Discontinuous Integral Control for Systems in Controller Form

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Abstract: In this paper we provide a discontinuous integral controller for systems in normal form and having relative degrees two and three. This controller is insensitive to matched Lipschitz perturbations, i.e. with bounded derivative. The closed loop system is designed to be homogeneous and we prove the global finite time stability of the equilibrium point by constructing explicitly a homogeneous and smooth Lyapunov function.

Keywords: Control Lyapunov functions, Integral control

1. INTRODUCTION

In the literature there exist several techniques Khalil (2002) to design (continuous) state feedback controllers $v = k(x)$, which are able to stabilize the origin of the system

$$\dot{x} = f(x) + g(x)(v + \rho(t)), \tag{1}$$

in the absence of persistently acting perturbations or uncertainties $\rho(t)$. An important approach uses Lyapunov functions for the design. In the presence of non vanishing perturbations, asymptotic stability of the origin is not achievable and the best to be expected is to obtain "practical" stability or Input-to-State Stability (ISS) with respect to $\rho(t)$. This kind of uncertainties/perturbations appear naturally in the modeling, since it is essentially impossible to know the exact value of all parameters of and even the exact form of the model. Moreover, the parameters can vary with time, as e.g. the gravity term in mechanical systems Fujishiro et al. (2016).

Full compensation of (bounded and matched) persistent perturbations/uncertainties can be achieved by means of classical or Higher Order Sliding Mode control Fridman L. and A. Levant (2002); Levant (2005)]. However, the main disadvantage of this discontinuous controllers is the presence of the "chattering" phenomenon, caused by the high frequency switching.

An alternative classical tool in control theory to deal with (constant) perturbations is the use of integral action, as e.g. in the classical PID control Khalil (2002). Based on this tool, Moreno (2016) and Kamal et al. (2016) propose a discontinuous integral controller, characterized by the fact that the integral action is discontinuous, and that is able to perfectly compensate not only constant perturbations, but a much more general class of perturbations: Lipschitz perturbations. The control signal is continuous in these algorithms and therefore the effect of chattering caused by the discontinuity is strongly attenuated. The aim of this paper is to extend this idea to an arbitrary order. The main difficulty of this extension is the technical proof of its convergence. In this work we construct explicitly a homogeneous, strong and smooth Lyapunov function for systems in the normal form with relative degrees two and three. However, the technique can be extended without problem to an arbitrary relative degree.

1.1 Problem Statement

In this paper, we consider a system in the (nonlinear) controller of the form

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, ..., n-1, \\ \dot{x}_n &= f(x,t) + \rho(t) + v, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ are the states, $v \in \mathbb{R}$ is the control variable, $f(x,t)$ is known and corresponds to the nominal system, while the term $\rho(t)$ represents uncertainties and/or coupled perturbations.

Our aim is to asymptotically stabilize the origin of system (2), despite of the perturbations. We can first cancel the known dynamic terms, what can be achieved with the feedback control law $v = u - f(x,t)$. After this the system becomes

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, ..., n-1, \\ \dot{x}_n &= u + \rho(t), \end{aligned} \tag{3}$$

where $u \in \mathbb{R}$ is the new control input. Note that, since $\rho(t)$ is an unknown perturbation term, it cannot be (fully) canceled. The presence of this perturbation does not permit to asymptotically stabilize the origin of system with a static continuous feedback control, especially in the case it is not vanishing, i.e. $x = 0 \nrightarrow \rho(t) = 0$.

In the absence of perturbation, i. e. $\rho = 0$, a memoryless continuous state feedback $u = k(x)$ can stabilize the origin, but this it is not possible with perturbation, because the control at the origin is zero, i.e. $k(0) = 0$, while the perturbation is still acting. So we add the integral action, which compensates the perturbation allowing to stabilize the origin of system (3), despite of Lipschitz perturbations.

We consider the control law as

$$
\begin{aligned}
u =& \phi(x) + z, \\
\dot{z} =& \psi(x),
\end{aligned} \tag{4}
$$

where $\phi(x)$ and $\psi(x)$ are homogeneous. Likewise, homogeneity degree of $\psi(x)$ is zero, where we obtain a discontinuous integral control, note that despite to this, the control law (4) is a continuous signal.

1.2 Overview

In this paper we present a controller able to stabilize the origin of system (2) in finite time, compensating matched Lipschitz perturbations, for systems of relative degrees (order) 2 and 3. In section 2, we present some tools to obtain the main result. In Section 3 we present the main result, consisting in two theorems 3 and 4, which propose a controller able to stabilize the system (3) for $n = 2, 3$ respectively. In the section 4 we will prove the theorems based on the explicit construction of Lyapunov functions. At the end, we will present a simulation example for a system of order 3 in section 5, where we will show the advantage to add the integral action on the nominal controller.

