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On The IDA-PBC Method With Dynamic Friction: Lagrangian Formulation [1]

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Abstract: In this paper, we present an extension of the interconnection and damping assignment passivity-based control (IDA-PBC) method to control a class of underactuated mechanical systems with dynamic friction using a Lagrangian framework. The IDA-PBC method was introduced some years ago but with a Hamiltonian framework. A feature of our extension in both frameworks is a new damping injection term –achieved through a nonlinear observer– added into the control law to compensate friction. Friction at the actuated coordinates is assumed to be captured by one of the simplest dynamic friction models: the Dahl model. Our main contribution is a novel stability analysis, which by invoking the Lyapunov direct method and a particular Lyapunov function, we give conditions to guarantee zero position error convergence. Simulations on a ball and beam system show the effectiveness of the proposed scheme when friction is compensated.

Keywords: Stabilization, Underactuated mechanical systems, Controlled Lagrangian, Dahl friction model.

1. INTRODUCTION

Friction is a complex physical phenomenon that occurs in many mechanical systems, e.g. servomechanisms, robots, and it is well-known that the friction can introduce undesirable behavior in experiments, such as a limit cycle, see [1] and references therein. Nowadays, friction models are able to predict an extense variety of friction phenomena, e.g. presliding displacement, frictional lag and varying break away force, which have all been observed in practice. The nature of the models is quite different but all approximate the real friction. Accuracy of approximation is mostly given by complexity of the model either static or dynamic. Dynamic friction models, such as the Dahl model [6] and LuGre model [4], have been used to compensate friction in control systems (see [11] and references therein).

On the other hand, the interconnection and damping assignment passivity-based control (IDA-PBC) method, introduced by Ortega et al. [12] is a control design methodology that assigns a desired dynamic in closed-loop. The problem of stabilization of a class of underactuated mechanical systems —those that have more degree-of-freedom than inputs of control— has been solved successfully via the IDA-PBC method being the Hamiltonian formulation the natural framework of this methodology. In the Lagrangian formalism, the related technique *Controlled Lagrangian* shares a reasoning similar to the IDA-PBC method and conditions of equivalence between techniques both are presented in [2, 5]. Further details on the *Controlled Lagrangian* method are described in the Bloch's book [3]. Moreover, a detailed analysis of the IDA-PBC method applied to the stabilization of underactuated mechanical systems in presence of natural damping, e.g. viscous friction and Coulomb friction, has been shown in Lagrangian formulation [14] as well as in Hamiltonian formulation [7]. In these works are shown how the natural damping can compromise the system's stability when this is not considered in the framework of the IDA-PBC methodology. We have carried out simulations to verify the performance of the IDA-PBC control system with dynamic friction on a ball and beam system, such that they revealed an oscillatory behavior which continued in spite of modifying the parameters of the control system. The latter is a scenario common in many underactuated mechanical systems where the presence of friction in the actuated coordinates is larger than in those underactuated coordinates. This behavior motivated us to analyze friction compensation in the framework of the IDA-PBC methodology to solve this problem, in [15], we extended the

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formulation of this methodology to compensate friction of a class of underactuated mechanical systems. The result of that extension has brought the possibility of getting better performance in a class of IDA-PBC control systems of underactuated mechanisms reported in the literature (such as those given in [13]). In this paper, our main contributions are the extension of the IDA-PBC method to control a class of underactuated mechanical systems with dynamic friction using a Lagrangian framework, and the proposal of a detailed stability analysis based on strict Lyapunov functions which originally were used for the stability study in motion control of robot manipulators (see e.g. Santibáñez and Kelly [17]). We give some conditions to ensure zero position error convergence. Simulation results on a ball and beam system show the effectiveness of the proposed scheme. Throughout this paper, we use the notation $\lambda_{\min}\{A\}$ and $\lambda_{\text{Max}}\{A\}$ to indicate the smallest and largest eigenvalues respectively of a symmetric positive-definite bounded matrix $A(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^n$. The norm of a vector $\boldsymbol{x}$ is defined as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$.

