Discontinuous Dynamic Feedback for Nonlinear Dynamic Systems: A Lyapunov Approach ⋆

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Abstract: In this work, a dynamic feedback control law based on the well known "Twisting" algorithm is under study. Dynamic compensation is added to the Twisting algorithm, in order to use position feedback only, and keep properties such as finite time stability. In some mechanical applications, an observer or a differentiator design is required for control purposes when the whole state space is not available for measurement. An alternative solution for this problem is proposed: a finite time stable algorithm that uses dynamic position feedback. Indeed, this new proposal does not require to measure or estimate another signal but the position of the mechanical system. In the stability analysis, strict nonsmooth Lyapunov functions are studied in order to show finite time stability and robustness. Based on the proposed algorithm, a control law for a Two Rotor Aerodynamical System affected by bounded external perturbations is designed.

Keywords: Robust control; Lyapunov method; Stability analysis.

1. INTRODUCTION

In the last decades, sliding mode algorithms (SOSM) have become very important for Variable Structure Systems (VSS) theory because of their properties (see for example Fridman and Levant [2002], Shtessel et al. [2007], SV et al. [1986]). One of the first SOSM algorithms, the twisting algorithm, became very popular due its advantage to consider Coulomb friction of a mechanical system as part of the controller, and it is well known that this algorithm has properties such as finite time stability (FTS) and robustness against bounded external perturbations (see for example Orlov [2008]). In order to design a control law based on this algorithm, all state space variables must be available for measurement. This restriction, in some cases, could be a disadvantage. In some mechanical systems, for example, there is no physical space for a velocity sensor, or the sensor is too expensive, among others. This situation can be solved using the derivative of the position, *i.e.* an observer or a differentiator design must be considered (J. Davila and Levant [2005], Bhat and Bernstein [2005], Drakunov and Utkin [1995], J.J. Slotine and Misawa [1987]).

With this in mind, the stabilization problem of a nonlinear dynamical system is more complex, since two algorithms must be designed, the observer/differentiator algorithm and the control law. Many problems arise when an observer design is used, for example: an exact or partial copy of the plant is needed, or the proof of a separation principle theorem has to be considered, among others (see J. Davila and Usai [2009], Floquet and Barbot [2007], Levant [2005]).

Some sliding mode based algorithms propose an alternative to this problem, like suboptimal algorithm (G. Bartolini and Usai [1997], Bartolini et al. [1998]). This algorithm use only position data and is based on a contraction principle and the time optimal bang-bang control method. However, the stability analysis only ensures finite time convergence to the sliding manifold of the state trajectories.

In (H. Sira-Ramirez [2010]) a dynamic feedback is considered in order to design a linear observer-linear controller-based robust output feedback scheme for output reference trajectory tracking tasks in a class of fully actuated nonlinear mechanical systems whose generalized position coordinates are measurable. In (H. Sira-Ramirez [2010]) the idea of dynamic feedback is considered in order to apply an observer for the accurate linear estimation of nonlinear disturbances inputs affecting the creation of local sliding regimes, on a given sliding manifold, for Single-Input Single-Output (SISO) systems with limited control authority (for more information (H. Sira-Ramirez [2012])). Moreover, the idea of dynamic feedback is used to stabilize the buck converter in (H. Sira-Ramirez [2013]).

In this work, a dynamic feedback design is considered in order to use only position data, and is applied to non linear systems of relative degree two, affected with external bounded perturbations. This algorithm (MTA: Modified Twisting Algorithm) increase the order of the system but ensures finite time stability of the point $(x, y, z) = (0, 0, 0)$. An advantage of this proposed algorithm is that an observer design or an algorithm to estimate the derivative is not necessary, constituting an interesting alternative to an observer based control law, for example twisting algorithm or even PID control law.

In applications for fully and under actuated mechanical systems affected by Coulomb friction, these proposed control law provide the desired performance in spite of significant uncertainties in the system description and external perturbations, as it is typically the case in control of electromechanical systems with complex hard-to-model nonlinear phenomena. Strict non smooth Lyapunov functions will be used to prove the stability

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of the proposed algorithm (for example, see Bacciotti and Ceragioli [1999], Moreno and Osorio [2008]).