2. PRELIMINARIES

Let vector $x \in \mathbb{R}^n$, its dilation operator is defined $\Delta_\epsilon^{\mathbf{r}} := (\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n)$, $\forall \epsilon > 0$, where $r_i > 0$ are the weights of the coordinates and $\mathbf{r}$ is the vector of weights. A function $V : \mathbb{R}^n \to \mathbb{R}$ (respectively, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, or vector-set $F(x) \subset \mathbb{R}^n$) is called $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if the identity $V(\Delta_\epsilon^{\mathbf{r}}) = \epsilon^m V(x)$ holds (or $f(\Delta_\epsilon^{\mathbf{r}} x) = \epsilon^m \Delta_\epsilon^{\mathbf{r}} f(x)$, $F(\Delta_\epsilon^{\mathbf{r}} x) = \epsilon^m \Delta_\epsilon^{\mathbf{r}} F(x)$), [Baccioti and Rosier (2005)], [Moreno (2016)]. Suppose that the vector $\mathbf{r}$ and dilation $\Delta_\epsilon^{\mathbf{r}}$ are fixed. The homogeneous norm is defined by $||x||_{\mathbf{r},p} := \left(\sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}$, $\forall x \in \mathbb{R}^n$, for any $p \geq 1$. The set $S = \{x \in \mathbb{R}^n : ||x||_{\mathbf{r},p} = 1\}$ is the unit sphere. Homogeneous functions and vector fields have interesting properties. Consider V_1 and V_2 two $\mathbf{r}$-homogeneous functions (respectively, a vector field f_1) of degree m_1, m_2 (and l_1), then [Baccioti and Rosier (2005)]: (i) $V_1 V_2$ is homogeneous of degree $m_1 + m_2$, (ii) There exist a constant $c_1 > 0$, such that $V_1 \leq c_1 ||x||_{\mathbf{r},p}^{m_1}$, moreover if V_1 is positive definite, there exists c_2 such that $V_1 \geq c_2 ||x||_{\mathbf{r},p}^{m_1}$, (iii) $\partial V_1(x)/\partial x_i$ is homogeneous of degree $m_1 - r_i$, (iv) $L_f V_1(x)$ is homogeneous of degree $m_1 + l_1$. Likewise, we recall the following well-known property of continuous homogeneous functions

Lemma 1. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ be two continuous homogeneous functions, with weights $\mathbf{r} = (r_1, ..., r_n)$ and degrees m, with $\gamma(x) \geq 0$, such that the following holds

$$\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subseteq \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\},$$

then, there exists a real number λ^* such that, for all $\lambda > \lambda^*$, for all $x \in R^n \setminus \{0\}$ and some $c > 0$, $\eta(x) - \lambda\gamma(x) < -c ||x||_{\mathbf{r},p}^m$. [Andrieu et al. (2008)], [Moreno (2016)]

The last lemma can be extended to discontinuous case, where it is possible to bound the discontinuous homogeneous function by a continuous homogeneous function and applying the lemma 1.

Homogeneous systems also have important properties, as local stability implies global stability and its homogeneity degree says the kind of stability [Baccioti and Rosier (2005)]: (i) $l < 0$ implies finite time stability, (ii) $l = 0$ exponential stability, (iii) $l > 0$ rational stability.

Finally we recall Young´s inequality

Lemma 2. For any positive real numbers $a > 0$, $b > 0$, $c > 0$, $p > 1$ and $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality is always satisfied [Hardy et al. (1951)]

$$ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q}.$$

Along this paper we use the following notation. For a real variable $z \in \mathbb{R}$ and a real number $p \in \mathbb{R}$ the symbol $\lceil z \rfloor^p = |z|^p \operatorname{sign}(z)$ is the signed power p of z. According to this $\lceil z \rfloor^0 = \operatorname{sign}(z)$, $\frac{d}{dz} \lceil z \rfloor^m) = m |z|^{m-1}$ and $\frac{d}{dz} |z|^m = m \lceil z \rfloor^{m-1}$. Note that $\lceil z \rfloor^2 = |z|^2 \operatorname{sign} \neq z^2$, and if p is an odd number then $\lceil z \rfloor^p = z^p$ and $|z|^p = z^p$ for any even integer p. Moreover $\lceil z \rfloor^p \lceil z \rfloor^q = |z|^{p+q}$, $\lceil z \rfloor^p \lceil z \rfloor^0 = |z|^p$ and $\lceil z \rfloor^0 \lceil z \rfloor^p = |z|^p$.

