On the Leader-following Consensus of Distributed Order Multi-agent Systems

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Abstract: This work presents an extension of recent results about the distributed coordination of multi-agent systems. The novelty of the present paper lies in the inclusion of the distributed order derivative in the dynamics of the studied problems, something which has not been investigated yet, to our best knowledge. By applying a generalization of the Lyapunov direct method and properties of inequalities, we give sufficient conditions to guarantee leader-following consensus of certain distributed order multi-agent systems with single and double integrator dynamics. As particular cases of our main results, we get back the already known fractional counterpart of the theory discussed in this article.

Keywords: Fractal systems; Lyapunov method; Dynamic systems; Asymptotic stability; Leader-following consensus; Distributed fractional calculus.

1. INTRODUCTION

Recently, the study of multi-agent systems and their distributed coordination has gained the attention of researchers. This field analyzes the accomplishment of group objectives through the local interaction of the agents. A key concept in distributed coordination is consensus, which determines if the dynamics of the agents will converge to a certain desired value. The relevance of consensus of multi-agent systems is clear when noticing its diverse applications: description of sensor networks, formation of drones, interaction of various satellites, etc. See Bosse and Pantke (2013); Wu et al. (2010).

Consensus of multi-agent systems has been mostly studied from the framework of integer order calculus. The consensus problem of integer order systems with single and double integrator dynamics was analyzed in Djaidja and Wu (2015) and Liu et al. (2015). Furthermore, research has explored models with time delays (Qin et al. (2015)) or external disturbances (Hu et al. (2015)), and controllers have been designed accordingly (Yang et al. (2015)).

Some of the ideas discussed in the previous paragraph have been extended to systems with fractional order dynamics. For instance: Yu et al. (2015) explored the leader-following consensus of fractional order nonlinear multi-agent systems described by directed graphs; and Cao et al. (2010) investigated the distributed coordination of networked multi-agent systems with fractional dynamics.

In addition to the articles already cited, it is worth highlighting the work done in Ren et al. (2015), which presents a method for determining if nonlinear fractional multi-agent systems, with single and double integrator dynamics, achieve leader-following consensus. That paper is the foundation of the present work, where we will generalize leader-following consensus results for systems with distributed order derivatives.

In summary, distributed order calculus generalizes the fractional and integer order differential operators. Besides being mathematically interesting, the distributed order derivative has been proposed to model certain physical systems in a more adequate fashion (see Naber (2004)). A conceptual interpretation of this new operator is provided in Lorenzo and Hartley (2002); there, it is argued that the differentiation order in a given problem can depend on anisotropic properties of the physical system in question, where each differential element should have its own order of differentiation.

Motivated by the above discussion, we present methods that allow proving if certain distributed order multi-agent systems achieve leader-following consensus. We consider the cases where single and double integrator dynamics are involved. By choosing appropriately the weight function of the distributed derivative, we recover Theorems 11 and 13 of Ren et al. (2015). As far as we are concerned, the present work is the first one that studies the consensus problem of multi-agent systems using distributed order calculus.

2. PRELIMINARY CONCEPTS

In this section some basic definitions and properties of the distributed order derivative are presented, as well as some concepts of Lyapunov stability, graph theory, and consensus of multi-agent systems.

2.1 Distributed order calculus and Lyapunov stability

Definition 1. Aguila-Camacho et al. (2014). The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ is defined as follows

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\zeta)}{(t-\zeta)^{\alpha-m+1}} d\zeta, \quad t > t_0 \tag{1} $$

where $m = \min\{k \in \mathbb{N} \mid k > \alpha\}$ and $\Gamma(r) = \int_0^{\infty} t^{r-1} e^{-t} dt$.

Definition 2. Jiao et al. (2012). The distributed order derivative with respect to the density function (weight function) $b(\alpha) \geqslant 0$ is defined as follows:

$$ {}_{t_0}^{C}D_t^{b(\alpha)} x(t) = \int_{m-1}^{m} b(\alpha) x^{(\alpha)}(t) d\alpha, \tag{2} $$

where $m-1 < \alpha < m$ and $x^{(\alpha)}(t) = {}_{t_0}^{C}D_t^{\alpha} x(t)$.

The Laplace transform of a distributed order derivative is

$$ \mathcal{L}\{ {}_{0}^{C}D_t^{b(\alpha)} x(t)\} = \int_{m-1}^{m} b(\alpha) \left[s^{\alpha} X(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} \times x^{(k)}(0^+) \right] d\alpha = B(s)X(s) - \sum_{k=0}^{m-1} \frac{1}{s^{k+1}} B(s) x^{(k)}(0^+), \tag{3} $$

where $B(s) = \int_{m-1}^{m} b(\alpha) s^{\alpha} d\alpha$. Since we will work with $\alpha \in [0,1]$ then $m = 1$.