The rest of the paper is organized as follows: In Section 2, we show a new formulation of the IDA-PBC method to control underactuated mechanical systems with friction using the Lagrangian formulation. Section 3 presents the IDA-PBC control system using friction compensation to control a ball and beam system and simulation results are shown. Finally, we offer some concluding remarks in Section 4.

2. THE IDA-PBC METHOD APPLIED TO UNDERACTUATED MECHANICAL SYSTEMS WITH DYNAMIC FRICTION USING LAGRANGIAN FORMULATION

In this section, we extend the formulation of the IDA-PBC method to control a class of underactuated mechanical systems when dynamic friction in all system actuated joints is considered and using Lagrangian formulation. First, we introduce the following terms that will be used to describe the Dahl friction model given by:

$$\dot{\boldsymbol{z}} = -G\Psi(\dot{\boldsymbol{q}})G^T\Sigma_o\boldsymbol{z} + [GG^T]^T\dot{\boldsymbol{q}}, \qquad (1)$$

$$\boldsymbol{f}(\boldsymbol{z},\dot{\boldsymbol{q}}) = GG^T\Sigma_o\boldsymbol{z} + GF_vG^T\dot{\boldsymbol{q}}, \qquad (2)$$

where $\boldsymbol{z} \in \mathbb{R}^n$ is a vector of unmeasurable internal states, $G = \begin{bmatrix} 0_{k\times m} \\ I_{m\times m} \end{bmatrix} \in \mathbb{R}^{n\times m}$, such that $\text{rank}(G) = m$, $m < n$, where $k = n - m$, being n the total number of joints of mechanical systems considered in this paper and m is the number of system actuated joints. This structure of G assumes that the first k joints are passives or underactuated ones, and the remaining m joints are active or actuated ones [2].

$\Psi = \text{diag}\left\{\frac{1}{f_{c_{k+1}}}|\dot{q}_{k+1}|, \ldots, \frac{1}{f_{c_n}}|\dot{q}_n|\right\}$ is a diagonal positive semidefinite matrix, where f_{c_i} denotes the Coulomb parameter for each of the system actuated joints, with $i = (k+1), (k+2), \ldots, (k+m)$, where $k+m = n$, thus, $i = k+1, \ldots, n$. Also, $\Sigma_o = \text{diag}\{\sigma_{o_1}, \ldots, \sigma_{o_k}, \sigma_{o_{k+1}}, \ldots \sigma_{o_n}\}$ is a diagonal positive definite matrix where $\sigma_{o_j} = 0$ with $j = 1, \cdots k$, and σ_{o_i} denotes the 'stiffness' parameter for each of the system actuated joints. Moreover, $\boldsymbol{f}(\boldsymbol{z},\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ is the vector of forces due to the friction, $F_v \in \mathbb{R}^{m\times m}$ where $F_v = \text{diag}\{f_{v_{k+1}}, \ldots f_{v_n}\}$ is a diagonal positive definite matrix which contains the viscous friction coefficients of all system actuated joints.

[2] Without loss of generality, we can rewrite $G = \begin{bmatrix} I_{m\times m} \\ 0_{k\times m} \end{bmatrix} \in \mathbb{R}^{n\times m}$ to handle several underactuated mechanical systems, with changes in the sub indexes of the elements of Ψ, Σ_o and F_v as follows: 1 instead of $k+1$ and n by m, where now $i = 1, \cdots, k$ and $j = k+1, \cdots, n$.

