line (Davila et. al. , 2016) despite the control signal produced by the Twisting as filter is Lipschitz continuous respect time.

In recent works (Ventura and Fridman , 2016) had been shown that under some assumptions a Lipschitz control law could reduce the chattering produced by a sliding-mode control against discontinuous one (FOSM) and laws with infinite gain at the origin (STA).

Then in this paper we propose a control law based on the integral of discontinuous signals (based on Twisting as filter) that is continuous Lipschitz respect time at the origin and could deal with unknown control coefficient and disturbances for a system with sliding variable with relative degree one.

Section 2 presents the problem statement and the solution proposed as main result, a mechanical system with uncertain control coefficient and dynamics that are considered disturbances is simulated to realize exact velocity-tracking with the proposed algorithm and its behavior is shown in Section 3, the conclusions of this work are presented in Section 4, and finally Appendix section shows the proof to the theorem.

2. PROBLEM STATEMENT AND MAIN RESULT

Let the dynamical system

$$\dot{\chi} = F(\chi, t) + G(\chi, t)u, \tag{1}$$

where $\chi \in \mathbb{R}^n$ and $u \in \mathbb{R}$. An output (sliding variable) of relative degree one is chosen[1], and the dynamic of the sliding variable could be represented by

$$\dot{x} = \phi(t, x) + \gamma(t, x)u, \quad |\phi| \leq F, \; 0 < \gamma_m \leq \gamma \leq \gamma_M, \tag{2}$$

in order to drive the sliding variable to zero and holding it, a purely integral control law based on SMC will be proposed, then the state of the system must be extended as

$$x = x_1, \quad \dot{x} = x_2,$$

and considering the variable change

$$u = \frac{x_2 - \phi}{\gamma},$$

in order to describe the system just in the variables x_1 and x_2, the system obtained has the form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \dot{\phi} + \dot{\gamma}u + \gamma\dot{u} \\
&= (\phi'_t + \phi'_{x_1}x_2) + (\gamma'_t + \gamma'_{x_1}x_2)u + \gamma\dot{u} \\
&= \left(\phi'_t - \phi\frac{\gamma'_t}{\gamma}\right) + \left(\phi'_{x_1} + \frac{\gamma'_t}{\gamma} - \frac{\gamma'_{x_1}}{\gamma}\phi\right)x_2 + \frac{\gamma'_{x_1}}{\gamma}x_2^2 + \gamma\dot{u},
\end{aligned} \tag{3}$$

in this new system $\dot{u}$ is considered as a control law, the origin of the system (3) can be locally stabilized using the Twisting algorithm, however, if its desired that the closed-loop system has a globally stable equilibrium point at the origin linear and quadratic terms are required in the control law, specially because the quadratic term in x_2 in not globally Lipschitz. Therefore the controller proposed to achieve this task has the form

$$\dot{u} = -k_1\lfloor x_1\rceil^0 - c_1\lfloor x_2\rceil^0 - k_2x_1 - c_2x_2 - k_3\lfloor x_1\rceil^2 - c_3\lfloor x_2\rceil^2, \tag{4}$$

where $\lfloor\alpha\rceil^\beta = |\alpha|^\beta \text{sign}(\alpha)$, the control law (4) has terms with exponent zero that provides the properties of 2-SMC, linear and quadratic terms that allows the rejection of the uncertainties with greater exponents (quadratic terms in the control law are crucial in order to make the trajectories of the system well defined in all the space).

Assumption 1. It will be assumed that the partial derivatives $\phi'_t, \phi'_x, \gamma'_t, \gamma'_x$ are bounded.

Theorem 1. Let the disturbed system (1) with a sliding variable (2), then if the uncertain control coefficient is such that

$$3\gamma_m > \gamma_M,$$

there exists gains such that the control law (4) enforces a second order sliding-mode in finite time.

Proof. The proof of the Theorem 1 will be presented in the following subsections, and the gains of the control law (4) can be computed as in the equations (14), (16), (19), (23), (25), and fulfilling the LMI (24).

