Generalized Functional Observer Design for Descriptor Linear Systems ⋆

C. Ríos-Ruiz * **C. M. Astorga-Zaragoza** *
G. L. Osorio-Gordillo * **M. Darouach** **

* *Tecnológico Nacional de México-CENIDET, Cuernavaca, Mexico (e-mail: {crios, astorga, gloriaosorio}@cenidet.edu.mx).*
** *CRAN-CNRS, (UMR 7039), Université de Lorraine, IUT Longwy, France (e-mail: modar@ptlu).*

Abstract: In this paper a generalized functional observer for descriptor linear systems is presented. The stability conditions of the observer are given and its performance evaluation is made through numerical simulations. An algorithm is provided in order to highlight the main steps involved in the functional observer design. The originality of this functional observer lies in its capability to be configured like a simple proportional functional observer (PFO) or like a generalized functional observer (GFO), depending on the particular requirements of the functional estimation problem. Another important feature of the observer is that it can be used for both linear descriptor systems or conventional linear time invariant (LTI) systems.

Keywords: Functional observers, Linear estimation, Linear systems, Descriptor systems, Feedback control.

1. INTRODUCTION

An important variety of mathematical tools for descriptor systems have the advantage that they can be applied not only to processes modeled in a descriptor form but also to processes modeled in a conventional linear time invariant (LTI) state-space form by considering LTI systems as a particular case of descriptor systems. This advantage is the main motivation to design in a descriptor framework several modern observation and control algorithms which in time can be applied to a wide variety of physical processes such as hydraulic systems (Araujo et al., 2012), mechanical systems (Dang et al., 2015), biomechanical systems (Guelton et al.,2008), wastewater treatment processes (Kiss et al., 2011) and electronic circuits (Hou et al., 2017).

In the particular case of observer design for descriptor systems, several works propose different approaches to solve the problem of state estimation (Osorio-Gordillo et al., 2016), parameter estimation (Alma and Darouach, 2014, Arefinia et al., 2017), unknown input estimation (Osorio-Gordillo et al., 2014, Estrada-Manzo et al., 2015) and so on.

Functional observers are useful when one or more functions of the linear states (rather than the unknown states) are required to be estimated. This feature is particularly advantageous in state feedback control (Fernando et al., 2016) and can be exploited for other control applications such as fault detection (Emami et al., 2015) or fault tolerant control (Lan and Patton, 2015).

The case of functional observers for descriptor systems has been discussed by several authors. For instance, Feng et al. propose a functional observer (and special cases for state and reduced-order observers) for descriptor systems which is designed to converge in finite-time. The authors in Koenig et al. (2016) propose a functional observer for discrete-time switched Lipschitz nonlinear descriptor systems which is used for fault estimation purposes. Arefinia et al, 2017 propose a robust adaptive observer for singular systems which is compared with a proportional-derivative observer (PD) developed by Ren and Zhang (2010). However, an important amount of the proposed functional observers for descriptor systems considers only proportional terms of the observation error and rarely other configurations like proportional-integral or proportional-derivative observers.

In this work, a generalized functional observer for descriptor systems is proposed. The main contribution is to take advantage of the new structure of observers for descriptor systems proposed in Osorio-Gordillo et al. (2016) in order to extend this result to functional observers. The interest of this general structure is to exploit the multi functionality of the generalized observer, for the reason that it can be configured as a proportional functional observer (PFO) which is only a particular case.

2. NOTATION AND PRELIMINARIES

This section describes the notation used in this paper. The symbol Σ^+ denotes the generalized inverse of Σ and verifies $\Sigma\Sigma^+\Sigma = \Sigma$. The notation $E^\perp$ denotes a maximal row rank matrix such that $E^\perp E = 0$. When E is a full row rank matrix, $E^\perp = 0$ by convention. Now, consider a class of linear descriptor systems of the form:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the measured output

⋆ The authors would like to thank the CONACYT.

vector and $z(t) \in \mathbb{R}^q$ is a vector that is required to be estimated. $E \in \mathbb{R}^{n\times n}$ is a constant matrix such that $rank(E) = r \leq n$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $L \in \mathbb{R}^{q\times n}$ are known constant matrices. $E^\perp \in \mathbb{R}^{r_1\times n}$ is a full row rank matrix, i.e. $rank(E^\perp) = r_1 = n - r$.