In section 2 the problem statement is presented: the stabilization of a second order dynamical system affected by external bounded perturbations. In section 3, some background is shown in order to emphasize the main contribution. In section 4, the stability analysis of the homogeneous nominal closed loop system is shown, and the robustness of the law control is under study. In all cases, finite time stability for the closed loop system is concluded. In section 5 to support theoretical results, a numerical example is shown and in section 6, the conclusions of this work are presented.

2. PROBLEM STATEMENT

The dynamics of a mechanical system affected by external perturbations is governed by the following state space equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x,t) + \tau + \delta(x,y,t), \;\; x, y \in \mathbf{R}, \end{aligned} \tag{1}$$

The known part of the system dynamics is represented by the piece-wise function $f(x,t)$ (such as the inertia, gravity forces, among others) while $\delta(t,x,y)$ denotes the unknown part (such as uncertainties, external/parametric perturbations, among others) and the control signal is represented by variable τ. The uncertainty term is considered bounded by a positive constant M, *i.e*

$$|\delta(x,y,t)| < M \tag{2}$$

The solutions of all systems of differential equations are understood in the Filippov's sense Filippov [1988]. For system (1) the following control design is as follows

$$\tau = U - f(x,t) \tag{3}$$

where U is the proposed algorithm. A homogeneous control law is proposed in order to achieve finite time stability using a non smooth Lyapunov function. Consider the nominal system (1) and the control law

$$\begin{aligned} U &= -\alpha \text{sign}(x) - \beta \text{sign}(z) \\ \dot{z} &= -a \text{sign}(x) - b \text{sign}(z) \end{aligned} \tag{4}$$

Then, the closed loop system is as follows

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha \text{sign}(x) - \beta \text{sign}(z) + \delta(x,y,t) \\ \dot{z} &= -a \text{sign}(x) - b \text{sign}(z) \end{aligned} \tag{5}$$

It is straightforward to verify that the point of interest of the system (5) is the point $(x,y,z) = (0,0,0)$. In the following section some mathematical background is given in order to explain clearly the main contribution of this work.

3. BACKGROUND

In Santiesteban [2015] the following result is presented,

Theorem 1. Santiesteban [2015] Let the parameters of the switched system (5), be such that inequalities

$$\begin{aligned} &\alpha > a * \max\left\{2\frac{\beta}{b}, \frac{\beta}{b} + \varepsilon, 2\frac{b}{\beta}\right\} \\ &a > b + \varepsilon > 0 \end{aligned} \tag{6}$$

are satisfied. Then, the trajectories of system (5) converge locally uniformly in finite time to the point $(x,y,z) = (0,0,0)$.

Notice that in this theorem the stability of system (5) is *local*. In the following section the stability analysis is not only *global*, but an estimation of the convergence time of the trajectories of system (5) to the point $(x,y,z) = (0,0,0)$ is given.

4. STABILITY ANALYSIS

Consider the Lyapunov function for the disturbed system (5)

$$\begin{aligned} V(x,y,z) &= \frac{1}{2}\left(y^2 + \frac{\alpha\beta}{ab}z^2\right) - \frac{\beta}{b}yz \\ &\quad + \left(\alpha - \frac{\beta}{b}a - \gamma \text{sign}(xy)\right)|x| \\ &= \frac{1}{2}\rho^T P \rho + \left(\alpha - \frac{\beta}{b}a - \gamma \text{sign}(xy)\right)|x| \end{aligned} \tag{7}$$

where $\rho^T = [y \;\; z]$ and

$$P = \begin{pmatrix} 1 & -\frac{\beta}{b} \\ -\frac{\beta}{b} & \frac{\alpha\beta}{ab} \end{pmatrix} \tag{8}$$