Hypothesis 3. In this text we will focus on distributed order systems of the form

$$ {}_{t_0}^{C}D_t^{b(\alpha)} x(t) = f(x(t), t), \tag{4} $$

where $f(x(t),t) \in L^1[0,\infty]$, $x(t) \in \mathbb{R}$ is such that ${}_{t_0}^{C}D_t^{\alpha} x(t) < M$ for $t \in [0,\infty)$ and $\forall \alpha \in (0,1)$, $b(\alpha)$ is an absolutely integrable function on the interval $\alpha \in [0,1]$ and satisfies that $\int_0^1 b(\alpha) s^{\alpha} d\alpha \neq 0$, for $Re(s) > 0$. This kind of system has a unique solution given in the next Theorem.

Theorem 4. Ford and Morgado (2012). If Hypothesis 3 is satisfied, then system (4) has the unique solution:

$$ x(t) = x(0) + \left(f * \mathcal{L}^{-1} \left[\frac{1}{\int_0^1 b(\alpha)(s)^{\alpha} d\alpha} \right] \right)(t), \tag{5} $$

where $*$ is the convolution operator, i.e. $g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau$.

Note that Theorem 4 can be easily extended for the case when $x(t) \in \mathbb{R}^n$.

Definition 5. The constant x_0 is said to be an equilibrium point of the system (4), if $f(x_0, t) = 0$, $\forall t$.

The next Theorem generalizes the Lyapunov direct method for nonlinear time-varying distributed order systems.

Theorem 6. Fernández-Anaya et al. (2017). Let $x = 0$ be an equilibrium point for the system (4). Assume that there exists a Lyapunov function $V(x(t), t)$ satisfying

$$ \alpha_1 \|x\|^a \leqslant V(x(t), t) \leqslant \alpha_2 \|x\|^{ab}, \tag{6} $$

$$ {}_{0}^{C}D_t^{b(\alpha)} V(x(t), t) \leqslant -\alpha_3 \|x\|^{ab}, \tag{7} $$

where $\alpha \in (0,1)$ and $a, b, \alpha_i > 0$, $i = 1, 2, 3$. If the roots of $B(s) + \alpha_3/\alpha_2 = 0$ are in the open left-half complex plane, and $b(\alpha)$ is such that $\mathcal{L}^{-1}\{1/(B(s) + \frac{\alpha_3}{\alpha_2})\} \geqslant 0$, $\forall t \geqslant 0$, then the origin of the system is asymptotically stable.

A useful property for building appropriate Lyapunov functions for distributed order systems is presented next:

Lemma 7. Fernández-Anaya et al. (2017). Let $x(t) \in \mathbb{R}^n$. Then, for $t \geqslant t_0$ and $\forall \alpha \in (0,1]$, the following relationship holds:

$$ \frac{1}{2} {}_{t_0}^{C}D_t^{b(\alpha)} x^T(t) x(t) \leqslant x(t)^T \, {}_{t_0}^{C}D_t^{b(\alpha)} x(t). \tag{8} $$

2.2 Graph theory fundamentals

Now, we will discuss some concepts of graph theory (see Ren and Beard (2008)). Graphs can be used to describe the interaction topology of multi-agent systems. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$ is defined by its vertices, contained in the set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, and its edges, elements of the set $\mathcal{W} \subseteq \mathcal{V}^2$. In our case, each agent corresponds to a vertex. Throughout this text, we will consider undirected graphs, where each edge is an unordered pair $(v_i, v_j) = (v_j, v_i)$ which signifies that agent j can receive the state information of agent i and vice versa; on this situation it is said that agent i and agent j are neighbors. The neighbors of agent i are denoted by N_i. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ are used to represent algebraically the interaction graph; these are defined as $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{W}$, $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{W}$, $l_{ii} = \sum_{j \in N_i} a_{ij}$, and $l_{ij} = -a_{ij}$ for $i \neq j$.

The following Lemma presents a property of multi-agent systems that will be useful in the proofs of our main results.

Lemma 8. Ren and Cao (2010). If an interaction graph is undirected and connected, then the matrix $H = L + B$ will be symmetric and positive definite, where $B = \text{diag}\{b_1, b_2, \ldots, b_n\}$, $i \in \{1, 2, \ldots, n\}$, and $b_i \geqslant 0$ does not always equal 0.