Assumption 1. Descriptor (1) system is R-observable and impulse observable, i.e.

$$rank \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix} = n \tag{2}$$

this is equivalent to $rank \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = rank(E) + n$.

Lemma 2.1. The general solution to any equation of the form

$$XA = B \tag{3}$$

is given by $X = BA^+ - Z(I - AA^+)$, where Z is an arbitrary matrix of appropriate dimensions (Darouach, 2012).

Lemma 2.2. The necessary and sufficient condition for the existence of a solution to (3) is given by Shafarevich and Remizov (2012) where

$$rank \begin{bmatrix} A \\ B \end{bmatrix} = rank(A) \tag{4}$$

The following Lemma is presented by de Oliveira (2005) and will be used later in this paper.

Lemma 2.3. Let matrices $\mathcal{B}$ and $\mathcal{Q}$ be given. The following statements are equivalent:

i. There exists a matrix $\mathcal{X}$ satisfying
$$\mathcal{B}\mathcal{X} + (\mathcal{B}\mathcal{X})^T + \mathcal{Q} < 0$$
ii. The following condition holds $\mathcal{B}^\perp \mathcal{Q} \mathcal{B}^{\perp T} < 0$

Suppose the above statements hold and assume that $\mathcal{B}^\perp \mathcal{B} > 0$. Then matrix $\mathcal{X}$ in statement (i) is given by

$$\mathcal{X} = -\gamma \mathcal{B}^T + \sqrt{\gamma}\mathcal{L}\Gamma^{1/2}$$

where $\mathcal{L}$ is any matrix such that $\|\mathcal{L}\| < 1$ and $\gamma > 0$ is any scalar such that $\Gamma = \gamma \mathcal{B}\mathcal{B}^T - \mathcal{Q} > 0$.

3. GENERALIZED FUNCTIONAL OBSERVER

3.1 Problem statement

Consider the following generalized functional observer (GFO) of the form

$$\begin{aligned}
\dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\
\dot{v}(t) &= S\zeta(t) + Gv(t) + M \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \\
\hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix}
\end{aligned} \tag{5}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ is the state of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{z}(t) \in \mathbb{R}^q$ is the estimate of $z(t)$. N, J, F, H, S, G, M, P and Q are constant matrices of appropriate dimensions to be determined such that $\lim_{t\to\infty} (\hat{z}(t) - z(t)) = 0$.

The following Lemma gives the sufficient conditions for the existence of the observer (5).

Lemma 3.1. There exists an observer having the form given in (5) for the system (1) if the matrix $\begin{bmatrix} N & J \\ S & G \end{bmatrix}$ is Hurwitz and if there exists a matrix T such that the following conditions are satisfied:

a. $NTE + F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA = 0$

b. $H = TB$

c. $STE + M \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = 0$

d. $PTE + Q \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = L$

Proof 1. Consider a parameter matrix $T \in \mathbb{R}^{q_0\times n}$ and define the transformed error vector $\varepsilon(t) = \zeta(t) - TEx(t)$. Since $E^\perp E = 0$ it can be deduced that $E^\perp Ax(t) = -E^\perp Bu(t)$, such that the derivative of $\varepsilon(t)$ is given by

$$\begin{aligned}
\dot{\varepsilon}(t) = N\zeta(t) + \left(F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA \right) x(t) \\
+(H - TB)u(t) + Jv(t)
\end{aligned} \tag{6}$$

By using the definition of $\varepsilon(t)$, $\dot{v}(t)$ in (5) can be rewritten as

$$\dot{v}(t) = S\zeta(t) + M \left(F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA \right) x(t) + Gv(t) \tag{7}$$

Considering that conditions *a.-c.* are satisfied, equations (6) and (7) become

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Jv(t) \tag{8}$$
$$\dot{v}(t) = S\varepsilon(t) + Gv(t) \tag{9}$$

By defining an augmented state vector $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$, equations (8), (9) can be rewritten as:

$$\dot{\sigma}(t) = \mathbb{A}\sigma(t) \tag{10}$$

where

$$\mathbb{A} = \begin{bmatrix} N & J \\ S & G \end{bmatrix}$$

Define the estimation error $e(t) = z(t) - \hat{z}(t)$. If condition *c.* is satisfied, then $e(t) = P\varepsilon(t)$. It can be seen that if matrix $\mathbb{A}$ is Hurwitz, then $\lim_{t\to\infty} \varepsilon(t) = 0$ and $\lim_{t\to\infty} e(t) = 0$.