[1] There exist several methods to design the switching surface $x = 0$ such that the reduced order dynamics shows desired properties (Shtessel et. al., 2014).

2.1 Stability analysis of the nominal closed loop

In order to start the stability analysis of the closed-loop system with the proposed control law, we are going to consider the nominal system ($\phi = 0$, $\gamma_m = \gamma_M = 1$) with the proposed control law (4),

$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1\lfloor x_1\rceil^0 - c_1\lfloor x_2\rceil^0 - k_2x_1 - c_2x_2 - k_3\lfloor x_1\rceil^2 - c_3\lfloor x_2\rceil^2.\end{cases} \tag{5}$$

Then to proof the stability of the origin the candidate Lyapunov function is proposed as

$$V = x_1^2 + a_1x_1x_2 + a_2x_2^2 + a_3|x_1| + a_4|x_1|^3 \tag{6}$$

where the only one condition for it to be positive definite is that

$$4a_2 > a_1^2,$$

and can be easily verified. Now calculating the partial derivatives

$$\begin{aligned}\frac{\partial V}{\partial x_1} &= 2x_1 + a_1x_2 + a_3\lfloor x_1\rceil^0 + 3a_4\lfloor x_1\rceil^2, \\ \frac{\partial V}{\partial x_2} &= a_1x_1 + 2a_2x_2,\end{aligned} \tag{7}$$

and the derivative of the candidate Lyapunov function with respect to the trajectories of the system is

$$\begin{aligned}
\dot{V} = -W &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\
&= (2x_1x_2 + a_1x_2^2 + a_3\lfloor x_1\rceil^0x_2 + 3a_4\lfloor x_1\rceil^2x_2) \\
&\quad -(a_1k_1|x_1| + 2a_2k_1\lfloor x_1\rceil^0x_2 + a_1c_1x_1\lfloor x_2\rceil^0 + 2a_2c_1|x_2|) \\
&\quad -(a_1k_2x_1^2 + 2a_2k_2x_1x_2 + a_1c_2x_1x_2 + 2a_2c_2x_2^2) \\
&\quad -(a_1k_3|x_1|^3 + 2a_2k_3\lfloor x_1\rceil^2x_2 + a_1c_3x_1\lfloor x_2\rceil^2 + 2a_2c_3|x_2|^3).
\end{aligned}$$

In this derivative can be noted that it has terms with different homogeneity degrees with respect the homogeneity weights $r_1 = r_2 = 1$ (homogeneity weights with which it can be shown that a linear system is homogeneous), for that reason the function W will be separated as $W = W_1 + W_2 + W_3$, and the task to prove that the origin is stable become to make W_1, W_2, W_3 positive definite

$$\begin{aligned}
W_1 &= a_1k_1|x_1| + (2a_2k_1 - a_3)\lfloor x_1\rceil^0x_2 + a_1c_1x_1\lfloor x_2\rceil^0 + 2a_2c_1|x_2|, \\
W_2 &= a_1k_2x_1^2 + (2a_2k_2 + a_1c_2 - 2)x_1x_2 + 2a_2c_2x_2^2 - a_1x_2^2, \\
W_3 &= a_1k_3|x_1|^3 + (2a_2k_3 - 3a_4)\lfloor x_1\rceil^2x_2 + a_1c_3x_1\lfloor x_2\rceil^2 + 2a_2c_3|x_2|^3,
\end{aligned}$$

where it can be noticed that W_1 has homogeneity degree 1, W_2 has homogeneity degree 2 and W_3 homogeneity degree 3, so the three can be analyzed separately, and if the constants of the candidate Lyapunov function are chosen as