Now, if we consider the presence of friction in all system joints, the equations of motion of a class of underactuated mechanical systems can be written as:

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r}(\boldsymbol{z},\boldsymbol{q},\dot{\boldsymbol{q}}) = G\boldsymbol{u} \qquad (3)$$

where $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are the vectors of generalized position and velocity, respectively, $M(\boldsymbol{q}) = M(\boldsymbol{q})^T > 0$ is the inertia matrix, where $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is the matrix of the centripetal and Coriolis torques, and $\boldsymbol{g}(\boldsymbol{q})$ is the vector of gravitational torques vector given by $\boldsymbol{g}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}\mathcal{U}(\boldsymbol{q})$, being $\mathcal{U}(\boldsymbol{q})$ the potential energy, the G matrix has been defined below of (2), $\boldsymbol{u} \in \mathbb{R}^m$ is the vector of torque/force control inputs and $\nabla_{\boldsymbol{q}} = \frac{\partial}{\partial \boldsymbol{q}}$. The vector of torques/forces due to the friction is $\boldsymbol{r}(\boldsymbol{z},\boldsymbol{q},\dot{\boldsymbol{q}})$, which it can be split as follows

$$\boldsymbol{r}(\boldsymbol{z},\boldsymbol{q},\dot{\boldsymbol{q}}) = GG^T\Sigma_o\boldsymbol{z} + R(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad (4)$$

where

$$R(\boldsymbol{q},\dot{\boldsymbol{q}}) = GF_vG^T + R_u(\boldsymbol{q}) \qquad (5)$$

is a symmetric and positive definite matrix and $R_u(\boldsymbol{q}) = \text{diag}\{r_1, \ldots, r_k, r_{k+1}, \ldots r_n\}$ is a matrix that contains the viscous friction coefficient of all system underactuated joints, where $r_j = f_{v_j}$ and $r_i = 0$, with $j = 1, 2, \cdots, k$ and $i = k+1, \cdots, n$. Although the state $\boldsymbol{z}$ is not measurable, we assume that all parameters of the model are known. The system (3) can be written in state space form as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}[G\boldsymbol{u} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{r}(\boldsymbol{z},\boldsymbol{q},\dot{\boldsymbol{q}})] \\ -G\Psi(\dot{\boldsymbol{q}})G^T\Sigma_o\boldsymbol{z} + [GG^T]^T\dot{\boldsymbol{q}} \end{bmatrix} \qquad (6)$$

The class of underactuated mechanical systems with dynamic friction only in the system actuated joints considered in this paper is given by (6). Following the rationale of the IDA-PBC method, the desired closed-loop system is given by

$$M_c(\boldsymbol{q})\ddot{\boldsymbol{q}} + C_c(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_c(\boldsymbol{q}) + \boldsymbol{r}_c(\tilde{\boldsymbol{z}},\boldsymbol{q},\dot{\boldsymbol{q}}) = E_c(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \qquad (7)$$

where $M_c(\boldsymbol{q}) = M_c(\boldsymbol{q})^T > 0$, the matrix $C_c(\boldsymbol{q},\dot{\boldsymbol{q}})$ is obtained through the Christoffel's symbols associated to $M_c(\boldsymbol{q})$, and $\boldsymbol{g}_c(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}V_c(\boldsymbol{q})$, with $V_c(\boldsymbol{q})$ the desired potential energy function. Taking into account, $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$, being $\hat{\boldsymbol{z}}$ the estimation of the state $\boldsymbol{z}$, and

$$\boldsymbol{r}_c(\tilde{\boldsymbol{z}},\boldsymbol{q},\dot{\boldsymbol{q}}) = M_c(\boldsymbol{q})M(\boldsymbol{q})^{-1}[GG^T\Sigma_o\tilde{\boldsymbol{z}} + R(\boldsymbol{q})\dot{\boldsymbol{q}}] \qquad (8)$$