3.2 Observer parameterization

From Lemma 3.1, it can be deduced that the design of the observer is reduced to find the matrices N, J, F, H, S, G, M, P, Q and T such that conditions *a.-c.* are satisfied. We define now matrix $\Gamma = \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix}$ and let $R \in \mathbb{R}^{q_0\times n}$ be a full row rank matrix such that $rank \begin{bmatrix} R \\ \Gamma \end{bmatrix} = rank(\Gamma)$. In this case there always exists two matrices T and K such that

$$TE + K \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = R \tag{11}$$

which can be rewritten as

$$[T \;\; K]\, \Gamma = R \tag{12}$$

the general solution for (12) is

$$[T\ K] = R\Gamma^{+} - Z\left(I - \Gamma\Gamma^{+}\right) \quad (13)$$

which can be decomposed in

$$T = T_1 - ZT_2 \quad (14)$$
$$K = K_1 - ZK_2 \quad (15)$$

where Z is a constant matrix of appropriate dimension, and

$$T_1 = R\Gamma^{+}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad T_2 = \left(I - \Gamma\Gamma^{+}\right)\begin{bmatrix} I \\ 0 \end{bmatrix},$$
$$K_1 = R\Gamma^{+}\begin{bmatrix} 0 \\ I \end{bmatrix}, \quad K_2 = \left(I - \Gamma\Gamma^{+}\right)\begin{bmatrix} 0 \\ I \end{bmatrix}$$

Now we define matrix $\Sigma = \begin{bmatrix} R \\ E^{\perp}A \\ C \end{bmatrix}$. From condition *a.* from Lemma 3.1 and (11), we have

$$N\left(R - K\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix}\right) + F\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = TA \quad (16)$$

which can be written as

$$\left[N\ \tilde{K}\right]\Sigma = TA \quad (17)$$

where $\tilde{K} = F - NK$. The necessary and sufficient condition for the existence of a solution to (17) is

$$rank\begin{bmatrix} \Sigma \\ TA \end{bmatrix} = rank(\Sigma)$$

the general solution to (17) is

$$\left[N\ \tilde{K}\right] = TA\Sigma^{+} - Y_1(I - \Sigma\Sigma^{+}) \quad (18)$$

if we replace (14) in (18), we obtain

$$N = N_1 - ZN_2 - Y_1N_3 \quad (19)$$
$$\tilde{K} = \tilde{K}_1 - Z\tilde{K}_2 - Y_1\tilde{K}_3 \quad (20)$$

where

$$N_1 = T_1A\Sigma^{+}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad N_2 = T_2A\Sigma\begin{bmatrix} I \\ 0 \end{bmatrix},$$
$$N_3 = (I - \Sigma\Sigma^{+})\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \tilde{K}_1 = T_1A\Sigma^{+}\begin{bmatrix} 0 \\ I \end{bmatrix},$$
$$\tilde{K}_2 = T_2A\Sigma^{+}\begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \tilde{K}_3 = (I - \Sigma\Sigma^{+})\begin{bmatrix} 0 \\ I \end{bmatrix}$$

since $F = \tilde{K} + NK$, we have

$$F = \tilde{K}_1 + N_1K - Z(\tilde{K}_2 - N_2K) - Y_1(\tilde{K}_3 - N_3K)$$
$$F = F_1 - ZF_2 - Y_1F_3 \quad (21)$$

where

$$F_1 = T_1A\Sigma^{+}\begin{bmatrix} K \\ I \end{bmatrix}, \; F_2 = T_2A\Sigma^{+}\begin{bmatrix} K \\ I \end{bmatrix},$$
$$F_3 = (I - \Sigma\Sigma^{+})\begin{bmatrix} K \\ I \end{bmatrix}.$$

From (11) we have

$$\begin{bmatrix} TE \\ E^{\perp}A \\ C \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}\Sigma \quad (22)$$

Conditions *c.* and *d.* of Lemma 3.1 can be written as

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix}\begin{bmatrix} TE \\ E^{\perp}A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (23)$$

if we substitute equation (22) into (23), we obtain

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix}\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}\Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (24)$$

The necessary and sufficient condition for the existence of a solution to (24) is $rank\begin{bmatrix} \Sigma \\ L \end{bmatrix} = rank(\Sigma)$ and since $\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$ the general solution is given by

