Filtered PD Optimal Controller for Unstable First Order Plus Time Delay Systems

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Abstract: A typical problem in the PD controllers is their practical implementation. The solution of using a filter to the derivative action has been widely proposed in the literature. However, just high value of the filter coefficient is suggested in order to recuperate the inherent derivative action of the controller. In this way, this work avoids to assume high values for the filter coefficient and presents a stability analysis of a proposed filter to PD controller to the case of systems with time delay, particularly Unstable First Order plus Time Delay (UFOPTD) systems. This analysis becomes interesting since the results have not been presented in literature. As a second step of analysis, it is proposed a simple methodology to obtain the optimal gains $\boldsymbol{k_d}, \boldsymbol{k_p}$ from the stable known region. The proposed analysis is based on an approximation of the delay term, using the first and second order Padé approximations.

Keywords: PD controller, Stabilization, Time-Delay, Unstable Processes, Optimal control.

1. INTRODUCTION

Time-delay of control inputs is a common phenomenon in diverse application fields. Currently some systems are controlled remotely generating a time delay between the systems and the control systems. The control problem of time delay systems arises mainly by the induced transcendental term in the characteristic equation due to the feedback loop, which gives as result a characteristic equation having an infinite number of poles. Due to the complexity of the problem, the community control have devoted its efforts for designing control strategies that provides an adequate performance of the system; see for instance Seshagiri, et al., (2007). To the case of open-loop stable process, the well-known Smith Predictor Compensator has been used as a traditional control structure .By using a similar approach, some works have been reported in order to deal with unstable processes Liu et al., (2005), Seshagiri and Chidambaram (2005). As a first attempt to analyse a generalized class of delayed systems, the case of first order delayed system has been widely studied by using the basis provided by the SPC, see for instance Seshagiri, et al., (2007), Michiels et al., (2002), Marquez et al., (2012). With a simpler and different perspective, some authors have regarded to analyse the case when the system is controlled by a Proportional (P), Proportional-Integral (PI) and Proportional-Integral-Derivative (PID). Nesimioglu and Soylemez (2010) computed all stabilization proportional controllers. Hwang and Hwang (2004) used the D-partition technique to estimate the stabilization limits of PID compensation, showing that an UFOPTD system can be stabilized if $\theta < \theta_{un}$, where θ is the time-delay and θ_{un} is the unstable constant-time. Silva and Bhattacharyya (2005) provided a complete parameterization of the stabilizing P and PI controller in the case $\theta < \theta_{un}$ and the stabilizing PID controllers for the case $\theta < 2\theta_{un}$. Recently, Marquez et al., (2014) consider the stabilization of UFOPTD system by using a Proportional-Derivative (PD) controller, showing that the derivative term minimizes the effects of the time-delay. In Lee et al., (2010) some results about the stabilization of a delayed system with one unstable pole and several stable poles by using P, PI, PD and PID controller are provided, however, they do not consider the important issue in the PD controller: the practical implementation. In this work we analyze the stability conditions of the closed-loop system for different values of the filter coefficient. It is known that when a control strategy is applied to a system it is desired to obtain optimal performance of the system with respect to specific variables. In this work a simple and effective methodology to obtain de optimal gains k_p, k_d in order to minimize the control input and state is proposed. Such methodology is based on the approximation of the delay term as well as the knowledge of the stabilizing region $k_p - k_d$ assumption. This work is organized as follows. Section 2 presents the problem formulation. After this, in Section 3 some preliminary results are given. Then, Section 4 presents the main results of this work. In Section 5 a numerical example is given and finally in Section 6 some conclusions are provided.

2. PROBLEM FORMULATION

Consider an UFOPTD system given by,

$$\bar{G}(s) = \frac{Y(s)}{U(s)} = \frac{b}{s-a}e^{-\theta s}, \tag{1}$$

a traditional filtered PD controller of the form,

$$\bar{C}(s) = k_p + \frac{k_d s}{\frac{s}{N}+1} = \frac{Nk_p + (k_p + Nk_d)s}{s+N}, \tag{2}$$