$$2a_2k_1 = a_3, \quad 2a_2k_2 + a_1c_2 = 2, \quad 2a_2k_3 = 3a_4$$

that allows to cancel terms without definite sign and the functions become

$$\begin{aligned}
W_1 &= a_1k_1|x_1| + a_1c_1x_1\lfloor x_2\rceil^0 + 2a_2c_1|x_2| \\
&\geq a_1k_1|x_1| - a_1c_1|x_1| + 2a_2c_1|x_2| \\
W_2 &= a_1k_2x_1^2 + 2a_2c_2x_2^2 - a_1x_2^2 \\
W_3 &= a_1k_3|x_1|^3 + a_1c_3x_1\lfloor x_2\rceil^2 + 2a_2c_3|x_2|^3,
\end{aligned}$$

so it's easy to verify that the conditions to make W_1 positive definite are

$$k_1 > c_1, \quad c_1 > 0, \tag{8}$$

for making W_2 positive definite the conditions are

$$k_2 > 0, \quad c_2 > a_1/2a_2, \tag{9}$$

and for W_3 the Young inequality can be used in order to bound terms without definite sign and the conditions are

$$k_3 > c_3/3, \quad c_3 > 0, \quad a_2 > a_1/3.$$

The previous inequalities always have a solution, so a controller of the form (4) can to stabilize the system without disturbances and constant control coefficient. Moreover there exist a Lyapunov function of the form (6) whose derivative over the trajectories of the system is negative definite. In addition the derivative of the Lyapunov function can be bounded by

$$\begin{aligned}\dot{V} \leq & -a_1(k_1 - c_1)|x_1| - 2a_2c_1|x_2| \\ & -a_1k_2x_1^2 - (2a_2c_2 - a_1)x_2^2 \\ & -a_1\left(k_3 - \frac{c_3}{3}\right)|x_1|^3 - 2c_3\left(a_2 - \frac{a_1}{3}\right)|x_2|^3 \\ \leq & -\alpha V\end{aligned}$$

for some α positive, and the system (5) has in the origin a equilibrium point globally asymptotically stable.

Then the system (5) can be seen as a sum of homogeneous vector fields as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -k_1\lfloor x_1\rceil^0 - c_1\lfloor x_2\rceil^0\end{bmatrix} + f_e(x)$$

with homogeneous weights $r_1 = 2$, $r_2 = 1$, that makes the first part of the system (the Twisting algorithm) homogeneous of degree -1, and f_e a vector field with terms with greater homogeneity degree, since the Twisting algorithm has a finite time stable equilibrium point at the origin by quasi-homogeneity principle (Orlov , 2008) it can be concluded the origin of the system (5) is globally finite time stable.

2.2 Stability analysis of the disturbed closed loop

Now, considering the disturbed system of the form

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = f_1 + f_2x_2 + f_3x_2^2 \\ \qquad +\gamma(-k_1\lfloor x_1\rceil^0 - c_1\lfloor x_2\rceil^0 - k_2x_1 - c_2x_2 - k_3\lfloor x_1\rceil^2 - c_3\lfloor x_2\rceil^2),\end{cases} \tag{10}$$

where the functions f_i are defined as

$$f_1 = \left(\phi'_t - \phi\frac{\gamma'_t}{\gamma}\right),$$
$$f_2 = \left(\phi'_{x_1} + \frac{\gamma'_t}{\gamma} - \frac{\gamma'_{x_1}}{\gamma}\phi\right),$$
$$f_3 = \frac{\gamma'_{x_1}}{\gamma},$$

and are bounded by constant values $|f_1| \leq F_1, |f_2| \leq F_2, |f_3| \leq F_3$. In order to prove the stability of the origin of the disturbed system the same candidate Lyapunov function (6) is going to be used, and calculating the derivative along the trajectories of the system (10) the next functions are obtained and them have to be positive definite