2.3 Consensus of multi-agent systems

The definition of a Lipschitz function, relevant in the study of multi-agent systems, is presented next.

Definition 9. Podlubny (1999). Let $k(t,x)$ be a real-valued continuous function, defined in the domain $G \subseteq \mathbb{R}$. $k(t,x)$ is said to satisfy the Lipschitz condition with respect to x in the domain G if,

$$ |k(t, x_1) - k(t, x_2)| \leqslant \theta |x_1 - x_2|, \tag{9} $$

where θ is defined as the Lipschitz constant.

The next definitions present the concept of leader-following consensus for multi-agent systems with single and double integrator dynamics.

Definition 10. Ren et al. (2015). A multi-agent system of n agents with single integrator dynamics achieves leader-following consensus if, for any initial condition, its solution satisfies

$$\lim_{t\to\infty} \|x_i(t) - x_r(t)\| = 0, \ \ i \in \{1, 2, \ldots, n\}, \qquad (10)$$

where $x_i(t)$ is the state of the ith agent and $x_r(t)$ is the state of the virtual leader.

Definition 11. Ren et al. (2015). A multi-agent system of n agents with double integrator dynamics achieves leader-following consensus if it satisfies the following equalities, for any initial condition:

$$\lim_{t\to\infty} \|x_{i0}(t) - x_{r0}(t)\| = 0, \ \ \lim_{t\to\infty} \|x_{i1}(t) - x_{r1}(t)\| = 0, \\ i \in \{1, 2, \ldots, n\}, \qquad (11)$$

where $x_{i0}(t)$, $x_{i1}(t)$ are the states of the ith agent and $x_{r0}(t)$, $x_{r1}(t)$ are the states of the virtual leader.

Before presenting the main results of the present paper, we will make a couple of clarifications: hereinafter, we will consider that all agents are in a one-dimensional space; there is not loss of generality in doing so, since all results presented will be valid for m dimensions using the Kronecker product. Additionally, in this work we will consider the norms: $\|Z\| = \sqrt{z_1^2 + z_2^2 + \ldots + z_n^2}$, $Z \in \mathbb{R}^n$, and $\|B\|_1$ $=\max_{1\leqslant j\leqslant n}\{\sum_{i=1}^n |b_{ij}|\}$, $B = [b_{ij}] \in \mathbb{R}^{n\times n}$, for vectors and for matrices respectively.

3. MAIN RESULTS

In this section we present two Theorems that allow to determine if certain distributed order multi-agent systems, with corresponding fixed undirected graphs, achieve leader-following consensus. First, we study the case of systems with single integrator dynamics and then with double integrator dynamics.

A distributed order multi-agent system with single integrator dynamics can be described by the differential equations

$$\begin{aligned} &{}_0^C D_t^{b(\alpha)} x_i(t) = f(t, x_i(t)) + u_i(t), \quad i \in \{1, 2, \ldots, n\}, \\ &{}_0^C D_t^{b(\alpha)} x_r(t) = f(t, x_r(t)), \end{aligned} \qquad (12)$$

where $x_r(t)$ is the state of the virtual leader of the system, and $x_i(t)$, $u_i(t)$, $f(t, x_i(t))$ are the state, control input and inherent nonlinear dynamics of the ith agent, respectively.

In order to achieve leader-following consensus, we will use the following controller:

$$u_i(t) = -\beta\Bigg\{\sum_{j=1}^n a_{ij}\,(x_i(t) - x_j(t)) \\ + b_i\,(x_i(t) - x_r(t))\Bigg\}, \qquad (13)$$

where a_{ij} is the (i, j)th entry of the corresponding adjacency matrix $A \in \mathbb{R}^{n\times n}$, $\beta \geqslant 0$ and b_i is a nonnegative constant that does not always equal zero.

In the following, we will use the matrices $H = L + B$, where $B = \text{diag}\{b_i, b_2, \ldots, b_n\}$ and L is the Laplacian matrix associated with the interaction graph. We will refer to the eigenvalues of H as λ_i, $i \in \{1, 2, \ldots, n\}$. Note that the Jordan canonical form of H satisfies $\Lambda = P^{-1}HP$, where $P = [p_{ij}] \in \mathbb{R}^{n\times n}$ and $P^{-1} = [\bar{p}_{ij}] \in \mathbb{R}^{n\times n}$.

Sufficient conditions to guarantee the leader following consensus of distributed order multi-agent systems with single integrator dynamics are given next.