where N is known as coefficient filter, and the control scheme (fig 1). If the parameters k_p, k_d are known in the case of non-filtered PD controller, a traditional and heuristic way to implement in practice the controller given in (2) is setting N as $N \gg 0$. Notice that for $N \gg 0$ in (2) the properties of the non-filtered PD controller are recovered. However there is not a guideline to set of N or an explanation if the closed-loop system remains stable when N decreases. This work considers this problem to the proposed filtered PD controller i. e., it is provided the allowable value of the coefficient filter such that the closed loop is stable and it is shown how the stability properties are modified due to the values of N. Also, it considers the values such that the gains of PD controller will be optimum. Consider a system given by (1) and a filtered PD controller given by (2). The transfer function on open loop given by,

$$\bar{C}(s)\bar{G}(s) = \left(\frac{Nk_p + (k_p + Nk_d)s}{s+N}\right)\frac{b}{s-a}e^{-\theta s}, \quad (3)$$

can be separated as follows,

$$C(s)G(s) = (Nk_p + (k_p + Nk_d)s)\frac{b}{(s-a)(s+N)}e^{-\theta s} \quad (4)$$

where,

$$G(s) = \frac{b}{(s-a)(s+N)}e^{-\theta s}, \quad (5)$$

$$C(s) = \left(Nk_p + \left(k_p + Nk_d\right)s\right) \quad (6)$$

The closed-loop stability properties of the system given by (1) - (2) can be obtained from the closed-loop (5) - (6). The closed loop stability properties of (5) - (6) are studied in Lee et al., (2010).

3. PRELIMINARY RESULTS

Consider a system given by (5) controlled by (6). The following result establishes the closed-loop stability conditions.

[Lee et al., (2010)] **Lemma 1.** A necessary condition for a PD controller given by (6) stabilizes the system given by (5) is,

$$\theta < \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{N}\right)^2} + \frac{1}{a} - \frac{1}{N} \quad (7)$$

If this condition is satisfied then the range of k_d and k_p gains for which a solution exists to the PD stabilization problem are given by,

$$\theta - \frac{1}{a} < \frac{k_d}{k_p} < \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{N}\right)^2} - \frac{1}{N}, \quad (8)$$

$$\frac{a}{b} < k_p < \frac{a}{b}\sqrt{\frac{(1+\omega_c^2)\left(1+\frac{a^2\omega_c^2}{N^2}\right)}{1+\left(\frac{k_d}{k_p}+\frac{1}{N}\right)^2 a^2\omega_c^2}}, \quad (9)$$

where ω_c satisfies,

$$arctg(\omega_c) + arctg\left(\frac{k_d}{k_p}a\omega_c\right) - arctg\left(\frac{a\omega_c}{N}\right) - \theta a\omega_c = 0$$

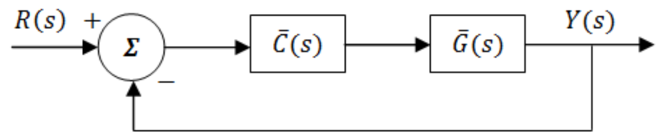

Fig 1. A control scheme UFOPTD system.

4. MAIN RESULTS

In this section the main results of this work are depicted. The following result provides the stability conditions such that the PD controller given by (2) stabilizes the system given by (1).

Theorem 1. Consider a system given by (1), a filtered PD controller of the form (2) and the control feedback (fig 1). A necessary condition for a filtered PD controller stabilizes the closed-loop system is:

$$\theta < \frac{2}{a} \text{ with } N \gg a \quad (10)$$

$$\theta < \frac{1}{a} \text{ with } N \ll a \quad (11)$$

Proof. i) From the necessary condition (7) given in Lemma 1, it should be that it $N \gg a$, the term $\frac{1}{N} \to 0$ which leads to the condition,

$$\theta < \sqrt{\left(\frac{1}{a}\right)^2} + \frac{1}{a} = \frac{2}{a},$$

Proof. ii) From the necessary condition (7) given by Lemma 1, if $N \ll a$ is considered i. e, $N = xa$ with $0 < x < 1$, the term $\left(\frac{1}{a}\right)^2 \ll \left(\frac{1}{xa}\right)^2$ and $\left(\frac{1}{a}\right)^2$ can be remove from the expression and we obtain,

$$\theta < \sqrt{\left(\frac{1}{xa}\right)^2} + \frac{1}{a} - \frac{1}{xa} = \frac{1}{a}$$