In order to show that P is a positive definite matrix, $det(P) = \frac{\alpha\beta}{ab} - \frac{\beta^2}{b^2} > 0$ must be satisfied at all time. From equation (7), notice that the coefficient of term $|x|$, $\frac{\alpha}{a} > \frac{\beta}{b} + \gamma$ also must be satisfied, then if inequalities (6) are satisfied then function (7) is positive definite. Now, let's calculate the time derivative of $V(x,y,z)$. The function $V(x,y,z)$ is locally Lipschitz, and it is differentiable at any point except on the set defined by $S = \{(x,y,z) \in \mathbb{R}^3 | x = y = 0\}$. **Notice** that the set S does not contain trajectories of system (5). This means that $\dot{V}(x,y)$ computed along the trajectory $(x(t), y(t), z(t))$ exists almost everywhere. The time derivative of equation (7) along the trajectories of system (5) is given by

$$\begin{aligned} \dot{V}(x,y,z) &= y\Big(-\alpha \text{sign}(x) - \beta \text{sign}(z) + M\Big) \\ &\quad + \frac{\alpha\beta}{ab}z\Big(-a\text{sign}(x) - b\text{sign}(z)\Big) \\ &\quad + \left(\alpha - \frac{\beta}{b}a - \gamma \text{sign}(xy)\right)\text{sign}(x)y \\ &\quad - \frac{\beta}{b}\Big\{y\big(-a\text{sign}(x) - b\text{sign}(z)\big)\Big\} \\ &\quad - \frac{\beta}{b}z\big(-\alpha\text{sign}(x) - \beta\text{sign}(z) + M\big) \end{aligned} \tag{9}$$

After some algebraic simplifications,

$$\begin{aligned} \dot{V}(x,y,z) &= -\beta|z|\left(\frac{\alpha}{a} - \frac{\beta}{b} - M\text{sign}(z)\right) \\ &\quad - (\gamma - M)|y| \end{aligned} \tag{10}$$

Notice that the inequalities

$$\frac{\alpha}{a} > \frac{\beta}{b} + M; and \;\; \gamma > M \tag{11}$$

must be satisfied in order to show that $\dot{V}$ is negative semi-definite. Then point $(x,y,z) = (0,0,0)$ of system (5) is stable if inequality (6) holds. Since no sliding motion appears on axis x or z, except the origin $x = y = z = 0$, where $\dot{V}(x(t), y(t), z(t)) = 0$, relation (10) remains in force for almost all t. Indeed, let $s_1 = x$ that implies $\dot{s_1} = y$, then $s_1\dot{s_1} < 0$ is not satisfied at all time. However, let $s_2 = z$ that implies $\dot{s_2} = -a\text{sign}(x) - b\text{sign}(z)$, then

$$s_2\dot{s_2} = -|z|\left(a\text{sign}(xz) + b\right) < 0, \tag{12}$$

i.e., if $b > a$, sliding motion appears at the surface $s_2 = z$. This is the reason that inequality $a > b > 0$ must be satisfied at all time. Moreover, as mentioned before, the trajectories of (4) cross the switching lines $x = 0$ and $z = 0$ everywhere except the origin $x = y = z = 0$ so that all the system trajectories are uniquely continuable on the right.

Remark 1. If $a > b > 0$ is not satisfied then sliding mode in the surface $s_2 = z$ can be present. When System (5) hits the sliding surface, *i.e.* $z = 0 \Rightarrow \dot{z} = 0$, it can be described as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha\text{sign}(x) \end{aligned} \tag{13}$$

If $x \neq 0$, the solution of system (13) is as follows

$$y^2 + \alpha|x| = C, \;\; C \in \mathbb{R} \tag{14}$$

It is well known that this system has two poles in the origin and generates a persistent oscillations. Indeed, system (13) is weak against external perturbations and it can be unstable/stable when it is affected by uncertainties.

The qualitative behavior of the nominal system (5) is depicted in Figure 1. Due to the parameter subordination (2), the velocity vectors of (5) point toward the same region in the switching lines

$$\begin{aligned} S_1 &= \{(x, y, z) \in \mathbb{R}^3 : x > 0, z = 0\} \\ S_2 &= \{(x, y, z) \in \mathbb{R}^3 : x < 0, z = 0\} \\ S_3 &= \{(x, y, z) \in \mathbb{R}^3 : x = 0, z > 0\} \\ S_4 &= \{(x, y, z) \in \mathbb{R}^3 : x = 0, z < 0\} \end{aligned} \tag{15}$$