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} = \left(\begin{bmatrix} 0 \\ L \end{bmatrix}\Sigma^{+} - Y(I - \Sigma\Sigma^{+})\right)\begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \quad (25)$$

where $Y = \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$ is an arbitrary matrix of appropriate dimensions. In this case the particular solutions for S, M, P and Q are given by

$$S = -Y_2N_3 \quad (26)$$
$$M = -Y_2F_3 \quad (27)$$
$$P = P_1 - Y_3N_3 \quad (28)$$
$$Q = Q_1 - Y_3F_3 \quad (29)$$

where

$$P_1 = L\Sigma^{+}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad Q_1 = L\Sigma^{+}\begin{bmatrix} K \\ I \end{bmatrix}.$$

Now, by using (19) and (26), the error dynamics (10) can be rewritten as

$$\dot{\sigma}(t) = (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\sigma(t) \quad (30)$$

where

$\mathbb{A}_1 = \begin{bmatrix} N_1 - ZN_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & J \\ Y_2 & G \end{bmatrix}$ and $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$.

The problem has been now reduced to find matrices $\mathbb{Y}$ and Z such that matrix $\mathbb{A}$ is Hurwitz. This can be reached by using the linear matrix inequality (LMI) approach.

4. OBSERVER DESIGN

Theorem 4.1. Under Assumption 1 there exist two matrices $\mathbb{Y}$ and Z such that (30) is asymptotically stable if there exists a matrix

$$X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$$

such that the following LMI is satisfied:

$$N_3^{T\perp}\left(N_1^TX_1 + X_1N_1 - N_2^TW^T - WN_2\right)N_3^{T\perp T} < 0. \quad (31)$$

Where $Z = X_1^{-1}W$ and matrix $\mathbb{Y}$ is determined as follows

$$\mathbb{Y} = -X^{-1}(-\gamma\mathcal{B}^T + \sqrt{\gamma}\mathcal{L}\Omega^{1/2})^T \quad (32)$$

where

$$\Omega = \gamma\mathcal{B}\mathcal{B}^T - \mathcal{Q} > 0 \quad (33)$$

with

$$\mathcal{Q} = \begin{bmatrix} X_1(N_1 - ZN_2) + (N_1 - ZN_2)^TX_1 & (N_1 - ZN_2^T)X_1 \\ X_1(N_1 - ZN_2) & 0 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} N_3^T & 0 \\ 0 & -I \end{bmatrix}$$

and matrix $\mathcal{L}$ is any matrix such that $\|\mathcal{L}\| < 1$ and $\gamma > 0$ is any scalar such that $\Omega > 0$.

Proof 2. We define the following Lyapunov function candidate

$$V(\sigma(t)) = \sigma(t)^TX\sigma(t) \quad (34)$$

its derivative is given by

$$\dot{V}(\sigma(t)) = \sigma(t)^T\left[(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX + X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\right]\sigma(t) \quad (35)$$

The asymptotic stability of (30) is guaranteed only if $\dot{V}(\sigma(t)) < 0$, this leas to the following LMI

$$\mathbb{A}_1^T X - \mathbb{A}_2^T \mathbb{Y}^T X + X\mathbb{A}_1 - X\mathbb{Y}\mathbb{A}_2 < 0 \tag{36}$$

which can be rewritten as

$$\mathcal{Q} + \mathcal{B}\mathcal{X} + (\mathcal{B}\mathcal{X})^T < 0 \tag{37}$$

where $\mathcal{X} = -\mathbb{Y}^T X$, $\mathcal{Q} = X\mathbb{A}_1 + \mathbb{A}_1^T X$ and $\mathcal{B} = \mathbb{A}_2^T$. According to Lemma 2.3, there exists a matrix $\mathcal{X}$ satisfying (37) if and only if the following condition holds

$$\mathcal{B}^{\perp}\mathcal{Q}\mathcal{B}^{\perp T} < 0 \tag{38}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} N_3^{T\perp} & 0 \end{bmatrix}$. By using the definitions of $\mathcal{Q}$ and W, we obtain (31). Matrix $\mathbb{Y}$ is obtained from (32) and (33).