On the other hand, if the condition i) or ii) is satisfied, then the stabilizing coefficient filter should be chosen from,

$$N > \frac{2 - 2a\theta}{a\theta^2 - 2\theta} \quad (12)$$

In what follows a simple and effective methodology to obtain the optimal k_p, k_d control parameters in order to minimize the energy of the input control and states system. It is important to note that the following optimization strategy assumes that the stabilizing region of the control parameters is known and such regions can be computed following Lemma 1 (Lee et al 2010). In order to obtain a rational representation on the complex variable "s" of the delay term the first step of the proposed methodology considers a first order Padé approximation, which can be expressed as,

$$e^{-\theta s} = \frac{-s + \frac{2}{\theta}}{s + \frac{2}{\theta}} \quad (13)$$

Then, by substituting the approximation (13) into the process given by (1), a rational representation on the complex variable "s" of the plant is obtained,

$$G_1(s) = \frac{b\left(-s + \frac{2}{\theta}\right)}{(s-a)(s+N)\left(s+\frac{2}{\theta}\right)} \tag{14}$$

Taking into account the approximated system (14), the proposed filtered PD (2), and the closed loop system shown in Figure 1, a closed-loop state space representation of the form,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u, \tag{15}$$

is obtained with,

$$A_1 = \begin{bmatrix} \frac{2}{\theta} & -b & 0 \\ \left(k_p + Nk_d\right)\frac{4}{\theta} & a + b\left(k_p + Nk_d\right) & -Nk_d \\ \frac{4}{\theta}N & bN & -N \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ k_p + Nk_d \\ N \end{bmatrix},\ C_1 = \begin{bmatrix} -\frac{4}{\theta} & -b & N \end{bmatrix},\ D_1 = [0], \tag{16}$$

for a second Padé approximation, the delay term is expressed by,

$$e^{-\theta s} = \frac{s^2 - \frac{6}{\theta}s + \frac{12}{\theta^2}}{s^2 + \frac{6}{\theta}s + \frac{12}{\theta^2}} \tag{17}$$

and his rational and state space representation are given by,

$$G(s) = \frac{b\left(s^2 - \frac{6}{\theta}s + \frac{12}{\theta^2}\right)}{(s-a)(s+N)\left(s^2 - \frac{6}{\theta}s + \frac{12}{\theta^2}\right)} \tag{18}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{12}{\theta^2} & -\frac{6}{\theta} & b & 0 & 0 \\ 0 & -\frac{12\,k_d}{\theta^2} & a - b\,k_d & k_i & k_p - Nk_d \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{12}{\theta} & -b & 0 & -N \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ k_d \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & -\frac{6}{\theta} & b & 0 & 0 \end{bmatrix},\ D = [0] \tag{19}$$

Now, based on the second method of Lyapunov an optimization process to obtain the optimal k_p and k_d control parameters is derived. The performance index defined to evaluate the behaviour of the system is given by,

$$J = \int_0^\infty \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}dt = \boldsymbol{x}^T(0)\boldsymbol{P}\boldsymbol{x}(0), \tag{20}$$

where:

$\boldsymbol{x}$ is the states vector.

$\boldsymbol{P}$ is a Define Positive Matrix.

$\boldsymbol{Q}$ is a Define Positive Matrix.

$\boldsymbol{A}$ is a nxn Matrix.

J is the performance index.

The main objective of the optimization process is to minimize the behavior index (20) assuring closed-loop stability. In this way, the solution of the following equation is required,

$$\boldsymbol{A^T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q} \tag{21}$$

From the second method of Lyapunov it is known that if the equation (21) has a unique solution, the closed-loop system given by (15) is stable. Equation (21) should be solved by proposing $\boldsymbol{Q}$ and solving to $\boldsymbol{P}$. This step requires that two of the control gains (for instance k_p and k_d, k_p and k_i or k_i and k_d) are given from the stable region computed with Lemma 1 (Lee et al. 2010). Notice that the resultant matrix $\boldsymbol{P}$ contains the control parameters of the control. Once that $\boldsymbol{P}$ is obtained, $\boldsymbol{P}$ is replaced into the performance index (20). Then, in order to minimize the performance index (20), the derivative of the performance index (20) is computed and the optimal gain (for instance $k_i, k_d\ or\ k_p$) is solved from,

$$\frac{dJ}{dx} = 0 \tag{22}$$

5. SIMULATION RESULTS

Example 1. Consider a UFOPTD system given by,

$$G(s) = \frac{1}{s-1}e^{-1.3s}$$

and a filtered PD controller by (2). Following the Theorem 2, the condition $\theta < \frac{2}{a}$ is satisfied due to $1.3 < 2$. From the iii), the range of N values is,

$$N > 0.65934066$$

Following the Lemma 1, the ranges of k_d and k_p values are,

$$0.3 < \frac{k_d}{k_p} < \sqrt{1 + \left(\frac{1}{N}\right)^2} - \frac{1}{N} \text{ and } 1 < k_p < \sqrt{\frac{\left(1+\omega_c^2\right)\left(1+\frac{\omega_c^2}{N^2}\right)}{1+\left(\frac{k_d}{k_p}+\frac{1}{N}\right)^2\omega_c^2}}$$