4.1 Main results

Theorem 2. Let the parameters of the nominal switched system (5) be such that conditions

$$\begin{aligned} &\alpha > \max\left(\beta; \frac{\beta}{b}a + \gamma + \gamma_2^{\frac{2}{3}}\right); \\ &(\gamma)^2\left(\alpha - \frac{\beta}{b}a - \gamma\right) > 4\gamma_2^2 \\ &a > b > 0 \end{aligned} \tag{16}$$

are satisfied, where $\gamma_2 > 0$. Then system (5) is globally uniformly finite time stable around the point $(x, y, z) = (0, 0, 0)$. Moreover, an estimation of the convergence time is given by

$$t_{reach} \leq \frac{4}{\lambda_4}\lambda_3^{\frac{3}{4}} W^{\frac{1}{4}}(x(0), y(0)) \tag{17}$$

with

$$\lambda_4 = \min\left\{\beta\left(\frac{\alpha}{\beta} - \frac{a}{b}\right)\left(\alpha - \frac{a}{b}\beta - \gamma\right), \right. \tag{18}$$

$$\left. \beta\left(\frac{\alpha}{\beta} - \frac{a}{b}\right)\frac{\alpha\beta}{ab}, \beta\left(\frac{\alpha}{\beta} - \frac{a}{b}\right)\frac{\beta}{b}, \gamma_2\left(\alpha - \beta\right), \frac{1}{4}\gamma\right\}$$

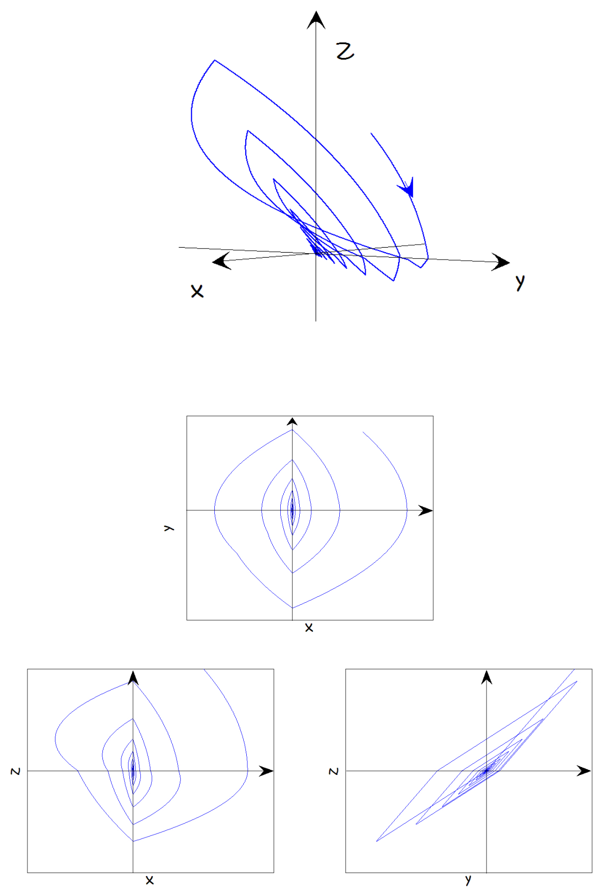

Fig. 1. Qualitative behavior of system (5).

and $\lambda_3 = \max\left\{1, \frac{\alpha\beta}{ab}, \frac{\beta}{b}, \left(\alpha - \frac{a}{b}\beta - \gamma\right), \gamma_2\right\}$

Based on the former theorem, the following result is presented for the perturbed system (5),

Theorem 3. Let the parameters of the disturbed switched system (5) be such that conditions

$$\begin{aligned} &\alpha > \max\left(\beta + M; \frac{\beta}{b}a + \gamma + \gamma_2^{\frac{2}{3}}\right); \\ &(\gamma - M)^2\left(\alpha - \frac{\beta}{b}a - \gamma\right) > 4\gamma_2^2 \\ &a > b > 0; \gamma > M \end{aligned} \tag{19}$$

are satisfied. Then system (5) is globally uniformly finite time stable around the point $(x, y, z) = (0, 0, 0)$ in spite of bounded external perturbations. Moreover, an estimation of the convergence time is given by

$$t_{reach} \leq \frac{4}{\lambda_4}\lambda_3^{\frac{3}{4}} W^{\frac{1}{4}}(x(0), y(0)) \tag{20}$$