The following algorithm summarize the observer design to obtain the corresponding matrices.
Algorithm 1:

(1) Choose a matrix R such that $rank \begin{bmatrix} R \\ \Gamma \end{bmatrix} = rank(\Gamma)$.
(2) Compute matrices N_1, N_2, N_3, T_1, T_2, K_1, K_2, P_1 and Q_1.
(3) Solve the LMI (31) to find X and Z.
(4) Choose a matrix $\mathcal{L}$ such that $\|\mathcal{L}\| < 1$, and a scalar $\gamma > 0$ such that $\Omega > 0$, then determinate matrix $\mathbb{Y}$ as in (32).
(5) Compute all the matrices gains of the observer (5) by using (19) to determinate N, (32) to determinate J and G, (26)-(29) to find S, M, P and Q taking matrix $Y_3 = 0$. F is given by (21) and matrix H could be determined with condition *b.* of Lemma 2.1.

5. PARTICULAR CASE

5.1 Proportional Functional observer

In order to obtain a Proportional Functional Observer (PFO) from the GFO, it corresponds to the parameter matrices $S = 0$, $J = 0$, $M = 0$, $G = 0$, $F = [0 \ F_a]$ and $Q = [0 \ Q_a]$, which generates the following observer (Trinh and Fernando, 2011):

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + F_a y(t) + Hu(t) \\ \hat{z}(t) &= P\zeta(t) + Q_a y(t) \end{aligned} \tag{39}$$

and the error dynamics (10) becomes

$$\dot{\varepsilon}(t) = \left(\tilde{\mathbb{A}}_1 - \tilde{\mathbb{Y}}\tilde{\mathbb{A}}_2\right)\varepsilon(t) \tag{40}$$

where $\tilde{\mathbb{A}}_1 = N_1 - ZN_2$, $\tilde{\mathbb{Y}} = Y_1$ and $\tilde{\mathbb{A}}_2 = N_3$. Matrices $\mathcal{Q}$, $\mathcal{B}$ and $\mathcal{X}$ of Theorem 1 become:

$$\mathcal{Q} = X(N_1 - ZN_2) + (N_1 - ZN_2)^T X, \ \mathcal{B} = N_3^T,$$
$$\mathcal{X} = -Y_1^T X$$

Matrices Γ and Σ are defined as $\Gamma = \begin{bmatrix} E \\ C \end{bmatrix}$ and $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$. With these matrices, the observer matrices can be obtained following the Algorithm 1.

6. NUMERICAL EXAMPLES

This section presents two numerical examples to illustrate the results obtained in this paper. The first one shows an unstable system which is stabilized by using a functional observer to estimate the control law signal. The second one is a stable system with uncertainties in the matrix A in which the not measured state is estimated.

6.1 Example 1

Consider an unstable descriptor system of the form (1),

$$E = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}, \quad A = \begin{bmatrix} 1&0&2&1 \\ 1&-1&1&0 \\ 1&1&-1&0 \\ 0&1&0&1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0&1 \\ 1&0 \\ 1&1 \\ 0&0 \end{bmatrix}, \quad C = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \quad eig(E, A) = \begin{bmatrix} 0.61 \\ -0.61 \\ -2 \end{bmatrix}$$

It can be seen that one of the eigenvalues of the descriptor system is positive, which means that the system is unstable. The aim is to estimate a control signal $u(t) = -Kx(t)$ that allows to stabilize the system, by using a functional observer. In this case, matrix L is determined using the matrix pencil $(\lambda E - A + BK)$, this gives us

$$L = -K = \begin{bmatrix} 0.34 & 1.13 & -0.19 & -0.26 \\ 1.43 & -0.01 & 0.24 & 0.39 \end{bmatrix}.$$

By choosing matrix $R = L$ and following Algorithm 1 the following matrices are obtained

$$N_1 = \begin{bmatrix} 0.14 & 0.59 \\ -0.18 & 0.32 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -0.02 & 0.46 \\ -0.11 & 0.62 \\ 0.27 & 0.18 \\ 0.21 & -0.09 \\ 0.02 & -0.46 \\ 0.09 & -0.16 \\ 0.02 & -0.46 \\ -0.09 & 0.16 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.42 & 0.09 \\ 0.09 & 0.11 \\ -0.27 & -0.18 \\ -0.21 & 0.09 \\ -0.21 & -0.19 \\ 0.28 & -0.11 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.58 & -0.09 \\ -0.09 & 0.89 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.73 & -0.18 & 0 & 0 \\ -0.18 & 0.55 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.27 & 0.18 & 0 & 0 \\ 0.18 & 0.45 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ -0.27 & -0.18 & 0 & 0 \\ 0.09 & -0.27 & 0 & 0 \\ -0.27 & -0.18 & 0 & 0 \\ -0.09 & 0.27 & 0 & 0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0.27 & -0.09 & 0.27 & 0.09 \\ 0.18 & 0.27 & 0.18 & -0.27 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -0.27 & 0.09 & -0.27 & -0.09 \\ -0.18 & -0.27 & -0.18 & 0.27 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.27 & -0.09 & 0.27 & 0.09 \\ -0.09 & 0.36 & -0.09 & -0.36 \\ 0.27 & -0.09 & 0.27 & 0.09 \\ 0.09 & -0.36 & 0.09 & 0.36 \end{bmatrix}$$