$$\begin{aligned}W_1 =& a_1k_1|x_1| + \left(2a_2k_1 - \frac{a_3}{\gamma}\right)\lfloor x_1\rceil^0x_2 + a_1c_1x_1\lfloor x_2\rceil^0 \\ & + 2a_2c_1|x_2| - \frac{a_1f_1}{\gamma}x_1 - \frac{2a_2f_1}{\gamma}x_2 \\ W_2 =& a_1k_2x_1^2 + \left(2a_2k_2 + a_1c_2 - \frac{2}{\gamma}\right)x_1x_2 + \left(2a_2c_2 - \frac{a_1}{\gamma}\right)x_2^2 \\ & - \frac{a_1f_2}{\gamma}x_1x_2 - \frac{2a_2f_2}{\gamma}x_2^2 \\ W_3 =& a_1k_3|x_1|^3 + \left(2a_2k_3 - \frac{3a_4}{\gamma}\right)\lfloor x_1\rceil^2x_2 + a_1c_3x_1\lfloor x_2\rceil^2 \\ & + 2a_2c_3|x_2|^3 - \frac{a_1f_3}{\gamma}x_1x_2^2 - \frac{2a_2f_3}{\gamma}x_2^3\end{aligned}$$

2.3 Positivity of W_1

The function W_1 can be bounded as

$$\begin{aligned}W_1 =& a_1k_1|x_1| + \left(2a_2k_1 - \frac{a_3}{\gamma}\right)\lfloor x_1\rceil^0x_2 + a_1c_1x_1\lfloor x_2\rceil^0 \\ & + 2a_2c_1|x_2| - \frac{a_1f_1}{\gamma}x_1 - \frac{2a_2f_1}{\gamma}x_2 \\ \geq & \left(a_1k_1 - a_1c_1 - \frac{a_1F_1}{\gamma}\right)|x_1| \\ & + \left(2a_2c_1 - \left|2a_2k_1 - \frac{a_3}{\gamma}\right| - \frac{2a_2F_1}{\gamma}\right)|x_2|\end{aligned}$$

that is gong to be positive definite if the next inequalities are satisfied

$$\gamma\alpha_1k_1 > \gamma a_1c_1 + a_1F_1, \quad \gamma 2a_2c_1 > |\gamma 2a_2k_1 - a_3| + 2a_2F_1, \tag{11}$$

if the constant a_3 is chosen as $a_3 = 2a_2k_1\bar{\gamma}$, where $\bar{\gamma} = (\gamma_M - \gamma_m)/2$ then the conditions for the positivity of W_1 are

$$\gamma(k_1 - c_1) > F_1, \quad \gamma c_1 > |\gamma k_1 - \bar{\gamma}k_1| + F_1 \tag{12}$$

in these can be observed that the worst case is that the uncertain coefficient of control $\gamma = \gamma_m$, also the term $|\gamma_m - \bar{\gamma}| = (\gamma_M - \gamma_m)/2$, so

$$\gamma_m(k_1 - c_1) > F_1, \quad \gamma_m c_1 > \frac{\gamma_M - \gamma_m}{2}k_1 + F_1 \tag{13}$$

choosing the gain k_1 as

$$k_1 = c_1 + \frac{F_1}{\gamma_m} + \beta_1, \quad \beta_1 > 0 \tag{14}$$

the first inequality in (13) is satisfied and substituting in the second one

$$\gamma_m c_1 > \frac{\gamma_M - \gamma_m}{2}c_1 + \left(\frac{\gamma_M - \gamma_m}{2\gamma_m} + 1\right)F_1 + \frac{\gamma_M - \gamma_m}{2}\beta_1 \tag{15}$$

where it could be verified that necessary condition is that

$$3\gamma_m > \gamma_M.$$

and finally the gain c_1 can be calculated as

$$c_1 = \frac{\gamma_M + \gamma_m}{\gamma_m(3\gamma_m - \gamma_M)}F_1 + \frac{\gamma_M - \gamma_m}{3\gamma_m - \gamma_M}\beta_1 + \beta_2, \quad \beta_2 > 0. \tag{16}$$

So, if the gains k_1 and c_1 are chosen as (14) and (16) respectively the function W_1 is positive definite.