Theorem 12. Consider a multi-agent system with a fixed interaction graph $\mathcal{G}$, which is undirected and connected. Suppose that the function $f(t, x)$ in (12) is Lipschitz in x with Lipschitz constant θ, and that $b(\alpha)$ fulfills the conditions stated in Theorem 6. Additionally, if the following inequality is satisfied

$$\frac{\beta}{l} > \frac{\|P\|_1 \left\|P^{-1}\right\|_1}{\min_{1\leqslant i\leqslant n}\{\lambda_i\}}, \qquad (14)$$

where $l = 2\theta$, then system (12) achieves leader-following consensus, using the control input (13).

Proof. Substituting (13) in (12) produces

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} x_i(t) =& f(t, x_i(t)) - \beta\Bigg[\sum_{j=1}^n a_{ij}\,(x_i(t) - x_j(t)) \\ &+ b_i(x_i(t) - x_r(t))\Bigg]. \end{aligned} \qquad (15)$$

Introducing the change of variables $y_i(t) = x_i(t) - x_r(t)$, $i \in \{1, 2, \ldots, n\}$, we obtain

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} y_i(t) =& f(t, y_i(t) + x_r(t)) - f(t, x_r(t)) \qquad (16) \\ &- \beta\left\{\sum_{j=1}^n a_{ij}\,(y_i(t) - y_j(t)) + b_i y_i(t)\right\}. \end{aligned}$$

We can rewrite (16) in vector form as

$${}_0^C D_t^{b(\alpha)} Y(t) = F(t, Y(t)) - \beta H Y(t), \qquad (17)$$

where $Y(t) = [y_1(t),\ y_2(t), \ldots,\ y_n(t)]^T$ and $F(t, Y(t)) = [f(t, y_1(t) + x_r(t)) - f(t, x_r(t)),\ \ f(t, y_2(t) + x_r(t)) - f(t, x_r(t)), \ldots,\ f(t, y_n(t) + x_r(t)) - f(t, x_r(t))]^T$.

Multiplying (17) by P^{-1} on the left hand side and defining $Z(t) = P^{-1}Y(t)$ yields

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} Z(t) &= P^{-1}F(t, Y(t)) - \beta P^{-1}HPP^{-1}Y(t) \\ &= P^{-1}F(t, PZ(t)) - \beta\Lambda Z(t), \end{aligned} \qquad (18)$$

where $Z(t) = [z_1(t),\ z_2(t), \ldots,\ z_n(t)]^T$, $F(t, PZ(t)) = [f(t, \sum_{k=1}^n p_{1k} z_k(t) + x_r(t)) - f(t, x_r(t)),\ f(t, \sum_{k=1}^n p_{2k} z_k(t) + x_r(t)) - f(t, x_r(t)), \ldots,\ f(t, \sum_{k=1}^n p_{nk} z_k(t) + x_r(t)) - f(t, x_r(t))]^T$. Assuming that system (18) satisfies the hypothesis of Theorem 4, it has a unique solution $Z(t)$.

Note that, since the undirected graph $\mathcal{G}$ is connected and considering that $b_i \geqslant 0$ $\forall i$, H is symmetric and positive definite (see Lemma 8). Hence $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i > 0$, $i \in \{1, 2, \ldots, n\}$.

Now, we will demonstrate that system (18) is asymptotically stable. Consider the following Lyapunov candidate function, which satisfies inequality (6), for $c \leqslant \frac{1}{2}$:

$$c\left\|Z(t)\right\|^2 \leqslant V(t,Z(t)) = \frac{1}{2}\sum_{i=1}^{2n} z_i^2 \leqslant \left\|Z(t)\right\|^2. \qquad (19)$$

We can analyze the distributed order derivative of $V(t,Z(t))$ by using Lemma 7:

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} V(t,Z(t)) &= \frac{1}{2}\sum_{i=1}^{n} {}_0^C D_t^{b(\alpha)} z_i^2(t) \qquad (20)\\ &\leqslant \sum_{i=1}^{n} z_i(t)\, {}_0^C D_t^{b(\alpha)} z_i(t) = \sum_{i=1}^{n} z_i(t)\Bigg\{ -\beta\lambda_i z_i(t) \\ &+ \sum_{j=1}^{n} \bar{p}_{ij}\left[f(t, \sum_{k=1}^{n} p_{jk} z_k(t) + x_r(t)) - f(t,x_r(t))\right]\Bigg\} \\ &\leqslant \sum_{i=1}^{n}\Bigg\{ -\beta\lambda_i z_i^2(t) + \sum_{j=1}^{n} |z_i||\bar{p}_{ij}|\Bigg| f(t, \sum_{k=1}^{n} p_{jk} z_k(t) \\ &+ x_r(t)) - f(t,x_r(t))\Bigg|\Bigg\}. \end{aligned}$$

