Fractional Order System Identification by a Genetic Algorithm

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Abstract: In this work we present a method to find the parameters of a fractional differential equation based on a Genetic Algorithm, considering only the knowledge of the structure of the equation and the input-output signals. Examples are given in order to show the effectiveness of the proposed method.

Keywords: Fractional systems, Systems Identification, Genetic Algorithms.

1. INTRODUCTION

Dynamical systems described by integro-differential fractional equations have been gaining attention in recent years, even though the concept was first proposed at the end of the 17th century and taken into account as a research subject until 1884. This interest relies on the fact that several classes of physical systems, especially those including diffusion dynamics or friction, as well as memory and hereditary properties in materials and systems can be better and more succinctly described by fractional derivatives and integrals, rather than using integer ones (Caponetto, 2010). As usually the integer integral or derivative are represented by operators J^n and D^m respectively, where $n \in \mathbb{N}$; so, fractional integral and derivative are typically described also as the operators J^β and D^α, where $\alpha, \beta \in \Re$, or even $\alpha, \beta \in \mathbb{C}$.

Moreover, the same tools used to analyse linear systems with integer differentials and integrals, such as the Laplace transform and the Fourier analysis can be extrapolated and used in fractional ones.

Some methods have been proposed to approximate the solution given by a fractional differential equation of fractional differential system (FOS), from a higher-order transfer function with integer derivatives (Mansouri et al., 2010; Oustaloup, 1991) to the analysis of the step response (Dorcák et al., 2002), similar to the case of first and second-order systems. However, in most cases the parameters of the FOS are assumed known or obtained from a physical analysis of the system, especially regarding the fractional values α and β.

Recently, some approaches have been proposed to identify the parameters of a FOS. For example, by expanding the fractional differential equation to a larger integer system (Sabatier et al., 2006), assuming that the fractional values α or β are known. When also this parameters are unknown, approximation methods have been proposed for the case of fractional-order chaotic systems (Yuan and Yang, 2012) by using a particle-swarm optimization and a numerical approximation of the solution of the FOS. In this sense, also Genetic Algorithms (GAs) have been proposed to tune the parameters of the $PI^\alpha D^\beta$ control (Cao et al., 2005). As it can be seen, the identification of a fractional order system is still an open and active research problem.

In this work we present a method to identify the parameters of a Fractional-Order System (FOS) based on GAs given that only the structure of the fractional differential equation is known as well as its input and output signals are available. The paper is organized as follows: In Section 2, a brief description of fractional calculus and systems are given. In Section 3, the algorithm for the identification of the parameters using a GA is presented, and in Section 4 results are shown in order to illustrate the effectiveness of the method. Finally, conclusions are discussed in Section 5.

2. FRACTIONAL ORDER SYSTEMS

From a mathematical point of view, a fractional order integral or derivative is defined as an extrapolation of the definition of the integer-order integral or derivative of a certain function $f(t)$, seen as a general fractional differential operator D^α. However, there exist different definitions of this operator, that in general do result in different solutions. Two of the main approaches and most generally used in control systems are the Riemann-Liouville and the Caputo fractional operator (Gorenflo and Mainardi, 1997).

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Recall that, for $n \in \mathbb{Z}^+ - \{0\}$, given the Cauchy's formula for the repeated integration

$$J^n f(t) \triangleq \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau) d\tau \cdots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t f(\tau)(t-\tau)^{n-1} d\tau \quad (1)$$

if n is changed from an integer value to any real (and even complex) value $\alpha \in \Re$, then the definition is extrapolated in the so called *Riemman-Liouville fractional integral*, defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau. \quad (2)$$

where $\Gamma(w)$ is the Gamma function of $w \in \mathbb{C}$. From the previous definition, the Riemann-Liouville fractional differential operator D^α is then defined as

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases} \quad (3)$$

Note that this operator is a left-inverse for (2) (Caponetto, 2010), i.e., $D^\alpha(J^\alpha f(t)) = f(t)$.

A slightly different, but also valid definition of the differential operator, is given by (Caputo, 1967) and called the *Caputo Fractional Differential Operator*:

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\frac{d^n f(\tau)}{dt^n}}{(t-\tau)^{\alpha+1-n}} d\tau, & \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases} \quad (4)$$

These two definitions are not always interchangeable. In the area of control systems, generally the Caputo's definition is preferred, rather than that of Riemann-Liouville, since in the first one the initial conditions typically associated with physical interpretation are involved, such as the integer derivative at $t = 0$. In the latter, the initial conditions involved do not have a clear physical interpretation (Podlubny, 1998). In this work, the Caputo's definition is used for the fractional derivative.

