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State Estimation for Uncertain Linear Systems: A Review ⋆

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Abstract: This paper provides a literature review about estimation strategies for uncertain, linear systems with emphasis on classic estimation strategies and strategies based on the moving horizon estimation (MHE) setting. The aim of this paper is to show the lack of estimation strategies handling hard constraints on the system variables. Although moving horizon schemes are mainly designed to include hard constraints to improve the estimates, available strategies for uncertain linear systems do not exploit this property. Finally, the game-theory approach to the $\mathcal{H}_\infty$ filtering is shown to be suitable to develop a MHE-based scheme for uncertain, linear systems to address directly hard constraints.

Keywords: moving horizon estimation, $\mathcal{H}_\infty$ filtering, soft sensors, robust estimation, uncertain linear systems

1. INTRODUCTION

As it is widely known, most industrial applications require the measurement of a large number of physical variables to monitor the variables/parameters indicating the product quality, or generally speaking, the overall process performance. However, the direct or on-line measurement of some variables is not always possible either because there is not a sensor for a specific variable, the sensor is not affordable, or the maintenance of the sensor is cumbersome. In some other cases, although the measurement of such variables may be available through off-line procedures like lab assays, it may prevent or difficult the opportune decision making. A reliable approach to overcome the above issues is to implement soft sensors and state estimation strategies to infer these inaccessible variables by using a process model and a set of easy-to-measure variables.

Regarding to the state estimation theory, a large number of contributions is available in the literature even if the search is restricted to linear systems. These contributions range from the Luenberger observer to the celebrated Kalman filter, going through robust approaches and optimization-based strategies. From the beginning of the 1980's, the attention of control theorists was focused on linear systems subject to significant uncertainty in parameters as well as on external inputs (Banavar and Speyer, 1991). The sensitivity of the Kalman-based filters to modeling errors led several works on robust state-space filters. Robustness is meant as the attempt to limit the effect of model uncertainties on the overall filter performance.

Incipient attempts to summarize relevant results around estimation of uncertain linear systems have been presented in (Simon, 2006; Shaked and Theodor, 1992). However, to the best of the authors' knowledge, there is no a single contribution describing the advantages and limitations of the available estimation strategies for uncertain linear systems in a systematic, ordered, and complete form. In this context, the present contribution provides a literature revision about estimation strategies for uncertain, linear systems with emphasis on classical strategies and strategies based on the moving horizon estimation (MHE) setting. With *classical approaches* the authors are referred to estimation schemes applied to uncertain linear systems in which analytic solutions are provided. Moving horizon approaches, on the other hand, state the estimation problem as an open optimization problem to be solved at every time step. The main aim of this paper is first to give a review on state-estimation strategies for uncertain linear systems, and consequently, to demonstrate the lack of estimation strategies able to handle useful insight about a process in the form of hard constraints.

2. PROBLEM STATEMENT

Consider the following uncertain discrete-time linear system

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$$x_{k+1} = (A + \Delta A_k)x_k + Gw_k,$$
$$y_k = (C + \Delta C_k)x_k + \nu_k. \qquad (1)$$

where k denotes the current discrete time and $x_k \in \mathbb{R}^n$ is the system state. x_0, the initial state, is a random variable with unknown statistics, $w_k \in \mathbb{R}^m$ is the uncertainty associated with the knowledge of the state equation, $y_k \in \mathbb{R}^p$ is the output or measured variable, $\nu_k \in \mathbb{R}^p$ is the measurement uncertainty, A, G, and C are known matrices, and ΔA_k and ΔC_k are the plant uncertainties affecting the matrices A and C, respectively.

Using (1), three different state estimation problems are posed as follows.

Problem 1. Systems with Unknown Inputs

Consider (1). Assume $\Delta A_k = 0$ and $\Delta C_k = 0$. Also, assume both w_k and ν_k as either deterministic variables of an unknown type or noises belonging to the space of square-summable sequences $\mathcal{L}_2$ on a specified time interval. Therefore the problem consists of designing a filter to produce $\hat{x}_k$, the estimate of x_k, using a specified performance criterion despite of the uncertainty from the inputs.