2.4 Positivity of W_3

In order to prove the positivity of W_3 the crossed terms of the variables x_1 and x_2 are going to be bounded using the Young's inequality

$$\lfloor x_1\rceil^2 x_2 \le \frac{2|x_1|^3}{3}+\frac{|x_2|^3}{3}, \quad x_1\lfloor x_2\rceil^2 \le \frac{|x_1|^3}{3}+\frac{2|x_2|^3}{3}. \tag{17}$$

Then choosing the constant a_4 such that $3a_4 = 2a_2k_3\gamma_M$ and using the Young's inequality to bound the crossed terms, the conditions for the positivity of W_3 are

$$\begin{aligned}
\gamma a_1 k_3 &> \frac{2}{3}(\gamma_M-\gamma)(2a_2k_3)+\frac{1}{3}a_1F_3,\\
\gamma a_1 k_3 &> \frac{1}{3}\gamma a_1 c_3+\frac{1}{3}a_1F_3,\\
\gamma 2a_2c_3 &> \frac{1}{3}(\gamma_M-\gamma)(2a_2k_3)+\frac{2}{3}a_1F_3+2a_2F_3,\\
\gamma 2a_2c_3 &> \frac{2}{3}\gamma a_1c_3+\frac{2}{3}a_1F_3+2a_2F_3,
\end{aligned} \tag{18}$$

if the gain k_3 is calculated as

$$k_3 = \frac{c_3}{3}+\frac{F_3}{3\gamma_m}+\beta_3, \tag{19}$$

the fisrt inequality in (18) is satisfied, now doing $d = a_2/a_1$ the other ones end as

$$\begin{aligned}
\gamma_m k_3 &> \frac{4}{3}(\gamma_M-\gamma_m)dk_3+\frac{F_3}{3}\\
\gamma_m c_3 &> \frac{1}{3}(\gamma_M-\gamma_m)k_3+\frac{F_3}{3d}+F_3\\
\gamma_m c_3 &> \frac{1}{3d}\gamma_m c_3+\frac{F_3}{3d}+F_3
\end{aligned}$$

of this new inequalities it's easy to see that the next conditions are required

$$d > \frac{1}{3}, \quad d < \frac{3\gamma_m}{4(\gamma_M-\gamma_m)} \tag{20}$$

but in the inequalities d appears multiplying and dividing, so a good choice to select d is as the average

$$d = \frac{1}{2}\left(\frac{1}{3}+\frac{3\gamma_m}{4(\gamma_M-\gamma_m)}\right) = \frac{4\gamma_M+5\gamma_m}{24(\gamma_M-\gamma_m)}, \tag{21}$$

then substituting in the inequalities

$$\begin{aligned}
k_3 &> \frac{6F_3}{13\gamma_m-4\gamma_M},\\
c_3 &> \frac{F_3}{\gamma_m}\left(\frac{12\gamma_M-3\gamma_m}{13\gamma_m-4\gamma_M}\right),
\end{aligned} \tag{22}$$

and taking the other inequality and substituting d y k_3 the gain c_3 can be computed as

$$\begin{aligned}
c_3 &= \frac{3(\gamma_M-\gamma_m)}{10\gamma_m-\gamma_M}\beta_3+\frac{F_3}{\gamma_m}d_1+\beta_4,\\
d_1 &= \frac{109\gamma_m\gamma_M+4\gamma_M^2-32\gamma_m^2}{35\gamma_m\gamma_M-4\gamma_M^2+50\gamma_m^2},
\end{aligned} \tag{23}$$

finally the gains should accomplish the inequalities (23), so the constants β_3, β_4 can be computed as

$$\begin{aligned}
\beta_3 &> \frac{F_3}{3\gamma_m}\left(\frac{-130\gamma_m\gamma_M+56\gamma_M^2+74\gamma_m^2}{32\gamma_m\gamma_M-16\gamma_M^2+65\gamma_m^2}\right),\\
\beta_4 &> \frac{F_3}{\gamma_m}\left(\frac{12\gamma_M-3\gamma_m}{13\gamma_m-4\gamma_M}-d_1\right),
\end{aligned}$$

and the gains k_3 and c_3 computed as (19) and (23) ensures the function W_3 is positive definite.