Following this definition, a fractional-single-order differential equation for an SISO system with input $u(t)$ and output $y(t)$ is defined as

$$D^\alpha y(t) + a y(t) = b u(t) \quad (5)$$

The solution for this equation can be found using the Laplace transform and, consequently, the transfer function in the form

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^\alpha + 1}, \quad (6)$$

and an analytical method or a numerical approximation given by an expansion of the transfer function (6) may be used.

3. IDENTIFICATION ALGORITHM

Consider a system given by the mapping $y(t) = H(u(t))$, and a fractional differential equation to approximate it given by

$$D^\alpha \hat{y}(t) + a\hat{y}(t) = bu(t). \quad (7)$$

where both the input and the ouput are bounded and continuous signals. The objective of the algorithm is to minimize, given finite time signals $u(t)$, $y(t)$ whose measurements are taken in $t \in [0, T_1]$, the function cost

$$C = \int_0^{T_1} (y(t) - \hat{y}(t))^2 dt, \quad (8)$$

where a, b and α are the unknown parameters.

In this work it is considered that bounds for α, a and b are given, that is, $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, $a \in [\underline{a}, \bar{a}]$, $b \in [\underline{a}, \bar{b}]$. Now, let each parameter be codified by binary genes. For presentation purposes, unsigned 8 bits are considered, but clearly it can be extrapolated to larger values and even signed values. If $\underline{\alpha}, \underline{a}, \underline{b} \geq 0$ and given that the quantization of each parameter $\gamma \geq 0$ such that $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ with quantization step $\Delta = \frac{\bar{\gamma} - \underline{\gamma}}{2^{\text{bits}} - 1}$, where *bits* is the number of bits, then the quantization and binary representation is given by

$$\gamma \to \gamma_b = Q_{2,\text{bits}}(\gamma) = \left\{ \left\lfloor \frac{\gamma}{\Delta} + \frac{1}{2} \right\rfloor \right\}_{\text{base}-2}. \quad (9)$$

So, the binary representation of each bound is, in list form:

$$\begin{aligned} \underline{\alpha} \to \underline{\alpha}_b &= \{0,0,0,0,0,0,0,0\} \\ \underline{a} \to \underline{a}_b &= \{0,0,0,0,0,0,0,0\} \\ \underline{b} \to \underline{b}_b &= \{0,0,0,0,0,0,0,0\} \\ \bar{\alpha} \to \bar{\alpha}_b &= \{1,1,1,1,1,1,1,1\} \\ \bar{a} \to \bar{a}_b &= \{1,1,1,1,1,1,1,1\} \\ \bar{b} \to \bar{b}_b &= \{1,1,1,1,1,1,1,1\} \\ \alpha \to \alpha_b &= Q_{2,\text{bits}}(\alpha) \\ a \to a_b &= Q_{2,\text{bits}}(a) \\ b \to b_b &= Q_{2,\text{bits}}(b) \end{aligned}$$

3.1 Genetic Algorithms

Genetic Algorithms (GAs) are methods that belong to a the so-called Evolutionary Algorithms (EAs), that are inspired on the processes present on natural evolution, where the different genes between individuals in a certain species are combined, inherited and even mutate from one generation to the next. Then the conditions of the environment, combined with the properties of each individual, determine which genes survive and are transferred to the next generation.

In general, a GA is focused on the optimization of a certain objective function of *fitness function*, and the objective is to find a solution (or a set of) that minimizes it sub-optimally via an iterative process, given a set of candidate solutions, called *population*. Its individuals, where the free parameters are coded in *genes*, are then classified by their fitness value. The properties or free parameters of each candidate are coded in *chromosomes*, sometimes in a binary fashion, and this set is called a *gene*.

In order to create the next generation, each new individual is created either by: copying the genes of the best ones (*elitism*), called the *elite genes*; randomly changing some of the code (*mutation*); combining the genes of different genes of the previous generation (called *parents* and the combination *crossover*), stochastically of by following a certain rule (for example, where the probability of crossover is a function of the fitness value); or by combining the previous methods.