Problem 2. Systems with Uncertainty in the Plant Parameters

Consider (1). Assume w_k and ν_k noises with known statistics. Also, assume both ΔA_k and ΔC_k belonging either to some defined compact sets or to a polytope. Therefore the problem consists of designing a filter to produce $\hat{x}_k$, the estimate of x_k, using a specified performance criterion despite of the uncertainty from the plant parameters.

Problem 3. State Estimation in Systems with Unknown Inputs and Uncertain Plant Parameters

Consider (1). Problem 3 combines both Problems 1 and 2. In this way, assume both w_k and ν_k as either deterministic variables of an unknown type or noises belonging to the space of square-summable sequences $\mathcal{L}_2$ on a specified time interval. Also, assume both ΔA_k and ΔC_k belonging either to some defined compact sets or to a polytope. Therefore the problem consists of designing a filter to produce $\hat{x}_k$, the estimate of x_k, using a specified performance criterion despite of the uncertainty from the inputs and the plant parameters.

In what follows, several state estimation schemes for uncertain linear systems are reviewed. A distinction between state-estimation strategies from a classical approach and strategies in a moving horizon approach is made.

3. STATE ESTIMATION FOR UNCERTAIN LINEAR SYSTEMS: THE CLASSICAL APPROACH

As it is well known, the solution to the optimal filtering problem for linear systems with known Gaussian inputs were first addressed by Kalman and coworkers for both discrete-time and continuous-time systems (Kalman, 1960; Kalman and Bucy, 1961). Since then, the Kalman filter turned out to be a valuable tool for monitoring a wide range of engineering systems from aerospace to industrial processes. Recent reviews on aerospace and industrial applications of the Kalman filter are found in (Grewal and Andrews, 2010) and (Auger et al., 2013), respectively.

There are many cases where the assumptions made by the Kalman filter cannot be fulfilled, i.e., the noises cannot be modeled by white-noises with known statistics and/or there are plant uncertainties. Studies to circumvent the above problem started in the 80's of the last century where significant effort has been spent.

For the sake of clearness, when an estimator is designed to face both unknown inputs and plant parameters, the filter is said to be robust. Otherwise the filter is named according to the used estimation approach: $\mathcal{H}_\infty$ filtering, set-valued approach, cost-guaranteed paradigm, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering, unknown input observers (UIO), etc., (Sayed, 2001; Xie et al., 1991). In the remainder of the Section, contributions tackling Problems 1-3 are particularly discussed.

3.1 Systems with Unknown Inputs

Consider the framework where the plant model is known but the inputs entering the system w_k and ν_k, i.e., Problem 1. To solve the above problem, three main approaches were identified:

- The inputs are assumed to be "Gaussian" noises with unknown covariances.
- Each input is determined as a sum of an unknown disturbance and a "white" noise.
- The inputs are assumed to be completely unknown and noisy, with a bounded feature, e.g., bounded energy or maximal amplitude.

When Gaussian inputs with unknown statistics are assumed, the adaptive Kalman-based filters and the M-estimators are well-accepted choices (Darouach et al., 1995; Durovic and Kovacevic, 1999; Liang et al., 2004). There also exist covariance estimation approaches to determine the true covariances of the noises before the Kalman filter design (Odelson et al., 2006). Useful contributions from the fault diagnosis theory allowed the use of unknown input observers (UIO) as an possibility when the nature of the input is unknown (Hsu et al., 2001). These observers assume that the state and measurement uncertainties are each the sum of a "white" noise and an unknown disturbance.

If non-white noises are assumed, two subdivisions for designing estimators were recognized:

- The set-valued approach.
- The $\mathcal{H}_\infty$ filtering theory

The main objective in the *set-valued* approach is to construct ellipsoids around the estimates that are consistent with observations and with certain norm constraints on the disturbances (Sayed, 2001). Contributions in this field are presented in (Becis-Aubry et al., 2008; Ra et al., 2004; Yang and Li, 2009). On the other hand, the $\mathcal{H}_\infty$ filtering theory looks for guaranteeing a bound over the $\mathcal{H}_\infty$-norm of the transfer function from the unknown inputs to the estimation error. In this context, the options to guarantee such a norm are:

- The polynomial approach (Grimble, 2006).
- The interpolation theory (Fu, 1991).