with

$$\lambda_4 = \min\left\{\beta\left(\frac{\alpha}{\beta} - \frac{a}{b} - M\right)\left(\alpha - \frac{a}{b}\beta - \gamma\right), \beta\left(\frac{\alpha}{\beta} - \frac{a}{b} - M\right)\frac{\alpha\beta}{ab}, \beta\left(\frac{\alpha}{\beta} - \frac{a}{b} - M\right)\frac{\beta}{b}, \gamma_2\left(\alpha - \beta - M\right), \frac{1}{4}\left(\gamma - M\right)\right\} \quad (21)$$

and $\lambda_3 = \max\left\{1, \frac{\alpha\beta}{ab}, \frac{\beta}{b}, \left(\alpha - \frac{a}{b}\beta - \gamma\right), \gamma_2\right\}$

A sketch of the proof of Theorem 3 is shown on Appendix I.

5. NUMERICAL EXPERIMENTS

In this section, the control problem known as tracking is considered, using a rigid body mechanical system as a test bed (Fig. 2). This system consists of a beam pivoted on its base that it can rotate freely in both horizontal and vertical planes. A mathematical model, based on Euler-Lagrange method, of a similar system with aerodynamic control inputs is presented in Mullhaupt et al. [2008]. Assuming control inputs as torques τ_1 and τ_2, the state equation of the system is given by

$$\begin{aligned} I_\psi \ddot{\psi} &= \tau_1 - C_\psi \dot{\psi} + \frac{1}{2} I_c \dot{\phi}^2 \sin(2\psi) \\ &+ G_s \sin(\psi) + G_c \cos(\psi) \end{aligned} \quad (22)$$

$$(I_\phi + I_c \sin^2(\psi))\ddot{\phi} = \tau_2 - C_\phi - I_c \dot{\psi}\dot{\phi}\sin(2\psi)\dot{\phi} \quad (23)$$

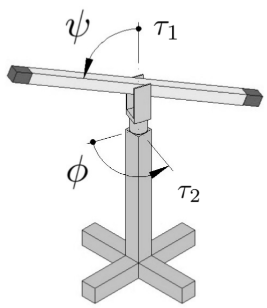

Fig. 2. The one-link pendulum system.

where q is the angle made by the pendulum with the vertical, m is the mass of the pendulum, l is the distance to the center of mass, J is the moment of inertia of the pendulum about the center of mass, g is the gravity acceleration, F is the friction force, τ is the control torque, and $\delta(t, q, \dot{q})$ is the external disturbance.

The control objective is to drive the given system to a known trajectory in exact finite time, *i.e.*

$$\psi(t) - r(t) = 0. \quad (24)$$
$$\phi(t) - r(t) = 0. \quad (25)$$

where $r(t) = \sin(t)$ even in the presence of an admissible external disturbance. Let the tracking error be given by

$$e_1(t) = \psi(t) - r(t). \quad (26)$$

$$e_2(t) = \phi(t) - r(t). \quad (27)$$

Considering (22) and (23) as independent systems, taking inertial couplings and friction forces as perturbations and using the control for system (22) in the form

$$\begin{aligned} \tau_1 &= I_\psi \ddot{r} - \alpha_1 sign(e_1) - \beta_1 sign(z_1) \\ &\quad - G_s \sin(\psi) - G_c \cos(\psi) \\ \dot{z}_1 &= -a_1 sign(e_1) - b_1 sign(z_1). \end{aligned} \quad (28)$$

and for system (23) in the form

$$\begin{aligned} \tau_2 &= -\alpha_2 sign(e_2) - \beta_2 sign(z_2) \\ &\quad + (I_\phi + I_c \sin^2(\psi))\ddot{r} \\ \dot{z}_2 &= -a_2 sign(e_2) - b_2 sign(z_2). \end{aligned} \quad (29)$$

the error dynamics can be written as follows, for system (22)

$$\begin{aligned} I_\psi \ddot{e}_1 &= -\alpha_1 sign(e_1) - \beta_1 sign(z_1) - C_\psi \dot{\psi} \\ &\quad + \frac{1}{2} I_c \dot{\phi} \sin(2\psi) + M \\ \dot{z}_1 &= -a_1 sign(e_1) - b_1 sign(z_1). \end{aligned} \quad (30)$$

and for system (23)

$$\begin{aligned} (I_\phi + I_c \sin^2(\psi))\ddot{e}_2 &= -\alpha_2 sign(e_2) - \beta_2 sign(z_2) \\ &\quad - I_c \dot{\psi}\dot{\phi}\sin(2\psi) - C_\phi \dot{\phi} + M \\ \dot{z}_2 &= -a_2 sign(e_2) - b_2 sign(z_2). \end{aligned} \quad (31)$$

In order to show the performance of the proposed algorithm, a comparison with twisting is considered. Parameters of Table 1 are considered.