Considering that $f(t,x)$ is Lipschitz with respect to x, with $\theta = \frac{l}{2}$, it follows from (20) that

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} V(t,Z(t)) &\leqslant \sum_{i=1}^{n}\Bigg\{ -\beta\lambda_i z_i^2(t) \qquad (21)\\ &+ \sum_{j=1}^{n} \frac{l}{2}|z_i||\bar{p}_{ij}|\left|\sum_{k=1}^{n} p_{jk} z_k\right|\Bigg\} \leqslant -\beta\sum_{i=1}^{n}\lambda_i z_i^2(t) \\ &+ \frac{l}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |z_i||\bar{p}_{ij}||p_{jk}||z_k|. \end{aligned}$$

Exchanging the indices i to j, j to k, k to i and doing some simple algebra produces

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} V(t,Z(t)) &\leqslant -\beta \min_{1\leqslant i\leqslant n}\{\lambda_i\}\sum_{i=1}^{n} z_i^2(t) \qquad (22)\\ &+ \frac{l}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |z_j||\bar{p}_{jk}||p_{ki}||z_i| \leqslant -\beta \min_{1\leqslant i\leqslant n}\{\lambda_i\}\sum_{i=1}^{n} z_i^2(t) \\ &+ \frac{l}{2}\max_{1\leqslant k\leqslant n}\left\{\sum_{j=1}^{n}|\bar{p}_{jk}|\right\}\max_{1\leqslant i\leqslant n}\left\{\sum_{k=1}^{n}|p_{ki}|\right\}\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j||z_i|. \end{aligned}$$

Using the properties $\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j||z_i| = \left(\sum_{i=1}^{n}|z_i|\right)^2$, $\frac{1}{2}\left(\sum_{i=1}^{n}|z_i|\right)^2 \leqslant \sum_{i=1}^{n}|z_i|^2$ and the definitions of $\|P\|_1$, $\left\|P^{-1}\right\|_1$ we obtain

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} V(t,Z(t)) &\leqslant \left[-\beta \min_{1\leqslant i\leqslant n}\{\lambda_i\} + l\left\|P\right\|_1\left\|P^{-1}\right\|_1\right]\times \\ \sum_{i=1}^{n} z_i^2 &\leqslant \left[-\beta \min_{1\leqslant i\leqslant n}\{\lambda_i\} + l\left\|P\right\|_1\left\|P^{-1}\right\|_1\right]\left\|Z(t)\right\|^2. \end{aligned} \qquad (23)$$

It is clear from (23) that inequality (7) is satisfied if $\beta/l > \left\|P\right\|_1\left\|P^{-1}\right\|_1/\min_{1\leqslant i\leqslant n}\{\lambda_i\}$. Under this hypothesis, we have shown that $Z(t)$ is asymptotically stable at its origin. Since $Y(t) = PZ(t)$, it follows that $\lim_{t\to\infty} Y(t) = \mathbf{0}_n$, where $\mathbf{0}_n$ is the zero vector of size n. Recalling that $y_i = x_i(t) - x_r(t)$, $i \in \{1,2,\ldots,n\}$, we have determined that $\lim_{t\to\infty}|x_i(t) - x_r(t)| = 0$, and consequently the multi-agent system achieves leader following consensus.

Our next result deals with the leader-following consensus of distributed order multi-agent systems with double-integrator dynamics. A multi-agent system with this kind of behavior can be described by the differential equations

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} x_{i0}(t) &= x_{i1}(t), \qquad (24)\\ {}_0^C D_t^{b(\alpha)} x_{i1}(t) &= f(t, x_{i0}(t), x_{i1}(t)) + u_i(t), \end{aligned}$$

where $i \in \{1,2,\ldots,n\}$, and $x_{i0}(t)$, $x_{i1}(t)$ are the states of the ith agent. In this case, the virtual leader double integrator dynamics is given by

$$\begin{aligned} {}_0^C D_t^{b(\alpha)} x_{r0}(t) &= x_{r1}(t), \qquad (25)\\ {}_0^C D_t^{b(\alpha)} x_{r1}(t) &= f(t, x_{r0}(t), x_{r1}(t)), \end{aligned}$$

where $x_{r0}(t)$ and $x_{r1}(t)$ are the states of the virtual leader.