- The Riccati-based approach.

Polynomial and interpolation approaches are mainly derived in the frequency domain. In the polynomial approach, the estimator is obtained from the solution of a linear equation and a spectral factorization calculation (Grimble, 2006). To solve the optimal estimation in the polynomial approach, the original problem is transformed to the $\mathcal{H}_2$ setting. The objective is to determine a weighting function such that, when substituted into a given $\mathcal{H}_2$ estimation problem satisfies an equivalence with the $\mathcal{H}_\infty$-filtering problem. The interpolation approach was found after writing the filtering problem as the so-called *optimal loop transfer recovery problem* (Fu, 1991). The idea behind the above approach is mainly focused on finding an estimator as a function of the output instead of the state.

Pole-zero properties of the optimal estimators are more easily handled in the *set-valued* and *interpolation* approaches since the analysis is made in the frequency domain. Among the advantages of such filters are: more direct and simpler solutions for optimal estimators if compared to the Riccati-based approach, easy adding and removing of frequency weights on the estimation error and disturbances, and the fact of avoiding unnecessary high observer gains (Fu, 1991).

A Riccati-based approach is a filter involving two Riccati equations used to guarantee the $\mathcal{H}_\infty$ condition. It is worth to mention that the number of contributions using a Riccati-based filter is considerably big if compared with the polynomial and interpolation approaches. This is due to the multiple design approaches making use of Riccati equations. Linear Matrix Inequalities (LMI), the game-theory, and frequency-domain tools are design approaches to design $\mathcal{H}_\infty$ filters by means of a Riccati-based approach (Banavar and Speyer, 1991; Simon, 2006; Sayed, 2001; Lu and Yang, 2009). More details on this $\mathcal{H}_\infty$ filters using the Riccati-based approach will be discussed later since they also give solution to the Problem 3.

3.2 Systems with Uncertainty in the Plant Parameters

The second framework is about models with uncertain or unknown plant parameters and known inputs entering the system dynamics, i.e., Problem 2. Particular solutions depend on the assumptions made over the nature of the uncertainty. For instance, if uncertainties on the parameters of the plant are modeled as non-Gaussian random variables, two approaches arise:

- The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering.
- The cost-guaranteed approach.

The $\mathcal{H}_2/\mathcal{H}_\infty$ filter takes advantage of both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering paradigms. The filters are designed to minimize the variance of the estimation error while respecting an upper bound over the frequency response of the transfer function from the estimation error to the unknowns (Wang and Unbehauen, 1999; Yang and Hung, 2000). The cost-guaranteed approach, on the other hand, is known to be similar to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering approach. However, there is a slight difference: the filter design is based on guaranteeing an upper bound over the steady-state variance of the estimation error, despite the variance itself, for all admissible uncertainties in the model (Milocco and Muravchik, 2005; Xie et al., 2003).

Robust Kalman filters (RKF) are used when the uncertainties over the plant parameters are modeled by random variables with Gaussian distributions. Given that the plant parameters are assumed to be unknown, the problem is addressed by designing a linear filter such that the variance of the estimation error is guaranteed to be within a certain bound for all admissible uncertainties (Zhu et al., 2002). When no information about the uncertain parameters is given, deterministic solutions are found to be useful (Muramatsu and Ikeda, 2011).

3.3 State Estimation in Systems with Unknown Inputs and Uncertain Plant Parameters

A more challenging problem comes up when combining Problems 1 and 2, i.e., uncertainty from some parameters of the plant model and the inputs need to be addressed in the filter design. As evidenced previously, the strategies revised here differ each other in the way the authors assume the nature of the uncertainty. For instance, the RKF and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters are useful when uncertainties are modeled as random variables with Gaussian distributions. On the other hand, the *set-valued* approach and the $\mathcal{H}_\infty$ theory are used when there is no assumption over the nature of the uncertainties.