Table 1. Model parameters

Parameter	Value	Unit
I_ψ	40e-3	$\mathrm{kg \cdot m^2}$
I_ϕ	6.7e-3	$\mathrm{kg \cdot m^2}$
I_c	31.7e-3	$\mathrm{kg \cdot m^2}$
C_ψ	6e-3	$\mathrm{N \cdot m \cdot s/rad}$
C_ϕ	2e-3	$\mathrm{N \cdot m \cdot s/rad}$
G_s	-60e-3	$\mathrm{N \cdot m}$
G_c	-0.31	$\mathrm{N \cdot m}$

The initial conditions for the model, selected for all experiments, are fixed as $\psi(0) = 1$ rad, $\phi(0) = 1$ rad, $\dot{\psi}(0) = 0$ $\mathrm{rad/s}$ and $\dot{\phi}(0) = 0$ $\mathrm{rad/seg}$ for the positions and velocities, respectively. the dynamics of the system in closed loop affected by the bounded external perturbations. The numeric exercise use $\alpha_i = 8.5$, $a_i = 2$, $\beta_i = 1$ and $b_i = 1$ as gains of the control law MTA. The positive constants $\alpha_{ti} = 8.5$, $\beta_{ti} = 6$ denote the gains of twisting algorithm, where $i = 1, 2$. Both simulations exercises are affected by $M = 0.5\ \mathrm{N \cdot m}$ as an uncertainty term from $t = 10$ onwards. Also parametric perturbations are considered, *i.e.* $\tilde{I}_\psi = 60\text{e-}3\ \mathrm{kg \cdot m^2}$, $\tilde{I}_\phi = 13.6\text{e-}3\ \mathrm{kg \cdot m^2}$ in the controller design.

Figure 4 shows the error signal $e_1(t)$ and $e_2(t)$ and the control input signal $\tau_1(t)$ and $\tau_2(t)$ for both cases. In both exercises there is no optimization criteria but the compensation of bounded external perturbations to fix gains controllers. This simulations shows that MTA performance is as good as twisting algorithm.

According to *Theorem 3*, conditions for globally finite time stability (17) of TRAS mechanism are satisfied taking $M = 1$, $\gamma = 6.08$ and $\gamma_2 = 0.21$. Then $\lambda_4 = 1.27$ and $\lambda_3 = 4.25$. Finally, the estimation of time converge (18) is $t_{reach} \leq 4.27$ seconds.