3. MAIN RESULT: INTEGRAL CONTROLLER

Let system (3) with homogeneity degree $d \geq -\frac{1}{n+1}$ and the vector of weights $\mathbf{r} = (r_1, ..., r_n)$, we consider $r_1 = 1$ and therefore $r_i = r_{i-1} + d = 1 + (i-1)d$, $i = 2, ..., n+1$. From [Cruz-Zavala and Moreno (2017)], we know that next nonlinear homogeneous state feedback control law

$$u = -k_n \left\lceil \lceil x_n \rfloor^{\frac{1}{r_n}} + \bar{k}_{n-1} \lceil x_{n-1} \rfloor^{\frac{1}{r_{n-1}}} + ... + \bar{k}_1 x_1 \right\rfloor^{r_{n+1}}, \tag{5}$$

stabilizes the origin of system (3), where $\bar{k}_i = \prod_{j=i}^{n-1} k_j^{\frac{1}{r_{j+1}}}$.

Note that if $d = 0$, we obtain a lineal state feedback controller, but this controller only achieves exponential stability.

For the main result, we consider $d = -\frac{1}{n+1}$. Also we just present the case $n = 2, 3$. However, the general case $n > 3$ it is analogous and the proofs will be showed in Section 4.

3.1 Integral Controller: Case $n = 2$

The system (3), for the case $n = 2$ becomes

$$
\begin{aligned}
\dot{x}_1 =& x_2, \\
\dot{x}_2 =& u + \rho(t),
\end{aligned} \tag{6}
$$

with homogeneity degree $d = -\frac{1}{3}$ and weights $\mathbf{r} = \left(1, \frac{2}{3}\right)$. From (5) we can build the nonlinear homogeneous state feedback control law

$$u = -k_2 \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} x_1 \right\rfloor^{\frac{1}{3}},$$

which stabilizes the origin of system (6) in finite time.

Theorem 3. Consider the plant (6) and a coupled Lipschitz continuous perturbation $\rho(t)$ with Lipschitz constant L. Then the control law

$$
\begin{aligned}
u =& -k_2 \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} x_1 \right\rfloor^{\frac{1}{3}} + z, \\
\dot{z} =& -k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^0,
\end{aligned} \tag{7}
$$

stabilizes the origin in finite time for any k_{I2} and appropriate gains k_1, k_2, k_{I1}.

The Theorem 3 shows that adding a discontinuous integral term, the controller compensates the Lipschitz perturbation. After a finite time, $z(t) = -\rho(t)$ cancels perturbation and therefore the designed controller is ables to stabilize the perturbed system like the nominal system.

3.2 Integral Controller: Case $n = 3$

The system (3), for the case $n = 3$ becomes

$$\begin{aligned} \dot{x}_1 &= x_1, \\ \dot{x}_2 &= x_2, \\ \dot{x}_3 &= u + \rho(t), \end{aligned} \tag{8}$$

with homogeneity degree $d = -\frac{1}{4}$ and weights $\mathbf{r} = \left(1, \frac{3}{4}, \frac{1}{2}\right)$. From (5) we can build the nonlinear homogeneous state feedback control law

$$u = -k_3 \left\lceil \lceil x_3 \rfloor^2 + k_2^2 \lceil x_2 \rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} x_1 \right\rfloor^{\frac{1}{4}}$$

which stabilizes the origin of system (8) in finite time.

Theorem 4. Consider the plant (8) and a coupled Lipschitz continuous perturbation $\rho(t)$ with Lipschitz constant L. Then the control law

$$\begin{aligned} u =& -k_3 \left\lceil \lceil x_3 \rfloor^2 + k_2^2 \lceil x_2 \rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} x_1 \right\rfloor^{\frac{1}{4}} + z, \\ \dot{z} =& -k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2 \rfloor^{\frac{4}{3}} + k_{I3} \lceil x_3 \rfloor^2 \right\rfloor^0, \end{aligned} \tag{9}$$

stabilizes the origin in finite time for any k_{I2}, k_{I3} and appropriate gains k_1, k_2, k_3, k_{I1}.

Again the Theorem 4, show that adding a discontinuous integral term, the controller compensates the Lipschitz perturbation. Note that in both controllers, the principal state to cancel the perturbation is the first state x_1, since other states converge to zero.

Note that by homogeneity of system in both cases, if the gains $(k_1, k_2, k_3, k_{I1}, k_{I2}, k_{I3})$ reach the objective for a perturbation with Lipschitz constant L, then the gains $(\lambda^{\frac{1}{r_1}} k_1, \lambda^{\frac{1}{r_2}} k_2, \lambda^{\frac{1}{r_3}} k_3, \lambda^{\frac{1}{r_1}} k_{I1}, \lambda^{\frac{1}{r_2}} k_{I2}, \lambda^{\frac{1}{r_3}} k_{I3})$ will also stabilize the system for a perturbation with Lipschitz constant λL, for any $\lambda > 0$.