As discussed before, the RKF arose as an extension of the Kalman filter to tackle the uncertain parameters and inputs (Simon, 2006). Recent contributions on RKF solving Problem 3 were found in (Simon, 2006; Xu and Mannor, 2009). According to Simon, the RKF have been tested in different scenarios showing an acceptable performance: measurements consisting entirely of noise, uncertainty in the measurement and transition matrices, and uncertainty due to unknown covariances in the inputs (Simon, 2006).

On the other hand, the $\mathcal{H}_\infty$ theory has shown to be suitable when unknown inputs and uncertain parameters in the plant model arise. As stated before, there are multiple ways to design a filter in the $\mathcal{H}_\infty$ setting. As stated earlier, there is a prevalence on the Riccati-based approaches to design filters when models are uncertain.

In any Riccati-based approach, the existence and design of the filter depend on the stabilizing solutions of a particular Riccati equation. The use of the Bounded Real Lemma (BRL) provides a link between the original robust filtering problem with the solution of an optimal estimation problem with unknown inputs, i.e., a link between Problem 3 and Problem 1, respectively, which in turn is easier to solve (de Souza and Xie, 1992; Vaidyanathan, 1985). Nevertheless, it is difficult to explain all Riccati-based approaches with a single methodology. In fact, as stated before, there are multiple ways to obtain a filter with the $\mathcal{H}_\infty$ performance under the Riccati approach: the game-theory approach (de Souza et al., 1995; Mangoubi, 1995), the robust filtering using the LMI theory (Xie et al., 2003), the robust filtering using frequency domain tools (Marquez, 2003), and others (Xie et al., 1991; Fu, 1991; Zhou, 2010).

The game-theory approach to the $\mathcal{H}_\infty$ filtering gives a formulation of the problem oriented to the explicit statement of a minimax optimization. The minimax optimization is analytically solved under specific conditions, avoiding the solution of a complex online optimization problem. Nevertheless, providing an analytic solution to the minimax optimization problem prevents the use of constraints. Further details about this approach are given in the following.

3.4 The Game-Theory Approach to the $\mathcal{H}_\infty$ Filtering

The game-theory addresses strategic interactions among multiple decision makers, called players and in some contexts agents (Başar and Bernhard, 1995). Each player has a preference among some alternatives. The set of players must bargain among themselves to achieve the purpose required by an objective function. In this context, the game-theory fits quite well to the time-domain formulation of the $\mathcal{H}_\infty$ filtering because the estimation problem can be posed as a bargain among the state estimation error and the worst-case disturbances of the dynamic system. Although the game-theory is quite vast, the zero-sum dynamical (differential) games provides the starting point to develop an estimation strategy based on the $\mathcal{H}_\infty$ theory. Formulations of $\mathcal{H}_\infty$ filtering under the game-theory approach are found in (Banavar and Speyer, 1991; Simon, 2006; Mangoubi, 1995) and references therein.

The best known formulation of the filtering problem under the game-theory approach is on Problem 1 (Simon, 2006). In this formulation, the estimate is made over a finite-horizon time window T; measurements up to and including the time $T-1$ are used to project the estimate at time T. In order to show this formulation, consider a dynamic system like (1) with $\Delta A_k = 0$ and $\Delta C_k = 0$. Assume that a linear combination of the state, e.g., $z_k = L_k x_k$ is to be estimated. To obtain the $\mathcal{H}_\infty$ performance, an objective function in form of a disturbance attenuation function is used:

$$J = \frac{\sum_{k=0}^{T-1} \|z_k - \hat{z}_k\|^2_{M_k^T M_k}}{\|x_0 - \hat{x}_0\|^2_{\Pi_0^{-1}} + \sum_{k=0}^{T-1}\left(\|w_k\|^2_{Q_k^{-1}} + \|\nu_k\|^2_{R_k^{-1}}\right)} \quad (2)$$