3.2 Proposed GA

For this work, the identification algorithm based on a GA is given by the following steps:

- i: Choose the number of genes N_g for each generation.
- ii: Initialize chromosomes for α, a, b by random binary values.
- iii: Choose the N_e genes that achieve the lower value of the cost function (8) and pass them to the next generation (elitism)
- iv: Generate the remanent genes by a 50% crossing of each chromosome, as shown in Fig. 1.
- v: Mutate the non-elite genes by shifting to the right each chromosome with a probability of %PM.
- vi: Repeat until a minimum design value for (8) is fulfilled or maximum generation number is reached.

In this method, the design parameters are N_g, N_e and $\%PM$.

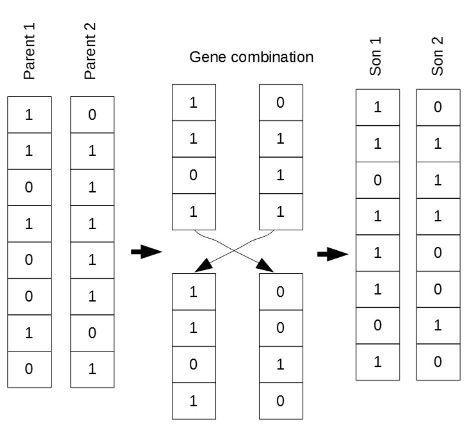


Fig. 1. Gene crossing algorithm.

4. RESULTS

In order to show the effectiveness of the proposed method, some simulation results are shown. In first place, it is considered an inherent fractional system. In second place, as fractional systems are usually approximated by high-order systems, it is considered a linear system with multiple poles. Lastly, a nonlinear system based on an anesthesia model is shown. In all cases the algorithm and required simulations were run on a PC with Intel i5 processor using the fractional differential algorithms reported by Diethelm and Freed (1998); Garrappa (2010) and considering 16 bits coding for all chromosomes in genes.

System	Parameters		
	α	a	b
a)	0.3	0.4	0.5
b)	0.8	0.2	0.6
c)	0.9	0.9	0.1

Table 1. Parameters for (10)

4.1 Fractional system.

In first place, consider a fractional system with initial conditions in the form

$$(D^{\alpha} + a)y(t) = bu(t), \ , y(0) = 0. \tag{10}$$

Three different parameter set were chosen, as shown in Table 1. In order to find the set of parameters $\hat{\alpha}$, $\hat{a}$, $\hat{b}$ that best fitted the given input-ouput data, the algorithm was run considering in all cases 30 genes, 30 generations, 4 elite genes and a mutation probability of 80%. The results for these simulations are shown in Fig. 2. As it can be seen, the identified fractional differential equations do effectively approximate the response of the given system.

4.2 High-order system.

For this case, it is considered a system to be identified with the transfer function with multiple poles:

$$G(s) = \frac{25}{(s+5)^3}.$$

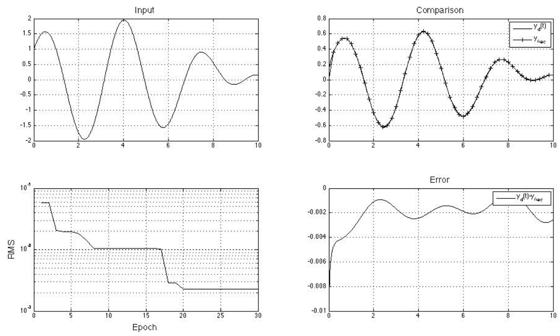
The identification results with 30 genes, 30 generations, PM=80% are shown in Fig. 3. The obtained fractional differential equation is

$$D^{0.8249}\hat{y}(t) + 0.6039\hat{y}(t) = 0.2902u(t). \tag{11}$$

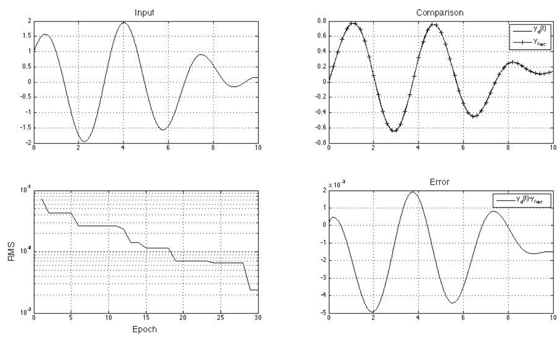
It can be seen that the fractional system does identify the dynamics of a higher-integer-order differential equation with a small error, inherent to the fact that the order is simplified by the model.