by using YALMIP toolbox to solve LMI (31), matrices X and Z are obtained as

$$X = \begin{bmatrix} 15.23 & 0 & 15.23 & 0 \\ 0 & 15.23 & 0 & 15.23 \\ 15.23 & 0 & 30.46 & 0 \\ 0 & 15.23 & 0 & 30.46 \end{bmatrix}, \; Z = \mathbf{0}.$$

Considering $\gamma = 1000$ and

$$\mathcal{L} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix},$$

to obtain

$$\mathbb{Y} = [\, \mathbb{Y}_1 \;\; \mathbb{Y}_2 \,]$$

where $\mathbb{Y}_1 = \begin{bmatrix} 52.3 & 8.18 & -38.4 & -30.08 \\ 12.87 & 15.54 & -22.11 & 13.0 \\ -29.41 & -7.35 & 15.94 & 11.78 \\ -9.71 & -11.05 & 7.78 & -9.77 \end{bmatrix}$ and

$$\mathbb{Y}_2 = \begin{bmatrix} -30.01 & 33.92 & 62.64 & -3.01 \\ -23.16 & -12.15 & 1.68 & 67.33 \\ 11.75 & -20.22 & -67.41 & -1.75 \\ 8.31 & 2.8 & -4.12 & -69.77 \end{bmatrix}.$$

With matrices $\mathbb{Y}$ and Z the observer matrices can be determined

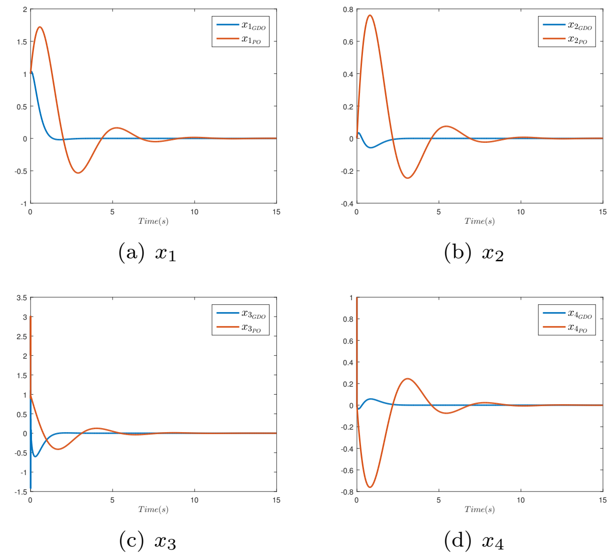
$$N = \begin{bmatrix} -54.85 & -11.2 \\ -11.55 & -13.2 \end{bmatrix}, \; J = \begin{bmatrix} 62.64 & -3.01 \\ 1.68 & 67.33 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.18 & 0.73 \\ 0.55 & -0.18 \end{bmatrix}, \; S = \begin{bmatrix} 27.84 & 5.24 \\ 6.04 & 6.11 \end{bmatrix},$$
$$G = \begin{bmatrix} -67.41 & -1.75 \\ -4.11 & -69.77 \end{bmatrix}, \; P = \begin{bmatrix} 0.58 & -0.09 \\ -0.09 & 0.89 \end{bmatrix},$$

$$F = \begin{bmatrix} 18.79 & 28.26 & 11.77\ 3-37.28 \\ 17.79 & -13.54 & 18.79 & 15.53 \end{bmatrix}$$
$$M = \begin{bmatrix} -8.71 & -15.28 & -4.39 & 20.4 \\ -8.11 & 5.25 & -8.36 & -6.07 \end{bmatrix}$$
$$Q = \begin{bmatrix} 0.41 & 0.13 & 0.34 & 0.21 \\ -0.32 & 0.17 & 0.33 & -0.15 \end{bmatrix}$$

In order to provide a comparison between the performances of the observers we have designed a Proportional Functional Observer with the parameter matrices being