where $J = J(\hat{z}_k, x_0, w_k, \nu_k)$ expresses the dependency of the cost function with respect to the optimization parameters, i.e., the estimate of z_k, the initial state, the model uncertainty, and measurement uncertainty, respectively. M_k, Q_k, R_k, and Π_0 are used-defined knobs to change the filter performance. The goal behind (2) is to find an estimate $\hat{z}_k$ minimizing J at the same time that other player, let us say nature, is trying to find the worst w_k, ν_k, and x_0 to maximize J. As the direct optimization of J in (2) is not tractable, a performance bound γ is used. The estimation problem is then transformed to fulfil the following condition:

$$J < \gamma, \quad \gamma > 0 \quad (3)$$

which is rewritten as:

$$\begin{aligned} J_a =& -\gamma\|x_0 - \hat{x}_0\|^2_{\Pi_0^{-1}} \\ &+ \sum_{k=0}^{T-1}\left[\|z_k - \hat{z}_k\|^2_{M_k^T M_k} - \gamma\left(\|w_k\|^2_{Q_k^{-1}} + \|\nu_k\|^2_{R_k^{-1}}\right)\right] < 0 \end{aligned} \quad (4)$$

Then, the optimal point, i.e., the saddle-point, is found by solving the following minimax optimization program:

$$J_a^* = \min_{z_k} \max_{x_0, w_k, \nu_k} J_a \quad (5)$$

The specific solution to (5) is omitted due to the space limitations.

The game-theory approach to the $\mathcal{H}_\infty$-filtering is considered to be a reliable strategy solving the state-estimation problem for uncertain linear systems. However, constraints handling is prevented since a closed analytic solution is given. In order to provide an optimization-based strategy able to address constraints, the game-theory approach to the $\mathcal{H}_\infty$-filtering should be manipulated in order to let the minimax optimization be open to be solved at each time step. In this way, hard constraints may be addressed.

4. STATE ESTIMATION FOR UNCERTAIN LINEAR SYSTEMS: THE MOVING HORIZON APPROACH

The moving horizon estimator (MHE) reformulates the $\mathcal{H}_2$-optimal state estimation as a moving, fixed-size, quadratic program (Rao et al., 2001). The success of the receding horizon estimation schemes is mainly focused on the statement of the estimation problem as an optimization program, allowing the direct handling of hard constraints of the variables involved in the system dynamics. The original work assumes the complete knowledge of the statistics of the noisy inputs affecting the system dynamics. Nevertheless, knowledge of the uncertainty is not always possible.

Few contributions have been reported about robust versions of the MHE to estimate the state of a system modeled by an uncertain linear system (Alessandri et al., 2003, 2004, 2005a,b, 2012). In (Alessandri et al., 2003, 2004), a robust receding horizon estimator is presented in the discrete-time linear framework. The filter design is addressed for time-invariant uncertainties on the transition and measurement matrices. Stability results are presented considering zero-input noises. The estimation problem is stated as an open minimax optimization:

$$\min_{\hat{x}_{k-N,k}} \max_{\Delta A \in \mathcal{A}, \Delta C \in \mathcal{C}} J_k(\bar{x}_{k-N,k}, \Delta A, \Delta C) \quad (6)$$

where $\bar{x}_{k-N,k}$ is a predicted (*a priori*) value of the state at the beginning of the time window, $\hat{x}_{k-N,k}$ is the parameter to be minimized, $\Delta A \in \mathcal{A}$ and $\Delta C \in \mathcal{A}$ are the uncertainties over the state and measurement matrices, respectively that are to be maximized, and $\mathcal{A}$ and $\mathcal{C}$ are assumed to be compact sets where ΔA and ΔC belong, respectively. The cost function is defined by

$$\begin{aligned} J_k =& \parallel \hat{x}_{k-N,k} - \bar{x}_{k-N} \parallel^2_M \\ &+ \sum_{i=k-N}^{k} \parallel y_i - (C + \Delta C)\bar{x}_{i,k} \parallel^2 \end{aligned} \quad (7)$$

where $J_k = J_k(\hat{x}^k_{k-N}, \Delta A, \Delta C)$ expresses the dependency of the cost function with respect to the optimization parameters, y_i is the $i-th$ measurement and N is the size of the estimation window. As noticed, there is a penalization term associated with the knowledge of the state at the beginning of the time window and a penalization associated with the difference between the real and the estimated measurements. As problem (7) is not directly solvable with known mathematical tools, the authors reformulated it as a regularized least-squares problem with uncertain data. This lead to a semi-explicit solution of the state estimate, i.e., an explicit solution of the estimate as a function of a new parameter λ_k^o that needs to be found by means of a new optimization problem. As a consequence, this new problem makes the filter nonlinear and time-variant because of the dependence on this new parameter.