Fig 2. shows the stability regions $\frac{k_d}{k_p} - k_p$ that each N value generates, Fig. 4 shows the stability regions for some N values. Notice that when $N = 50$ the stability region is seems to the regions with N larger values. With this N value, the ranges of k_d values and k_p values are,

$$0.3 < \frac{k_d}{k_p} < 0.982 \text{ and } 1 < k_p < 1.2173$$

To obtain the optimal values for these ranges, a $\frac{k_d}{k_p}$ stabilizers values sweep is performed to find the k_p values that they make the index to be the minimum, and vice versa.

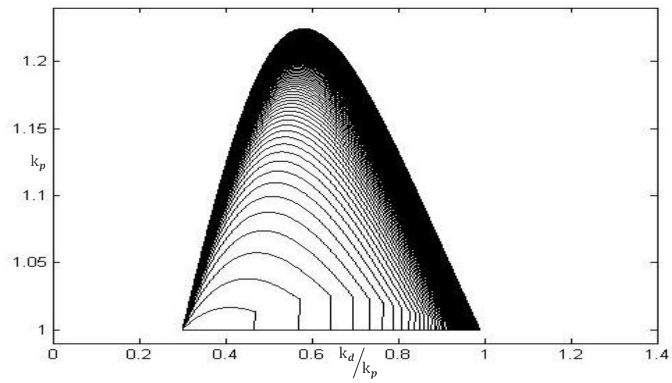

Fig 2. Stability regions to different N values.

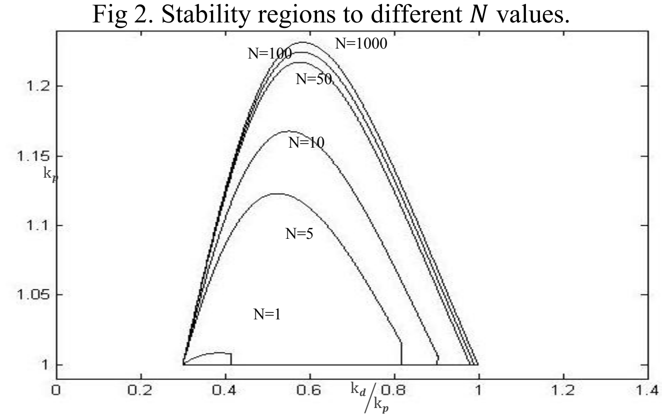

Fig 3. Stability region for N = 1, 5, 10, 50, 100, 1000.

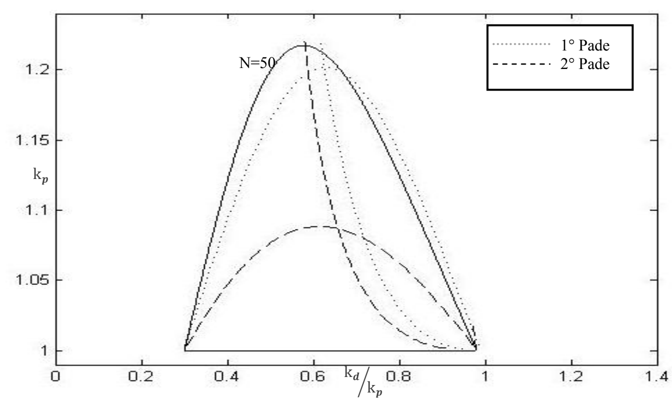

Fig 4. k_p and $\frac{k_d}{k_p}$ Optimal values with 1° and 2° Pade approx.