4. LYAPUNOV FUNCTION

We will proof, by using homogeneous and smooth Lyapunov Functions that Theorems 3 and 4 are valid.

4.1 Proof of Theorem 3

Consider the closed loop system of plant (6) with the controller (7), and a new variable $x_3 = z + \rho(t)$

$$\begin{aligned} \dot{x}_1 =& x_2, \\ \dot{x}_2 =& -k_2 \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} x_1 \right\rfloor^{\frac{1}{3}} + x_3, \\ \dot{x}_3 \in& -k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^0 + [-L, L], \end{aligned} \tag{10}$$

We define the variable

$$\xi_1 = x_1 - k_2^{-3} k_1^{-\frac{3}{2}} \lceil x_3 \rfloor^3,$$

and its derivative

$$\dot{\xi}_1 \in x_2 + 3k_2^{-3} k_1^{-\frac{3}{2}} |x_3|^2 \left(k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^0 - [-L, L] \right),$$

Consider the homogeneous and smooth Lyapunov function, proposed in [Cruz-Zavala and Moreno (2017)], but with ξ_1

$$\begin{aligned} V(x) =& \frac{\gamma_1}{m} |\xi_1|^m + \frac{2}{3m} |x_2|^{\frac{3m}{2}} + k_1^{\frac{3m-2}{2}} \lceil \xi_1 \rfloor^{m - \frac{2}{3}} x_2 + \\ & \left(1 - \frac{2}{3m}\right) k_1^{\frac{3m}{2}} |\xi_1|^m + \frac{1}{3m} |x_3|^{3m}, \quad \gamma_1 > 0, \end{aligned}$$

where $m \geq r_1 + r_2 = \frac{5}{3}$. From Lemma 2 it is easy to show that V is a positive definite function.

Its derivative along the trajectories of system (10) is

$$\dot{V}(x) = F_1(x) + F_2(x, L),$$

where

$$\begin{aligned} F_1(x) =& \Bigg[\left(\gamma_1 + \left(\frac{3m-2}{3}\right) k_1^{\frac{3m}{2}} \right) \lceil \xi_1 \rfloor^{m-1} + \\ & \left(\frac{3m-2}{3}\right) k_1^{\frac{3m-2}{2}} |\xi_1|^{\frac{3m-5}{3}} x_2 \Bigg] x_2 - \\ & k_2 \left(\lceil x_2 \rfloor^{\frac{3m-2}{2}} + k_1^{\frac{3m-2}{2}} \lceil \xi_1 \rfloor^{\frac{3m-2}{3}} \right) \alpha(x), \\ \alpha(x) =& \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} \xi_1 + k_2^{-3} \lceil x_3 \rfloor^3 \right\rfloor^{\frac{1}{3}} - k_2^{-1} x_3, \\ F_2(x, L) =& \Bigg(3k_2^{-3} k_1^{-\frac{3}{2}} \left(\gamma_1 + \left(\frac{3m-2}{3}\right) k_1^{\frac{3m}{2}} \right) \lceil \xi_1 \rfloor^{m-1} + \\ & (3m-2) k_2^{-3} k_1^{\frac{3m-5}{2}} |\xi_1|^{\frac{3m-5}{3}} x_2 - \lceil x_3 \rfloor^{3(m-1)} \Bigg) |x_3|^2 \times \\ & \left(k_{I1} \left\lceil \xi_1 + k_{I2} \lceil x_2 \rfloor^{\frac{3}{2}} + k_2^{-3} k_1^{-\frac{3}{2}} \lceil x_3 \rfloor^3 \right\rfloor^0 - [-L, L] \right), \end{aligned}$$

Consider first F_1, we recall the term $\alpha(x)$

$$\alpha(x) = \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} \xi_1 + k_2^{-3} \lceil x_3 \rfloor^3 \right\rfloor^{\frac{1}{3}} - k_2^{-1} x_3,$$

so we can write $\alpha(x)$ as

$$\alpha(x) = \left\lceil \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} \xi_1 + k_2^{-3} \lceil x_3 \rfloor^3 \right\rfloor^{\frac{1}{3}} - \left\lceil k_2^{-3} \lceil x_3 \rfloor^3 \right\rfloor^{\frac{1}{3}},$$

which satisfies (see Appendix A)

$$\operatorname{sign}\big(\alpha(x)\big) = \operatorname{sign}\Big(\lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} \xi_1 \Big).$$