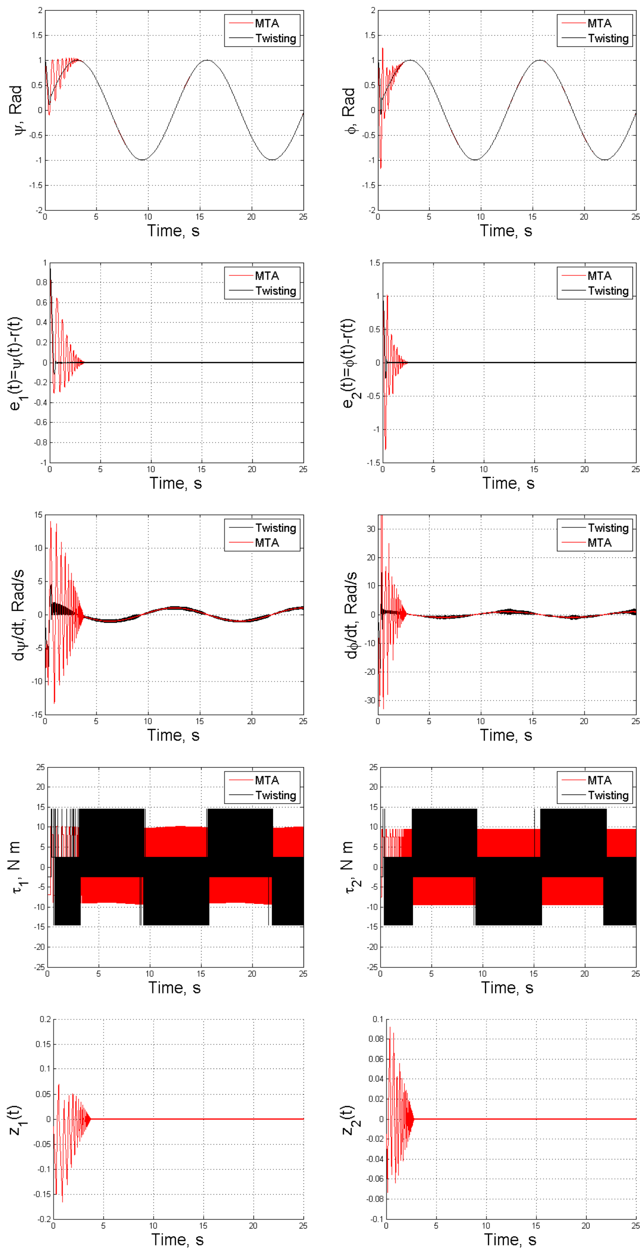

Fig. 3. Tracking stabilization of the one-link pendulum.

6. CONCLUSIONS

In this work, a globally uniformly finite time algorithm for relative degree two systems using dynamic position feedback was proposed. Moreover, a strict non-smooth Lyapunov function was proposed in order to estimate convergence time of the closed loop system. The performance of the proposed algorithm was shown by solving the tracking control problem of a Two Rotor Aerodynamical System in spite of bounded external and parametric perturbations. The closed loop mechanical system showed to be robust and provide nice performance in spite of unknown but bounded uncertainties. Moreover, a comparison with twisting algorithm is presented in order to show the closed loop behavior.

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7. APPENDIX I

Proof of theorem 1: In order to show finite time stability a non-smooth candidate Lyapunov function, based on the former one, is proposed. Let write the former function $V(x, y, z)$ as follows

$$V(x, y, z) = \eta + \left(\alpha - \frac{\beta}{b}a - \gamma\text{sign}(xy)\right)|x| \tag{32}$$

where $\eta = \frac{1}{2}\rho^T P\rho$, and

$$P = \begin{pmatrix} 1 & -\frac{\beta}{b} \\ -\frac{\beta}{b} & \frac{\alpha\beta}{ab} \end{pmatrix} \tag{33}$$

Consider the candidate Lyapunov function for system (5)

$$\begin{aligned} W(x, y, z) &= \frac{1}{2}V(x, y, z)^2 + \gamma_2|x|^{\frac{3}{2}}\text{sign}(x)y \\ &= \frac{1}{2}\eta^2 + \eta\left(\alpha - \frac{\beta}{b}a - \gamma\text{sign}(xy)\right)|x| \\ &+ \frac{1}{2}\left(\alpha - \frac{\beta}{b}a - \gamma\text{sign}(xy)\right)^2 x^2 \\ &+ \gamma_2|x|^{\frac{3}{2}}\text{sign}(x)y \end{aligned} \tag{34}$$

In order to show that $V(x, y, z)$ is positive definite, the following inequalities

$$\alpha > \max\left(\frac{\beta}{b}a; \frac{\beta}{b}a + \gamma + \gamma_2^{\frac{2}{3}}\right) \tag{35}$$

must be satisfied. Moreover, in order to find an upper bound for the candidate Lyapunov function, let $\eta \le \lambda_1(y+z)^2$, with $\lambda_1 = \max\{1, \frac{\alpha\beta}{ab}, \frac{\beta}{b}\}$, then

$$V(x, y, z) \le \lambda_2(|x|^{\frac{1}{2}} + |y| + |z|)^2 \tag{36}$$

where $\lambda_2 = \max\{\lambda_1, \alpha - \frac{\beta}{b}a - \gamma\}$ hence, an upper bound for $W(x, y, z)^2$ is as follows

$$W(x, y, z) \le \lambda_3(|x|^{\frac{1}{2}} + |y| + |z|)^4 \tag{37}$$

where $\lambda_3 = \max\{\lambda_2, \gamma_2\}$.