The k_p and $\frac{k_d}{k_p}$ optimal values for first and second Pade approximation are shown in fig 4. for $N = 50$. The cross point of the k_p and $\frac{k_d}{k_p}$ optimal values is the point where J is the minimum value for all k_p and $\frac{k_d}{k_p}$ vales. These values are shown in table 1. Consider the control design used in the system of the Example 3., i. e., $k_p = 1.087$, $k_d = 0.715$. The stability in closed-loop on the Nyquist Plot as it shown in fig 5.., this plot shows encircles the critical point $(-1,0)$ one time anticlockwise due there is one unstable pole. Fig. 6 shows the output of the controller with the values of table 1, note that second Pade approximation offers the best result. The output plots to different N values are shown in Fig 7.

Table 1. Optimal values and performance index.

Pade approximation	$\bar{k}_p$	$\bar{k}_d$
1° Pade approximation	1.201	0.753
2° Pade approximation	1.087	0.715

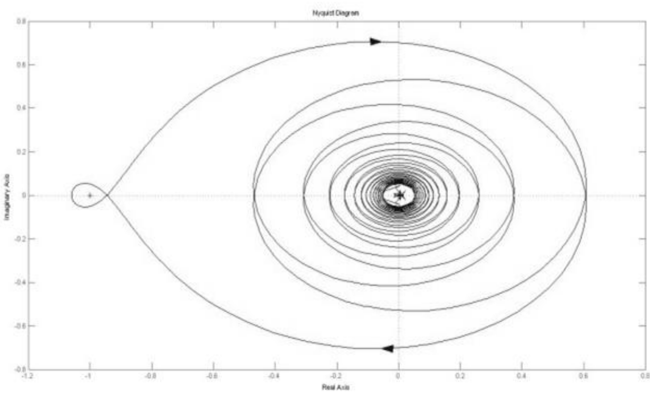

Fig 5. Nyquist plot of the closed-loop.

When N values are big, the output behaves as the output of a system controlled by a PD controller without filter, while for N values are small, the system output begins to oscillate stronger and its steady state time increases.

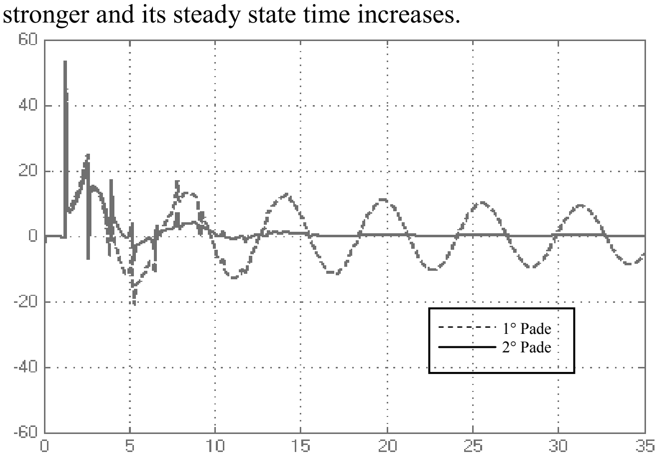

Fig 6. Controller output without reference and $x(0) = 0.1$

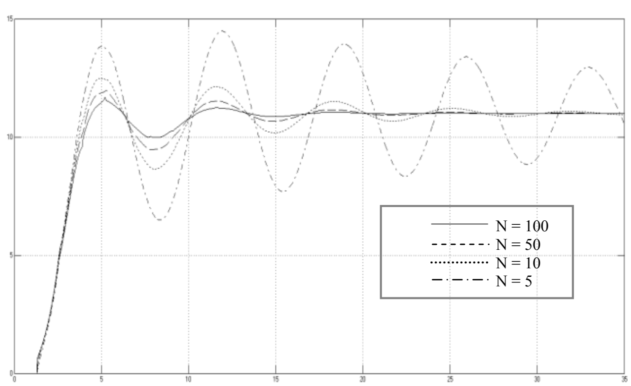

Fig 7. Output plots.

6. CONCLUSION

In this paper, the stabilization of UFOPTD systems is investigated. The stability conditions by the filtered PD controller are established to different N values. The analysis provides exact stability region in terms of control parameters and indicates that stability can be achieved as long as $\theta < \frac{2}{a}$ and we have illustrated it through an example and its simulation. Using a good approximation of the time delay and the proposed methodology of optimization for the controller parameters $\bar{k}_d - \bar{k}_p$ it is possible to compute approximate optimal gains $\bar{k}_d - \bar{k}_p$ for UFOPTD systems. The results in this work complement previously published results on the stabilization of unstable processes under classical compensation strategies.

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