In order to achieve leader-following consensus in system (24), we will use the following controller, based on the analysis presented in Ren et al. (2015):

$$\begin{aligned} u_i(t) = &-\sum_{j=1}^{n} a_{ij}\left[(x_i0(t) - x_j0(t)) + \beta(x_{i1}(t) - x_{j1}(t))\right] \\ &- b_i\left[(x_i0(t) - x_r0(t)) + \beta(x_{i1}(t) - x_{r1}(t))\right], \qquad (26) \end{aligned}$$

where a_{ij} (i,j = 1,2,..., n) is the (i,j)th entry of corresponding adjacency matrix $A \in \mathbb{R}^{n\times n}$, $\beta \geqslant 0$ and d_i is a nonnegative constant that does not always equal zero.

In the proof of the next result we will use the matrices $B = \text{diag}\{b_i, b_2, \ldots, b_n\}$, $H = L + B$, and $M = \begin{pmatrix} \mathbf{0}_{n\times n} & -I_n \\ H & \beta H \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$, where I_n and $\mathbf{0}_{n\times n}$ are the $n\times n$ identity matrix and zero matrix, respectively. We will denote the Jordan canonical form of matrix M as Δ, which satisfies $\Delta = Q^{-1}MQ$, where $Q = [q_{ij}] \in \mathbb{R}^{2n\times 2n}$ and $Q^{-1} = [\bar{q}_{ij}] \in \mathbb{R}^{2n\times 2n}$. To refer to the eigenvalues of matrices H and M we will use the notation λ_i and δ_i, $i \in \{1,2,\ldots,n\}$ respectively.

The following Theorem allows to determine if a multi-agent system with double integrator dynamics achieves leader-following consensus over a fixed undirected graph.

Theorem 13. Asume that in system (24) $f(x,t)$ is Lipschitz and the distribution function $b(\alpha)$ satisfies the hypothesis of Theorem 6. If the fixed undirected graph $\mathcal{G}$ is connected and the inequalities

$$\sqrt{1 + \frac{4}{\max_{1\leqslant i\leqslant n}\{\lambda_i\}}} \geqslant \beta \geqslant \frac{2}{\sqrt{\min_{1\leqslant i\leqslant n}\{\lambda_i\}}} \qquad (27)$$

$$2\theta < \frac{\min_{1\leqslant i\leqslant 2n}\{\delta_i\}}{\left\|Q\right\|_1\left\|Q^{-1}\right\|_1} \qquad (28)$$

are satisfied, then leader-following consensus is achieved, using the control input (26).

Proof. Substituting the expression of the controller (26) in (24) we obtain

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} x_{i0}(t) = x_{i1}(t), \\
&{}_0^C D_t^{b(\alpha)} x_{i1}(t) = f(t, x_{i0}(t), x_{i1}(t)) \\
&- \sum_{j=1}^{n} a_{ij}[(x_{i0}(t) - x_{j0}(t)) + \beta(x_{i1}(t) - x_{j1}(t))] \\
&- b_i[(x_{i0}(t) - x_{r0}(t)) + \beta(x_{i1}(t) - x_{r1}(t))].
\end{aligned} \tag{29}
$$

Making the change of variables $y_{i0}(t) = x_{i0}(t) - x_{r0}(t)$, $y_{i1}(t) = x_{i1}(t) - x_{r1}(t)$, $i \in \{1, 2, \ldots, n\}$ results in

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} y_{i0}(t) = y_{i1}(t), \\
&{}_0^C D_t^{b(\alpha)} y_{i1}(t) = f(t, x_{i0}(t), x_{i1}(t)) - f(t, x_{r0}(t), x_{r1}(t)) \\
&- \sum_{j=1}^{n} a_{ij}[(y_{i0}(t) - y_{j0}(t)) + \beta(y_{i1}(t) - y_{j1}(t))] \\
&- b_i(y_{i0}(t) + \beta y_{i1}(t)).
\end{aligned} \tag{30}
$$

In vector form, we can rewrite (30) as

$$
{}_0^C D_t^{b(\alpha)} \begin{pmatrix} Y_0(t) \\ Y_1(t) \end{pmatrix} = -M \begin{pmatrix} Y_0(t) \\ Y_1(t) \end{pmatrix} + \begin{pmatrix} 0_{n\times 1} \\ F(t, Y_0(t), Y_1(t)) \end{pmatrix}, \tag{31}
$$

where $Y_k(t) = [y_{1k}(t), y_{2k}(t), \ldots, y_{nk}(t)]^T$ $(k = 0, 1)$, and $F(t, Y_0(t), Y_1(t)) = [f(t, y_{10} + x_{r0}(t), y_{11}(t) + x_{r1}(t)) - f(t, x_{r0}(t), x_{r1}(t)), \ldots, f(t, y_{n0}(t) + x_{r0}(t), y_{n1}(t) + x_{r1}(t)) - f(t, x_{r0}(t), x_{r1}(t))]^T$.