Likewise, it is easy to see that

$$\operatorname{sign}\left(\lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} \xi_1 \right) = \operatorname{sign}\left(\lceil x_2 \rfloor^{\frac{3m-2}{2}} + k_1^{\frac{3m-2}{2}} \lceil \xi_1 \rfloor^{\frac{3m-2}{3}} \right),$$

for some $x, y \in R$ and any $\beta > 0$

$$\begin{aligned} x + y > 0 \Leftrightarrow& x^\beta + y^\beta > 0, \\ x + y = 0 \Leftrightarrow& x^\beta + y^\beta = 0, \\ x + y < 0 \Leftrightarrow& x^\beta + y^\beta < 0, \end{aligned}$$

Therefore, the last term in $F_1(x)$ is negative semidefinite and it is zero only on the set $S_1 = \left\{ x_2 = -k_1 \lceil \xi_1 \rfloor^{\frac{2}{3}} \right\}$. On S_1 the value of F_1 becomes

$$F_1|_{S_1} = -k_1 \gamma_1 |\xi_1|^{m - \frac{1}{3}},$$

which is negative for $k_1 > 0$. Using Lemma 1 it follows that $F_1 < -c \, ||(\xi_1, x_2)||_{\mathbf{r},p}^{\frac{3m-1}{3}}$ for k_2 sufficiently large.

Note that $F_1(x) = 0$ only on the set $S_2 = \{(\xi_1, x_2) = 0\}$. Then, the value of F_2 on S_2 is

$$F_2|_{S_2} = -(k_{I1} - L) \, |x_3|^{3m-1}$$

which is negative for $k_{I1} > L$. Again, Lemma 1 implies that $\dot{V} < 0$ for k_{I1} and L sufficiently small.

4.2 Proof of Theorem 4

The proof of Theorem 4 is analogous as well as for the case n arbitrary, since the methodology is the same.

Consider the closed loop system of plant (8) with the controller (9), and an additional variable $x_4 = z + \rho(t)$

$$
\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= -k_3 \left\lceil \lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} x_1 \right\rfloor^{\frac{1}{4}} + x_4,\\
\dot{x_4} &\in -k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2\rfloor^{\frac{4}{3}} + k_{I3} \lceil x_3\rfloor^2 \right\rfloor^{0} + [-L, L],
\end{aligned}
\tag{11}
$$

we define the variable

$$\xi_1 = x_1 - k_3^{-4} k_2^{-2} k_1^{-\frac{4}{3}} \lceil x_4\rfloor^4 ,$$

and its derivative

$$
\begin{aligned}
\dot{\xi}_1 \in& x_2 + 4k_\xi |x_4|^3 \left(k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2\rfloor^{\frac{4}{3}} + k_{I3} \lceil x_3\rfloor^2 \right\rfloor^{0} - [-L, L] \right),\\
k_\xi =& k_3^{-4} k_2^{-2} k_1^{-\frac{4}{3}},
\end{aligned}
$$

We define the function based in [Cruz-Zavala and Moreno (2017)],

$$
\begin{aligned}
V_2(\xi_1, x_2) =& \frac{\gamma_1}{m}|\xi_1|^m + \frac{2}{3m}|x_2|^{\frac{4m}{3}} + k_1^{\frac{4m-3}{3}} \lceil \xi_1\rfloor^{m-\frac{3}{4}} x_2 +\\
& \left(1 - \frac{3}{4m}\right) k_1^{\frac{4m}{3}} |\xi_1|^m, \quad \gamma_1 > 0,
\end{aligned}
$$

where these terms are equivalent to the four first terms in the Lyapunov function for the case $n = 2$. Now the full Lyapunov function is

$$
\begin{aligned}
V(x) =& \gamma_2 V_2(\xi_1, x_2) + \frac{1}{2m}|x_3|^{2m} + k_2^{2m-1} \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{m-\frac{1}{2}} x_3 +\\
& \left(1 - \frac{1}{2m}\right) k_2^{2m} \left| \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right|^{m} + \frac{1}{4m} |x_4|^{4m}, \quad \gamma_2 > 0.
\end{aligned}
$$

Again, from Lemma 2 it is easy to show that $V(x)$ is a positive definite function.