Remark 1. If the values of the uncertain control coefficient are $\gamma_M = \gamma_m$, then the constants should be

$$\beta_3 > 0, \quad \beta_4 > 0.$$

And, in the worst case the values of the uncertain control coefficient are $\gamma_M = 3\gamma_m$, and the constants should be

$$\begin{aligned}
\beta_3 &> \frac{188}{51}\frac{F_3}{\gamma_m} \approx 3.687\frac{F_3}{\gamma_m},\\
\beta_4 &> \frac{F_3}{\gamma}\left(33-\frac{331}{119}\right) \approx 30.22\frac{F_3}{\gamma_m}.
\end{aligned}$$

2.5 Positivity of W_2

In order to minimize the value of the crossed terms of x_1, x_2 without definite sign, some constants are chosen to achieve the equation

$$2 = \bar{\gamma}(2a_2k_2+a_1c_2),$$

then, considering the worst framework in the values of the uncertain control coefficient W_2 will be positive definite if the LMI is satisfied

$$\begin{bmatrix} \gamma_m a_1 k_2 & \frac{\gamma_M-\gamma_m}{4}(2a_2k_2+a_1c_2)+\frac{1}{2}a_1F_2 \\ \star & \gamma_m 2a_2c_2 - a_1 - 2a_2F_2 \end{bmatrix} > 0$$

and from the previous subsection $d = a_2/a_1$, $a_1 > 0$, then the LMI become

$$\begin{bmatrix} \gamma_m k_2 & \frac{\gamma_M-\gamma_m}{4}(2dk_2+c_2)+\frac{1}{2}F_2 \\ \star & \gamma_m 2dc_2 - 1 - 2dF_2 \end{bmatrix} > 0, \tag{24}$$

where the condition $3\gamma_m > \gamma_M$ appear again, then the gains that always accomplish the LMI (24) should be

$$k_2 = \frac{c_2}{2d}, \tag{25}$$

$$4c_2\gamma_m\left(\gamma_m c_2 - \frac{1}{2d} - F_2\right) > F_2^2 + c_2(\gamma_M-\gamma_m)(2F_2+(\gamma_M-\gamma_m)c_2).$$

and the function W_2 is positive definite with these gains.

Finally the system (10) has in the origin a equilibrium point globally asymptotically stable. Then, considering again the system as a sum of homogeneous terms, the homogeneous approximation at the origin with homogeneous weights $r_1 = 2$, $r_2 = 1$ is the Twisting algorithm which is finite time stable itself and it can be concluded by quasi-homogeneity that the system (10) has an globally finite time equilibrium point at the origin.

3. SIMULATIONS AND NUMERICAL EXAMPLES

For this section the motivational example shown in (Castillo et. al., 2016) is considered

$$(1+\cos^2(q))\ddot{q}+g\sin(q)+b(\dot{q}+\arctan(\dot{q})) = u \tag{26}$$

where $q, \dot{q} \in \mathbb{R}$ are the state variables and $u \in \mathbb{R}$ the control input. The terms of the differential equation represent a varying inertia moment, gravitational term and viscous and dry friction. It is desired to realize exact velocity-tracking to a desired trajectory $\dot{q}_d$. By defining the error variable $e_1 = \dot{q}-\dot{q}_d$, the velocity error dynamics are

$$\dot{e}_1 = [u-g\sin(q)-b(\dot{q}+\arctan(\dot{q}))](1+\cos^2(q))^{-1}-\ddot{q}_d$$

then, extending the system with $\dot{e}_1 = e_2$, the total error dynamics are

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= \gamma(f_{11} + f_{21}e_1 + f_{22}e_2 + f_{31}e_1^2 + f_{32}e_1e_2 + \dot{u}) + f_{12}
\end{aligned}$$