Now, let's calculate the time derivative of $W(x, y, z)$. Note that $W(x, y, z)$ is locally Lipschitz, and it is differentiable at any point except on the set defined by $S = \{(x, y, z) \in \mathbb{R}^3 | x = y = 0\}$. **Notice** that the set S does not contain trajectories of system (5). This means that $\dot{W}(x, y)$ computed along the trajectory $(x(t), y(t), z(t))$ exists almost everywhere. The time derivative of equation (7) along the trajectories of system (5) is given by

$$\begin{aligned} \dot{W}(x, y, z) &= V(x, y, z)\dot{V}(x, y, z) \\ &- \gamma_2|x|^{\frac{3}{2}}(\alpha + \beta\text{sign}(xz) + M\text{sign}(x)) \\ &+ \gamma_2|x|^{\frac{1}{2}}y^2 \end{aligned} \tag{38}$$

In order to show that $\dot{W}(x, y, z) < 0$, it is clearly that the terms $-V(x, y, z)|z|$ and $-V(x, y, z)|y|$ are negative definite if

$$\frac{\alpha}{a} > \frac{\beta}{b} + M; \;\; \gamma > M \tag{39}$$

and the terms

$$\begin{aligned} &(\gamma - M)\left(\alpha + \frac{\beta}{b}a - \gamma\right)|x||y|; \;\; \gamma_2|x|^{\frac{1}{2}}y^2 \\ &\frac{1}{4}(\gamma - M)|y|^3(\alpha + \beta\text{sign}(xz) + M\text{sign}(x)) \end{aligned} \tag{40}$$

are written as follows

$$\begin{aligned} -|y|&\left(\frac{1}{4}(\gamma - M)|y|^3 - \gamma_2|x|^{\frac{1}{2}}y + \zeta|x|\right) \\ &= -\frac{1}{2}|y|\rho_c^T P_c\rho_c \end{aligned} \tag{41}$$

where $\zeta = (\gamma - M)\left(\alpha + \frac{\beta}{b}a - \gamma\right)$, $\rho^T = [|x|^{\frac{1}{2}} \;\; y]$ and

$$P_c = \begin{pmatrix} \frac{1}{8}(\gamma - M) & -\gamma_2 \\ -\gamma_2 & 2\zeta \end{pmatrix} \tag{42}$$

then if $det(P_c) > 0$, the function $\dot{W}(x, y, z)$ is negative definite, *i.e.* $(\gamma - M)^2\left(\alpha + \frac{\beta}{b}a - \gamma\right) > 4\gamma_2^2$

In order to show finite time stability, $\dot{W}(x, y, z)$ is written as follows

$$\begin{aligned} \dot{W}(x, y, z) &= -\beta\left(\frac{\alpha}{a} - \frac{\beta}{b} - M\right)V(x, y, z)|z| \\ &- \gamma_2|x|^{\frac{3}{2}}(\alpha + \beta\text{sign}(xz) + M\text{sign}(z)) \\ &- \frac{1}{4}(\gamma - M)|y|^3 \end{aligned} \tag{43}$$

it is easy to see that

$$\dot{W}(x, y, z) \le -\lambda_4\left(|x|^{\frac{1}{2}} + |y| + |z|\right)^3 \tag{44}$$

where λ_4 is as inequality (21). Finally, writing $\dot{W}(x, y, z)$ in terms of $W(x, y, z)$ is

$$\dot{W}(x, y, z) \le -\lambda_4\left(\frac{W(x, y, z)}{\lambda_3}\right)^{\frac{3}{4}} \tag{45}$$

Then finite time stability of system (5) can be concluded. To estimate an upper bound for convergence time , let us consider the following comparison system

$$\dot{\omega} = -a\omega^{\frac{3}{4}} \tag{46}$$

The solution of this system is $\omega(t) = (\omega^{\frac{1}{4}}(0) - \frac{1}{4}at)^4$, and thus the estimation for reaching time is $t_{reach} = \frac{4}{a}\omega^{\frac{1}{4}}(0)$. Summing up, an estimation of an upper bound for the reaching time of the system (5) can be calculated as equation (20).

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