The matrix $E_c(\boldsymbol{q},\dot{\boldsymbol{q}})$ is defined as

$$E_c(\boldsymbol{q},\dot{\boldsymbol{q}}) = S_c(\boldsymbol{q},\dot{\boldsymbol{q}}) - D_c(\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad (9)$$

where $S_c(\boldsymbol{q},\dot{\boldsymbol{q}})$ is skew-symmetric, which is used as free parameter in the solution of a set of partial differential equations (PDE's). Also, $D_c(\boldsymbol{q},\dot{\boldsymbol{q}})$ is given by

$$D_c(\boldsymbol{q},\dot{\boldsymbol{q}}) = M_c(\boldsymbol{q})M^{-1}(\boldsymbol{q})GK_v(\boldsymbol{q},\dot{\boldsymbol{q}})G^T \quad (10)$$

with $K_v(\boldsymbol{q},\dot{\boldsymbol{q}}) = K_v(\boldsymbol{q},\dot{\boldsymbol{q}})^T > 0$, which is used as free parameter for the damping injection. In state space form (7) yields [3]

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{q}\\ \dot{\boldsymbol{q}}\\ \tilde{\boldsymbol{z}}\end{bmatrix} = \begin{bmatrix}\dot{\boldsymbol{q}}\\ M_c^{-1}[S_c\dot{\boldsymbol{q}} - D_c\dot{\boldsymbol{q}} - C_c\dot{\boldsymbol{q}} - \boldsymbol{g}_c - \boldsymbol{r}_c]\\ -G\Psi G^T\Sigma_o\tilde{\boldsymbol{z}} + [GG^T]^T M^{-1}M_c\dot{\boldsymbol{q}}\end{bmatrix}. \quad (11)$$

Now, for this new class of systems (11) the main challenge of the IDA–PBC method still consists in solving the following set of PDE's, called *matching equations* [12]

$$G^{\perp}\left\{C\dot{\boldsymbol{q}} - MM_c^{-1}C_c\dot{\boldsymbol{q}} - S_c\dot{\boldsymbol{q}}\right\} = \boldsymbol{0}, \quad (12)$$

$$G^{\perp}\left\{\nabla_{\boldsymbol{q}}\mathcal{U} - MM_c^{-1}\nabla_{\boldsymbol{q}}V_c\right\} = \boldsymbol{0}, \quad (13)$$

where $G^{\perp} \in \mathbb{R}^{k\times n}$ is a full-rank left annihilator of G, so that $G^{\perp}G = 0$, whose solutions M_c and V_c define the control law given by

$$\boldsymbol{u} = \underbrace{[G^TG]^{-1}G^T\left[MM_c^{-1}[S_c\dot{\boldsymbol{q}} - C_c\dot{\boldsymbol{q}} - \boldsymbol{g}_c] + C\dot{\boldsymbol{q}} + \boldsymbol{g}\right]}_{\boldsymbol{u}_{es}} - \underbrace{K_vG^T\dot{\boldsymbol{q}}}_{\boldsymbol{u}_{di}} + \underbrace{G^T\Sigma_o\hat{\boldsymbol{z}}}_{\boldsymbol{u}_{fric}} \quad (14)$$

being $\boldsymbol{u}_{es}$ the *energy shaping* control and $\boldsymbol{u}_{di}$ the *damping injection* [12], while the new term given by $\boldsymbol{u}_{fric}$ define friction compensation. It is worth remarking that in our proposal we introduce a new damping injection term given by $\boldsymbol{u}_{di}^{\text{new}} = \boldsymbol{u}_{di} + \boldsymbol{u}_{fric}$.

2.1 Friction Observer

Since the Dahl model is dynamic and has an unmeasurable state $\boldsymbol{z}$, some kind of observer could be used. Inspired by Kelly *et al.* [9] and how we did in [15], in this paper we present a nonlinear friction observer

$$\dot{\hat{\boldsymbol{z}}} = [GG^T]^T[\dot{\boldsymbol{q}} - M^{-1}M_c\dot{\boldsymbol{q}}] - G\Psi G^T\Sigma_o\hat{\boldsymbol{z}} \quad (15)$$

which provides an estimate of the unmeasurable state $\boldsymbol{z}$ of the Dahl model described by (1) and (2). An advantage from the inclusion of the Dahl model (1)-(2) in (6) is the simplicity to design the observer. To clarify this procedure, notice that, from the definition $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$, the time derivative can be written as

$$\dot{\tilde{\boldsymbol{z}}} = \dot{\boldsymbol{z}} - \dot{\hat{\boldsymbol{z}}}. \quad (16)$$

Thus, the proposed observer (15) is achieved by substituting $\dot{\boldsymbol{z}}$ and $\dot{\tilde{\boldsymbol{z}}}$ from (6) and (11), respectively, in (16).