where f_{ij}, γ are defined by

$$\begin{aligned}
\gamma =& \frac{1}{1+\cos^2(q)}, \\
f_{11} =& \frac{\sin(2q)}{1+\cos^2(q)}\dot{q}_d(g\sin(q) + b(\dot{q}_d + \arctan(e_1 + \dot{q}_d))) \\
& - g\cos(q)\dot{q}_d - b\ddot{q}_d\frac{2+(e_1+\dot{q}_d)^2}{1+(e_1+\dot{q}_d)^2}, \\
f_{12} =& -\dddot{q}_d, \\
f_{21} =& \frac{\sin(2q)}{1+\cos^2(q)}(2b\dot{q}_d + g\sin(q) + b\arctan(e_1+\dot{q}_d) \\
& - g\cos(q), \\
f_{22} =& \sin(2q)\dot{q}_d - b\frac{2+(e_1+\dot{q}_d)^2}{1+(e_1+\dot{q}_d)^2}, \\
f_{31} =& \frac{\sin(2q)}{1+\cos^2(q)}b, \\
f_{32} =& \sin(2q),
\end{aligned}$$

It should be noted that even the existence of damping b makes the original system stable, in the tracking problem damping generates terms that should be compensated, if the dry friction term is considered as a sign function, then the controller could not maintain the trajectory every time since the term to compensate is discontinuous and the control law is designed to be continuous. The desired speed is considered $\dot{q}_d = a\sin(\omega t)$ and the parameters are $b = 1$, $g = 10$, $a = 2$, $\omega = 2$ then f_{ij} and γ are bounded as

$$\begin{aligned}
0.5 \leq \gamma &\leq 1, \\
|f_{11}| &\leq 46, \\
|f_{12}| &\leq 8, \\
|f_{21}| &\leq 20.4, \\
|f_{22}| &\leq 4, \\
|f_{31}| &\leq 1, \\
|f_{31}| &\leq 1,
\end{aligned}$$

then the gains were selected as

$$\begin{aligned}
k_1 = 150 \;\; k_2 = 80 \;\; k_3 = 2, \\
c_1 = 100 \;\; c_2 = 80 \;\; c_3 = 2,
\end{aligned} \tag{27}$$

and in the Figure 1 is shown the behavior of the tracking of the speed of the mechanical system considering that it starts at rest ($q(0) = \dot{q}(0) = 0$).

4. CONCLUSIONS

A second-order sliding mode controller was designed to ensure the convergence of a system with relative degree one sliding variable despite having bounded disturbances and uncertain control coefficient under certain constraints if the derivative of the sliding variable is known, in this case it has the advantage that the control law is continuous Lipschitz with respect to time and therefore the generated *chattering* could be less than the generated by a control law with infinite gain at the origin (STA) or a discontinuous one (FOSM) [2], as disadvantage has that in case the derivative is unknown, it has to be estimated.

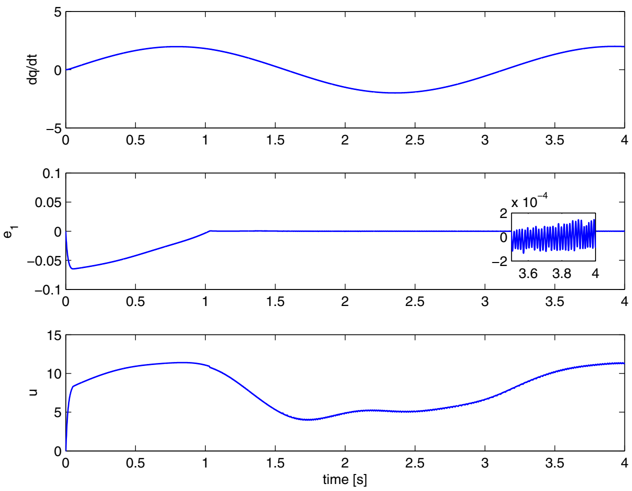

Fig. 1. Behavior of the system (26)with gains (27).

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