4.3 Nonlinear system: Anesthesia

The modeling of the dynamics of the anesthesia administered to a patient, concretely by propofol, can lead to a better knowledge of its stability, evolution, and design of better controllers. In this sense, one of the most used models is a pharmacokinetic and pharmacodynamic 4th order model with unknown parameters (Copot and



a)



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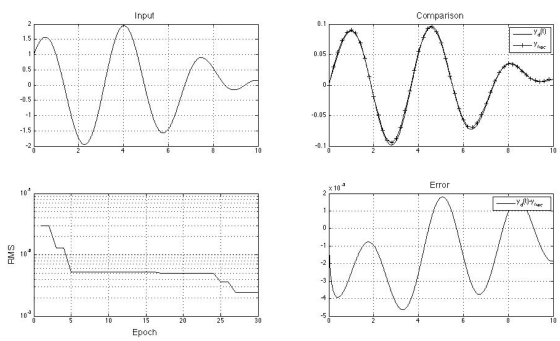


c)

Fig. 2. Identification of a fractional system. Simulation results. In each case, the upper-left figure is the input signal, the upper-right is the output of the best gene, the lower-left is the RMS evolution in each generation, and the lower-right is the error $y_d - \hat{y}$.

Ionescu, 2014). As the structure and the parameters become quite complex, a reduced FOS could lead to a simpler model, given that some delays and dynamics could be grasped by it. In this sense, simulations were run for a fourth-order compartmental model (Schnider et al., 1998; Marsh et al., 1991). The identification algorithm was run with 20 genes and 30 generations. The results are shown in Fig. 4, obtaining a final RMS error of 0.373, and FOS parameters $\hat{\alpha} = 0.9650$, $\hat{a} = 0.1922$ and $\hat{b} = 0.4941$. It can be seen that, although a reduced order model is obtained, it still captures the essential dynamics of the system.

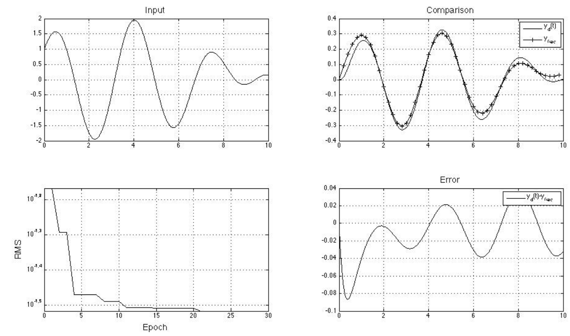
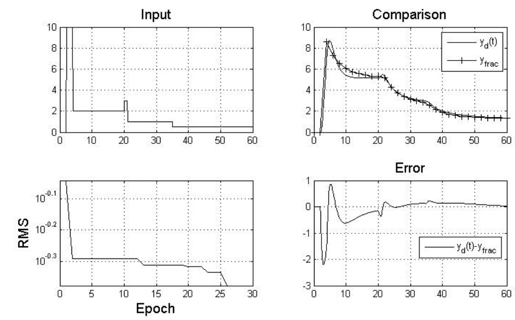
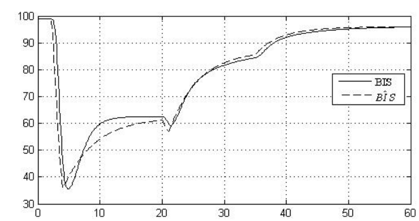


Fig. 3. Identification results for a high-order transfer function



a)



b)

Fig. 4. Identification for an anesthesia system

5. CONCLUSIONS

In this work we have shown a simply method for identification of a single-fractional order differential equation. In first place, each of the parameters of the linear fractional equation are coded into genes, and by using both elitism and mutation during a number of generations, an identified set of parameters is achieved. Simulations for a fractional order equation, a high-order integer-order differential equation and a nonlinear system were shown in order to illustrate the effectiveness of the proposed method. Further work will focus on also identifying the nonlinear parameters in the case of Wiener or Hammerstein models with a fractional-order equation.

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