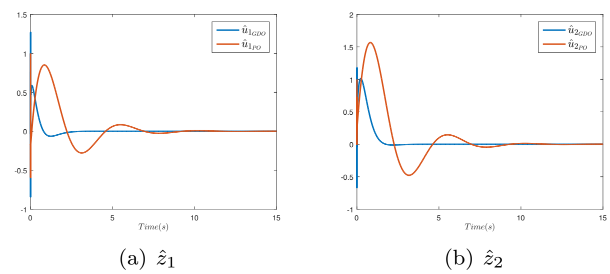
$$N = \begin{bmatrix} -0.5 & -0.13 \\ 0.13 & -0.5 \end{bmatrix}, \; F_a = \begin{bmatrix} 3.5 & 1.17 \\ 0.67 & 0.17 \end{bmatrix} \; J = \begin{bmatrix} -1.5 & -0.5 \\ 1.33 & 0.33 \end{bmatrix}$$
$$P = \begin{bmatrix} 0.34 & 1.13 \\ 1.43 & -0.01 \end{bmatrix} \; Q_a = \begin{bmatrix} -0.19 & -0.26 \\ 0.24 & 0.39 \end{bmatrix}$$

The results of the simulation are shown in Figs. 1-2. The input is the estimate control signal $\hat{z}(t)$ and initial conditions are $x(0) = [1 \;\; 0 \;\; 2 \;\; 1]$. Fig. 1 shows the dynamical behavior of the states and Fig. 2 shows the estimated control signals.



(a) x_1 (b) x_2 (c) x_3 (d) x_4

Fig. 1. Stabilization of the state.



(a) $\hat{z}_1$ (b) $\hat{z}_2$

Fig. 2. Estimate of z

6.2 Example 2

This case is a stable descriptor system of the form (30) to determinate the not measured state. The main purpose of this example is to show the performance of a minimum order GFO and a minimum order PFO in presence of parametric uncertainties. The matrices of the system are

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 & 2 & -3 \\ 0 & -2 & -1 & -2 \\ 1 & 0 & -1 & -3 \\ 2 & 0 & 6 & -2 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = [1 \;\; 0 \;\; 0 \;\; 0]$$

In order to make a minimum order observer, we choose $R = L$. The GFO parameter matrices are determined by following Algorithm 1, which gives us

$$N = -262.17, \; J = 1451.6, \; Q = [0.11 \;\; 0 \;\; -0.47 \;\; 0.22],$$
$$H = 0.85, \; S = 145.78, \; V = -1493.9, \; P = 0.91,$$
$$F = [111.13 \;\; 0 \;\; -723.09 \;\; 220.37],$$
$$M = [-62.09 \;\; 0 \;\; 404.96 \;\; -124.19].$$

For the proportional case we choose $R = L$, the observer matrices are

$$N = -0.69, \; J = 1, \; F_a = [0 \;\; 2.79 \;\; -3.37]$$
$$P = 1, \; Q_a = [0 \;\; -0.03 \;\; 0]$$

To evaluate the performances of the observers, an uncertainty $\Delta A(t)$ is added to the matrix A, then we obtain the matrix $(A + \Delta A(t))$ where $\Delta A(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \delta(t)$.

The results of the simulation are shown in Figs 4. The input is taken constant with $u(t) = 1$ and initial conditions $x(0) = [1 \;\; 0 \;\; 2 \;\; 4]$.

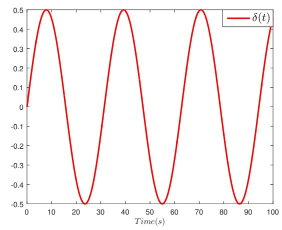
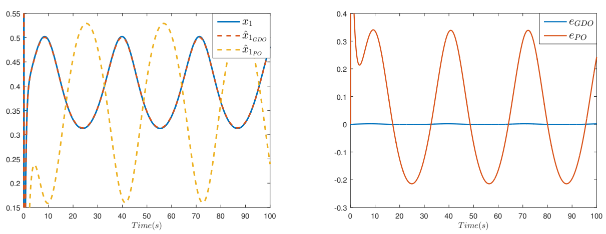


Fig. 3. Uncertainty factor $\delta(t)$



(a) Estimate of x_1

(b) Estimation error

Fig. 4. Results for example 2

7. CONCLUSION

A generalized functional observer for descriptor systems is proposed. Stability conditions are given in order to ensure asymptotically convergence of the estimation error. The observer gains are easily computed by solving a set of linear matrix inequalities. An algorithm is provided in order to highlight the main steps involved in the functional observer design. The main contribution of this work is that the proposed functional observer can be configured as a proportional functional observer for descriptor systems or for conventional LTI systems by considering $E = I$ in system (1). A great advantage of this general representation is the variety of applicability in different tasks. For instance, a proportional observer can be enough for stabilizing unstable systems via feedback control. However the integral term of the observer could be necessary when steady state errors affects the desired estimated functions.