Its derivative along the trajectories of system (11) is

$$\dot{V}(x) = F_1(x) + F_2(x, L),$$

where

$$
\begin{aligned}
F_1(x) =& \gamma_2 \left(\frac{\partial}{\partial \xi_1} V_2(\xi_1, x_2) x_2 + \frac{\partial}{\partial x_2} V_2(\xi_1, x_2) x_3 \right) +\\
& \left(m - \frac{1}{2}\right) k_2^{2m-1} \left| \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right|^{m-\frac{3}{2}} \times\\
& \left(\frac{4}{3}|x_2|^{\frac{1}{3}} x_3 + k_1^{\frac{4}{3}} x_2\right) \left(x_3 + k_2 \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{\frac{1}{2}} \right) -\\
& k_3 \left(\lceil x_3\rfloor^{2m-1} + k_2^{2m-1} \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{m-\frac{1}{2}} \right) \alpha(x)\\
\alpha(x) =& \left\lceil \lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} \xi_1 + k_3^{-4} \lceil x_4\rfloor^4 \right\rfloor^{\frac{1}{4}} - k_3^{-1} x_4\\
F_2(x, L) =& \left(\gamma_2 4 k_\xi \frac{\partial}{\partial \xi_1} V_2(\xi_1, x_2) + \left(4m - 2\right) k_3^{-4} k_2^{2m-3} \left| \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right|^{m-\frac{3}{2}} \right.\\
& \left. \left(x_3 + k_2 \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{\frac{1}{2}} \right) - \lceil x_4\rfloor^{4(m-1)} \right) |x_4|^3 \times\\
& \left(k_{I1} \left\lceil x_1 + k_{I2} \lceil x_2\rfloor^{\frac{4}{3}} + k_{I3} \lceil x_3\rfloor^2 \right\rfloor^{0} - [-L, L] \right)
\end{aligned}
$$

Consider first F_1, where we recall $\alpha(x)$

$$\alpha(x) = \left\lceil \lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} \xi_1 + k_3^{-4} \lceil x_4\rfloor^4 \right\rfloor^{\frac{1}{4}} - k_3^{-1} x_4,$$

we can write as

$$\alpha(x) = \left\lceil \lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} \xi_1 + k_3^{-4} \lceil x_4\rfloor^4 \right\rfloor^{\frac{1}{4}} - \left\lceil k_3^{-4} \lceil x_4\rfloor^4 \right\rfloor^{\frac{1}{4}}$$

and therefore

$$\operatorname{sign}\big(\alpha(x)\big) = \operatorname{sign}\Big(\lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} \xi_1 \Big).$$

Again, it is easy to see that

$$
\begin{aligned}
&\operatorname{sign}\left(\lceil x_3\rfloor^2 + k_2^2 \lceil x_2\rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} \xi_1 \right) =\\
&\operatorname{sign}\left(\lceil x_3\rfloor^{2m-1} + k_2^{2m-1} \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{m-\frac{1}{2}} \right).
\end{aligned}
$$

Therefore, the last term is negative semidefinite. It is zero only on the set $S_1 = \left\{ x_3 = -k_2 \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{\frac{1}{2}} \right\}$. On S_1 the value of F_1 becomes

$$F_1|_{S_1} = \gamma_2 \left(\frac{\partial}{\partial \xi_1} V_2(\xi_1, x_2) x_2 - k_2 \frac{\partial}{\partial x_2} V_2(\xi_1, x_2) \left\lceil \lceil x_2\rfloor^{\frac{4}{3}} + k_1^{\frac{4}{3}} \xi_1 \right\rfloor^{\frac{1}{2}} \right)$$

Note that x_3 in S_1 becomes into controller for the reduced system of order 2, so it is easy to show that $F_1|_{S_1}$ is negative for k_2 large enough. Therefore, using Lemma 1 it follows that $F_1 < -c\,||(\xi_1, x_2, x_3)||_{\mathbf{r},p}^{m-\frac{1}{4}}$ for k_3, k_2 sufficiently large and $k_1 > 0$.

Again $F_1(x) = 0$ only on the set $S_2 = \{(\xi_1, x_2, x_3) = 0\}$. Then the value of F_3 on S_2 is

$$F_3|_{S_2} = -(k_{I1} - L)\,|x_4|^{4m-1}$$

which is negative for $L < k_{I1}$. Again, Lemma 1 implies that $\dot{V} < 0$ for k_{11} and L sufficiently small.

We can see that it is possible to extend this result for any n.

5. SIMULATION EXAMPLE

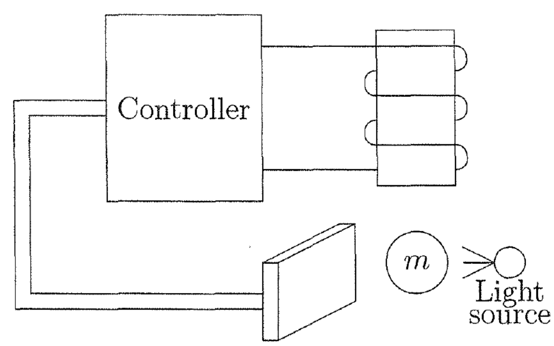

Fig. 1. Magnetic Supension

Consider the dynamic of a magnetic suspension system [Khalil (2002)]