Based on the analysis done in Song et al. (2010) the eigenvalues of M are

$$
\begin{aligned}
\delta_i &= \frac{\lambda_i + \sqrt{\beta^2 \lambda_i^2 - 4\lambda_i}}{2}, \\
\delta_j &= \frac{\lambda_{j-n} - \sqrt{\beta^2 \lambda_{j-n}^2 - 4\lambda_{j-n}}}{2},
\end{aligned} \tag{32}
$$

where $i \in \{1, 2, \ldots, n\}$, $j \in \{n+1, n+2, \ldots, 2n\}$. It is clear from (32) that all eigenvalues of M will be positive real numbers if $\beta^2 \lambda_{j-n}^2 - 4\lambda_{j-n} \geqslant 0$ and $\lambda_{j-n} - \sqrt{\beta^2 \lambda_{j-n}^2 - 4\lambda_{j-n}} \geqslant 0$, which are equivalent to $\sqrt{1 + \frac{4}{\max_{1\leqslant i\leqslant n}\{\lambda_i\}}} \geqslant \beta \geqslant 2/\sqrt{\min_{1\leqslant i\leqslant n}\{\lambda_i\}}$. Therefore Δ is a diagonal positive definite matrix, specifically $\Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_{2n}\}$, $\delta_i > 0$, $i \in \{1, 2, \ldots, 2n\}$.

Multiplying (31) by Q^{-1} on the left hand side and defining $Z(t) = [z_1(t), z_2(t), \ldots, z_{2n}(t)]^T = Q^{-1} \begin{pmatrix} Y_0(t) \\ Y_1(t) \end{pmatrix}$, yields

$$
{}_0^C D_t^{b(\alpha)} Z(t) = -\Delta Z(t) + Q^{-1} \begin{pmatrix} 0_{n\times 1} \\ F(t, Y_0(t) Y_1(t)) \end{pmatrix}. \tag{33}
$$

If system (33) fulfills the conditions stated in Theorem 4, then it will have a unique $Z(t)$, which we will prove to be asymptotically stable at its origin. Consider the following Lyapunov candidate function, which clearly satisfies inequality (6), for $c \leqslant \frac{1}{2}$:

$$
c\,\|Z(t)\|^2 \leqslant V(t, Z(t)) = \frac{1}{2}\sum_{i=1}^{2n} z_i^2 \leqslant \|Z(t)\|^2. \tag{34}
$$

Using the property introduced in Lemma 7 in (34), produces

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} V(t, Z(t)) = \frac{1}{2}\sum_{i=1}^{2n} {}_0^C D_t^{b(\alpha)} z_i^2(t) \\
&\leqslant \sum_{i=1}^{2n} z_i(t)\, {}_0^C D_t^{b(\alpha)} z_i(t) = \sum_{i=1}^{2n} z_i(t) \Bigg\{ -\delta_i z_i(t) \\
&+ \sum_{j=1}^{n} \bar{q}_{i,j+n} \Big[f(t, y_{j0}(t) + x_{r0}(t), y_{j1}(t) + x_{r1}(t)) \\
&- f(t, x_{r0}(t), x_{r1}(t)) \Big] \Bigg\}.
\end{aligned} \tag{35}
$$

Since $f(t, x_0, x_1)$ is assumed to be Lipschitz, with $\theta = \frac{l}{2}$, we can simplify (35) as follows

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} V(t, Z(t)) \leqslant \sum_{i=1}^{2n} \Bigg[-\delta_i z_i^2(t) \\
&+ \sum_{j=1}^{n} |\bar{q}_{i,j+n}| \frac{l}{2} |y_{j0}(t) + y_{j1}(t)| |z_i(t)| \Bigg] \\
&= \sum_{i=1}^{2n} \Bigg[-\delta_i z_i^2(t) + \sum_{j=1}^{n} |\bar{q}_{i,j+n}| \frac{l}{2} \Bigg| \sum_{k=1}^{2n} q_{jk} z_k(t) \\
&+ \sum_{k=1}^{2n} q_{j+n,k} z_k(t) \Bigg| |z_i(t)| \Bigg].
\end{aligned} \tag{36}
$$