REFERENCES

Alma, M. and Darouach, M. (2014). Adaptive observers design for a class of linear descriptor systems. *Automatica*, 50(2), 578–583.

Araujo, J.M., Barros, P.R., and Dorea, C.E. (2012). Design of observers with error limitation in discrete-time descriptor systems: A case study of a hydraulic tank system. *IEEE Transactions on Control Systems Technology*, 20(4), 1041–1047.

Arefinia, E., Talebi, H.A., and Doustmohammadi, A. (2017). A robust adaptive observer for a class of singular nonlinear uncertain systems. *International Journal of Systems Science*, 48(7), 1404–1415.

Dang, Q.V., Vermeiren, L., Dequidt, A., and Dambrine, M. (2015). Design of stabilizing controller for uncertain nonlinear systems under input saturation. In *American Control Conference (ACC), 2015*, 5974–5979. IEEE.

Darouach, M. (2012). On the functional observers for linear descriptor systems. *Systems and Control Letters*, 61(3), 427–434.

de Oliveira, M.C. (2005). A robust version of the elimination lemma. *IFAC Proceedings Volumes*, 38(1), 310–314.

Emami, K., Fernando, T., Nener, B., Trinh, H., and Zhang, Y. (2015). A functional observer based fault detection technique for dynamical systems. *Journal of the Franklin Institute*, 352(5), 2113–2128.

Estrada-Manzo, V., Lendek, Z., and Guerra, T.M. (2015). Unknown input estimation for nonlinear descriptor systems via lmis and takagi-sugeno models. In *Decision and Control (CDC), 2015 IEEE 54th Annual Conference on*, 6349–6354. IEEE.

Fernando, T., Emami, K., Yu, S., Iu, H.H.C., and Wong, K.P. (2016). A novel quasi-decentralized functional observer approach to lfc of interconnected power systems. *IEEE Transactions on Power Systems*, 31(4), 3139–3151.

Guelton, K., Delprat, S., and Guerra, T.M. (2008). An alternative to inverse dynamics joint torques estimation in human stance based on a takagi–sugeno unknown-inputs observer in the descriptor form. *Control Engineering Practice*, 16(12), 1414–1426.

Hou, Y., Zhu, F., Zhao, X., and Guo, S. (2017). Observer design and unknown input reconstruction for a class of switched descriptor systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*.

Kiss, A.N., Marx, B., Mourot, G., Schutz, G., and Ragot, J. (2011). State estimation of two-time scale multiple models. application to wastewater treatment plant. *Control Engineering Practice*, 19(11), 1354–1362.

Koenig, D., Marx, B., and Varrier, S. (2016). Filtering and fault estimation of descriptor switched systems. *Automatica*, 63, 116–121.

Lan, J. and Patton, R.J. (2015). Robust fault-tolerant control based on a functional observer for linear descriptor systems. *IFAC-PapersOnLine*, 48(14), 138–143.

Osorio-Gordillo, G.L., Darouach, M., and Astorga-Zaragoza, C.M. (2014). H_∞ dynamical observer design for linear descriptor systems. In *American Control Conference (ACC), 2014*, 4829–4833. IEEE.

Osorio-Gordillo, G.L., Darouach, M., Astorga-Zaragoza, C.M., and Boutat-Baddas, L. (2016). New dynamical observers design for linear descriptor systems. *IET Control Theory & Applications*, 10(17), 2223–2232.

Ren, J. and Zhang, Q. (2010). Pd observer design for descriptor systems: an lmi approach. *International Journal of Control, Automation and Systems*, 8(4), 735–740.

Shafarevich, I.R. and Remizov, A. (2012). *Linear algebra and geometry*. Springer Science & Business Media.

Trinh, H. and Fernando, T. (2011). *Functional observers for dynamical systems*, volume 420. Springer Science & Business Media.