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -\frac{k}{m} x_2 - \frac{k_L}{2m} \frac{x_3^2}{(a + x_1)^2} + g\\
\dot{x}_3 &= \frac{1}{L(x_1)} \Big(-R x_3 + k_L \frac{x_2 x_3}{(a + x_1)^2} + u \Big)
\end{aligned}
$$

where $x_1 = y \in R_+$ is the vertical distance of the ball measured from the coil, $x_2 = \dot{y}$ is the velocity, m is the mass of the ball, g is the gravity acceleration, K is a viscous friction coefficient, $L(x_1) = L_1 + \frac{K_L}{a+x_1}$ is the inductance of the coil (where k_L, L_1 and a are positive constants), $x_3 = i$ is the electric current, R is the electric resistance on the circuit and the control u is the voltage applied. Note that model is local.

We consider the input

$$u = Rx_3 - k_L \frac{x_2 x_3}{(a+x_1)^2} + L(x_1)v,$$

and we obtain

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{k_L}{2m}\frac{x_3^2}{(a+x_1)^2} + g,$$
$$\dot{x}_3 = v,$$

Let $h(x) = x_1$ output of the system and the diffeomorphism well defined for $x_3 > 0$

$$z = T(x) = \begin{bmatrix} h(x) \\ \dot{h}(x) \\ \ddot{h}(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -\frac{k}{m}x_2 - \frac{k_L}{2m}\frac{x_3^2}{(a+x_1)^2} + g \end{bmatrix},$$

Therefore, the system in these coordinates becomes

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 \\ -\frac{k}{m}z_3 - \sqrt{\frac{2k_L}{m}}\frac{\lceil g - \frac{k}{m}z_2 - z_3 \rfloor^{\frac{1}{2}}}{(a+z_1)}v \end{bmatrix},$$

We define the input

$$v = -\sqrt{\frac{m}{2k_L}}\frac{(a+z_1)}{\lceil g - \frac{k}{m}z_2 - z_3 \rfloor^{\frac{1}{2}}}\left(\frac{k}{m}z_3 + w\right),$$

and the obtained system is

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 \\ w + \rho(t) \end{bmatrix},$$

where $\rho(t)$ means a perturbation, due to uncertainties about model of system like gravity, mass, et al. Likewise, it is possible to suppose that there exists a external force on environment.

Now, we can apply the integral control to system in z. We consider the following parameter values $m = 1[kg]$, $g = 9.815[\frac{m}{s^2}]$, $k = 0.1[\frac{N \cdot s}{m}]$, $L_1 = 0.1[H]$, $k_L = 10[mH \cdot m]$, $a = 0.05[m]$, $R = 10[\Omega]$, with a perturbation $\rho(t) = 0.15t$ and the initial conditions $x_1(0) = 0.001[m]$, $x_2 = 0[\frac{m}{s}]$, and $x_3 = 2.2[A]$. Then integral control becomes

$$w = -k_3 \left\lceil \lceil z_3 \rfloor^2 + k_2^2 \lceil z_2 \rfloor^{\frac{4}{3}} + k_2^2 k_1^{\frac{4}{3}} z_1 \right\rfloor^{\frac{1}{4}} + z_4,$$
$$\dot{z} = -k_{I1} \left\lceil z_1 + k_{I2} \lceil z_2 \rfloor^{\frac{4}{3}} + k_{I3} \lceil z_3 \rfloor^2 \right\rfloor^0,$$

and system in closed loop is

$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = z_3,$$
$$\dot{z}_3 = u + z_4,$$
$$\dot{z}_4 = -k_{I1} \left\lceil z_1 + k_{I2} \lceil z_2 \rfloor^{\frac{4}{3}} + k_{I3} \lceil z_3 \rfloor^2 \right\rfloor^0 + \dot{\rho}(t),$$

where $z_4 = z + \rho(t)$.

For simulations we use gains $k_1 = 2$, $k_2 = 9$, $k_3 = 45$, $k_{I1} = 0.5$, $k_{I2} = k_{I3} = 0$.

The behavior of the states is presented in Figures 2-4. In Figure 2 the evolution of the position is showed, where the integral controller brings the position to zero in finite time, later the position change to $x_1 = 10$ cm, and finally position follow a sinewave, while the state feedback has problems with the rising perturbation and it is not able to keep the ball on track.

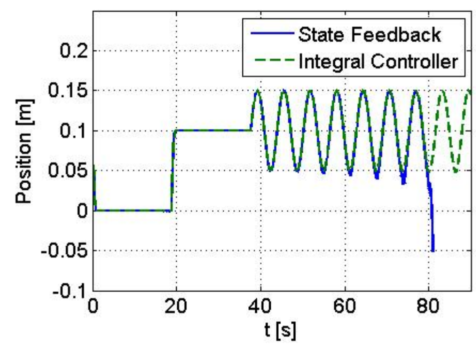

Fig. 2. Time evolution of position (x_1)

Likewise, Figure 3 shows the behavior of the tracking error, where the integral controller converges to zero. while the state feedback diverges to infinity after a time.