In (Alessandri et al., 2005a), Alessandri and co-workers improved the results presented in (Alessandri et al., 2003, 2004). The alternative description of uncertainty allowed a robust estimation technique to address time-varying uncertainties,

$$\begin{aligned} [\,\Delta A_k \;\; \Delta B_k\,] &= D\Delta_k\,[\,E \;\; F\,], \\ \Delta C_k &= G\bar{\Delta}_k H. \end{aligned} \tag{8}$$

where D, E, F, G, and H are known matrices, and Δ_k and $\bar{\Delta}_k$ are arbitrary contractions:

$$\parallel \Delta_k \parallel \leq 1, \quad \parallel \bar{\Delta}_k \parallel \leq 1 \tag{9}$$

x_0, w_k, and ν_k are assumed to be unknown deterministic variables. The following cost function is proposed:

$$\begin{aligned} J_k =& \parallel \hat{x}_{k-N,k} - \bar{x}_{k-N} \parallel_M^2 + \sum_{i=k-N}^{k-1} \parallel y_i - (C + \Delta C_i)\bar{x}_{i,k} \parallel_R^2 \\ &+ \sum_{i=k-N}^{k-1} \parallel \hat{x}_{i+1,k} - (A + \Delta A_i)\bar{x}_{i,k} - (B + \Delta B_i)\bar{u}_i \parallel_Q^2 \end{aligned} \tag{10}$$

where $J_k = J_k(\hat{x}_{k-N}^k, \Delta A, \Delta B, \Delta C)$ expresses the dependency of the cost function with respect to the optimization parameters, and the remainder parameters are as defined in (7). Matrices M, Q, and R are assumed to be definite positive. Unlike previous contributions from the same authors, there is a new term penalizing the distance from the estimate to the state projection at each sample in the time window. The problem is then posed to find the optimal estimates $\hat{x}_{k-N,k}^o, ..., \hat{x}_{k,k}^o$ minimizing the maximum of cost (10) over all the possible uncertainties, i.e., the solutions of the minimax optimization problem:

$$\min_{\hat{x}_{k-N}^k} \max_{\Delta_{k-N}^{k-1};\hat{\Delta}_{k-N}^k} J_k(\hat{x}_{k-N}^k, \Delta_{k-N}^{k-1}, \bar{\Delta}_{k-N}^k) \tag{11}$$

fulfilling (9) for $i = k-N, ..., k-1$ and $i = k-N, ..., k$, respectively.

The specific solutions to (6) and (11) are omitted due to the space limitations.

5. CONCLUDING REMARKS AND FUTURE PROSPECTS

In this document, a synthesis about the main state-estimation schemes for uncertain linear systems is provided. From the classic point of view, the reviewed strategies were classified with respect to the faced problem, i.e., Problems 1-3. A prevalence of Riccati-based approaches using the game-theory and LMI's was evidenced and commented. Although some MHE-based schemes for uncertain linear systems were reviewed, little attention about constraint handling was evidenced. Therefore, efforts should be directed to develop an estimation strategy able to handle hard constraints over the system variables.

In order to provide an estimation strategy for uncertain linear systems with constraints, a game-theory $\mathcal{H}_\infty$ filter based on the solution of an open optimization problem is foreseen. Regarding to the numerical solutions of such a moving horizon estimation scheme, it is evident that a high computational burden arises even if known uncertainties are considered. When uncertainties arise, the optimization problem turns to be a minimax optimization with constraints which is far away harder to solve than the usual quadratic programming problem. To give a solution to the aforementioned problem, suitable solutions to the posed minimax optimization should be found or developed.

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