Doing some simple algebra and switching the indices i to k, and k to i:

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} V(t, Z(t)) \leqslant -\sum_{i=1}^{2n} \delta_i z_i^2(t) + \frac{l}{2} \sum_{i=1}^{2n}\sum_{j=1}^{n}\sum_{k=1}^{2n} |\bar{q}_{i,j+n}| \times \\
&|q_{jk}(t) + q_{j+n,k}|\, |z_k(t)||z_i(t)| \leqslant -\min_{1\leqslant i\leqslant 2n}\{\delta_i\} \sum_{i=1}^{2n} z_i^2(t) \\
&+ \frac{l}{2} \max_{1\leqslant j\leqslant n} \left\{ \sum_{k=1}^{2n} |\bar{q}_{k,j+n}| \right\} \max_{1\leqslant i\leqslant 2n} \left\{ \sum_{j=1}^{n} |q_{ji}(t) + q_{j+n,i}| \right\} \times \\
&\sum_{i=1}^{2n}\sum_{k=1}^{2n} |z_k(t)||z_i(t)|.
\end{aligned} \tag{37}
$$

Considering that $\max_{1\leqslant j\leqslant n}\left\{\sum_{k=1}^{2n} |\bar{q}_{k,j+n}|\right\} = \|Q^{-1}\|_1$, $\max_{1\leqslant i\leqslant 2n}\left\{\sum_{j=1}^{n} |q_{ji}(t) + q_{j+n,i}|\right\} = \|Q\|_1$ and $\sum_{i=1}^{n} |z_i|^2 \geqslant \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |z_j||z_i|$, it can be obtained form (37) the following inequality:

$$
\begin{aligned}
&{}_0^C D_t^{b(\alpha)} V(t, Z(t)) \leqslant -\min_{1\leqslant i\leqslant 2n}\{\delta_i\} \sum_{i=1}^{2n} z_i^2(t) \\
&+ l\,\|Q\|_1 \|Q^{-1}\|_1 \sum_{i=1}^{2n} z_i^2(t) \\
&\leqslant -\left(\min_{1\leqslant i\leqslant 2n}\{\delta_i\} - l\,\|Q\|_1 \|Q^{-1}\|_1 \right) \|Z(t)\|^2.
\end{aligned} \tag{38}
$$

If $l < \min_{1\leqslant i\leqslant 2n}\{\delta_i\} / \|Q\|_1 \|Q^{-1}\|_1$, the coefficient of $\|Z(t)\|^2$ in the right hand side of (38) will be negative and consequently inequality (7) will be satisfied. Hence, according to Theorem 6, system (33) is asymptotically stable at its origin. Furthermore, it follows from $\begin{pmatrix} Y_0(t) \\ Y_1(t) \end{pmatrix} = QZ(t)$ that $\lim_{t\to\infty} |x_{i1}(t) - x_{r1}(t)| = 0$, given that $y_{i0}(t) = x_{i0}(t) - x_{r0}(t)$, $y_{i1}(t) = x_{i1}(t) - x_{r1}(t)$, $i \in \{1, 2, \ldots, n\}$. This means that system (24) accomplishes second-order leader-following consensus.

Remark 14. Note that Theorems 12 and 13 extend the main results of Ren et al. (2015); the contribution of our work is that it includes distributed order dynamics, which generalizes the fractional order case. If the distribution function is chosen as $b(\alpha) = \delta(\alpha - \beta)$ we get back the fractional versions of our results, which allow to determine if fractional multi-agent systems achieve leader-following consensus.

Remark 15. As far as we are concerned, research has not found numerical methods to solve the following problems with distributed order derivatives: linear non-autonomous differential equations, non-linear differential equations, systems of differential equations. Simulations for some simple distributed order equations can be found in Diethelm and Ford (2009); Katsikadelis (2014); Mashayekhi and Razzaghi (2016). Considering this, we leave for future work the numerical solution of examples that verify the usefulness of the results presented in this work.

4. CONCLUSION

In this paper we introduced a couple of consensus results for distributed order multi-agent systems. In Theorems 12 and 13, we presented methods to determine if leader-following consensus is accomplished in linear or nonlinear multi-agent systems with associated fixed undirected graphs for the cases of single and double integrator dynamics, respectively.

As future work, a possible path would be to obtain similar results for more general families of distributed order multi-agent systems, perhaps using different kinds of control inputs. Another potential objective would be to attend the issue discussed in Remark 15 by designing numerical methods to solve unexplored distributed order problems. Consequently we would be able to verify, with computer simulations, the theoretical results introduced in the present article.

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