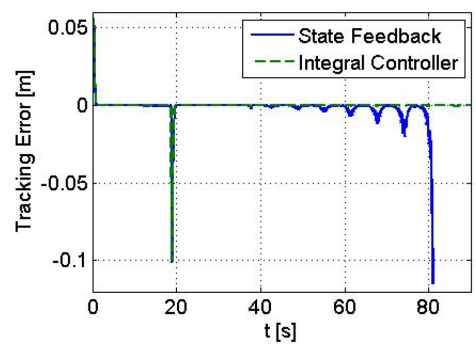

Fig. 3. Time evolution of tracking error

Figure 4 shows the behavior of the velocity, for the integral controller which remains close to zero in regulation case and before it has a sinusoidal behavior. However, in state feedback case the velocity is unstable.

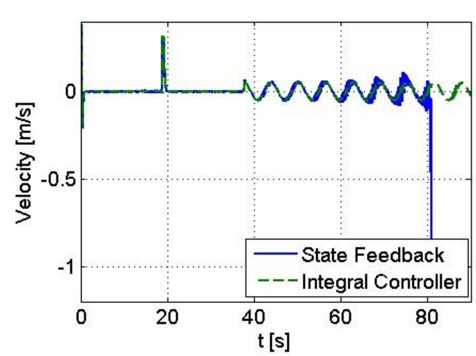

Fig. 4. Time evolution of velocity (x_2)

Figure 5 presents the evolution of the current, which converges to necessary values to lift the ball and again has a sinusoidal behavior for up-down the ball. In other hand, the state feedback controller is not able to keep the ball on track, this implies a variation in current.

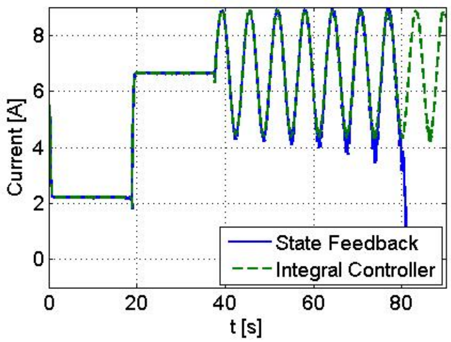

Fig. 5. Time evolution of current (x_3)

Finally, Figure 6 shows the behavior of input. For the integral controller, we can see that the input converges to necessary voltage to generate the current and an additional value to compensate the perturbation. While the state feedback can not compensate it, which causes a input unstable for a large value of the perturbation.

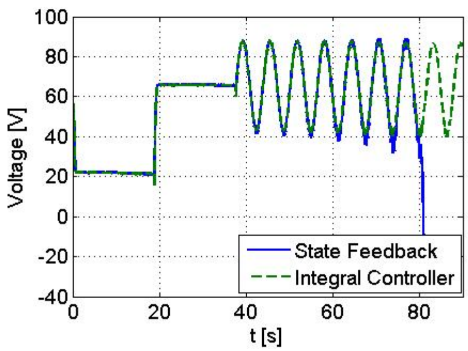

Fig. 6. Time evolution of input (u)

6. CONCLUSION

We present an integral controller, which allows to deal with coupled perturbations unknown with bounded derivative (Lipschitz). The homogeneous discontinuous integral control is able to cancel Lipschitz nondecreasing perturbations.

The approach of Lyapunov helped us to design a controller insensitive to Lipschitz perturbations. So Extending the idea of the integral action in the classical PID controller.

The dynamic controllers are able to compensate not vanishing perturbations with a continuous control signal while the static or memoryless controller can not reach it.

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Appendix A. AUXILIARY PROOF

Consider the function of two real variables

$$F(x,y) = \lceil x+y \rfloor^{\beta} - \lceil y \rfloor^{\beta}, \quad \beta > 0 .$$

Suppose that

$$F(x,y) = 0,$$

and this implies

$$\lceil x+y \rfloor^{\beta} - \lceil y \rfloor^{\beta} = 0 \Leftrightarrow \lceil x+y \rfloor^{\beta} = \lceil y \rfloor^{\beta} \Leftrightarrow x+y = x \Leftrightarrow y = 0.$$

Therefore we can conclude that

$$F(x,y) = 0 \Leftrightarrow y = 0.$$

Analogously, it is possible to show that

$$F(x,y) > 0 \Leftrightarrow y > 0, \text{ and } F(x,y) < 0 \Leftrightarrow y < 0.$$