2.2 Stability analysis

Inspired in the work of Santibáñez and Kelly [17], we present below the closed-loop stability analysis. First, we can to verify that in (11) the point $[\boldsymbol{q}^T\ \ \dot{\boldsymbol{q}}^T\ \ \tilde{\boldsymbol{z}}^T]^T = [\boldsymbol{q}^{*T}\ \ \boldsymbol{0}^T\ \ \boldsymbol{0}^T]^T$ is an equilibrium point, where

$$\boldsymbol{q}^* = \arg\min\{V_c\}.$$

Now, in accordance with [14], we establish the following condition on the R_d matrix:

$$[M_cM^{-1}R_d + (M_cM^{-1}R_d)^T] > 0 \quad (17)$$

where

$$R_d(\boldsymbol{q},\dot{\boldsymbol{q}}) = GK_vG^T + R \quad (18)$$

is a symmetric and positive definite matrix, with R given by (5). To carry out the stability analysis, we show that a Lyapunov function can be constructed for the class Lagrangian systems (11), whose desired potential energy function in closed-loop system satisfies the following two conditions:

$$V_c(\boldsymbol{q}^*,\tilde{\boldsymbol{q}}) - V_c(\boldsymbol{q}^*,\boldsymbol{0}) \geq \beta\|\tilde{\boldsymbol{q}}\|^2 \text{ if } \|\tilde{\boldsymbol{q}}\| < \varepsilon, \quad (19)$$

$$\tilde{\boldsymbol{q}}^T\frac{\partial V_c(\boldsymbol{q}^*,\tilde{\boldsymbol{q}})}{\partial\tilde{\boldsymbol{q}}} \geq \beta'\|\tilde{\boldsymbol{q}}\|^2 \text{ if } \|\tilde{\boldsymbol{q}}\| < \varepsilon', \quad (20)$$

where $\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}^* \in \mathbb{R}^n$ denotes the position error and $\beta, \beta', \varepsilon, \varepsilon'$ are strict positive constants.

Now, it's convenient to define a function and some of her properties, which will be instrumental in our analysis.

Definition 1. The tangent hiperbolic vectorial function is defined as:

$$\boldsymbol{h}(\boldsymbol{x}) = [\tanh(x_1), \cdots, \tanh(x_n)]^T \quad (21)$$

with $\boldsymbol{x} \in \mathbb{R}^n$. This function satisfies the following properties for all $\boldsymbol{x}, \dot{\boldsymbol{x}} \in \mathbb{R}^n$:

- $\|\boldsymbol{h}(\boldsymbol{x})\| \leq \alpha_1\|\boldsymbol{x}\|$, $\|\boldsymbol{h}(\boldsymbol{x})\| \leq \alpha_2$, $\|\dot{\boldsymbol{h}}(\boldsymbol{x})\| \leq \alpha_3\|\dot{\boldsymbol{x}}\|$,

with $\alpha_1, \alpha_2, \alpha_3 > 0$. In according at (21), the α_i constants yield as $\alpha_1 = 1, \alpha_2 = \sqrt{n}, \alpha_3 = 1$.

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The following are two important properties of the Lagrangian dynamic (see e.g. Kelly *et al.* [10]).

Property 1. The matrix $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ and the time derivative of the inertia matrix satisfy

$$\dot{\boldsymbol{q}}^T\left[\frac{1}{2}\dot{M}(\boldsymbol{q}) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\right]\dot{\boldsymbol{q}} = 0\ \forall \boldsymbol{q},\dot{\boldsymbol{q}} \in \mathbb{R}^n.$$

Property 2. There exist a positive constant k_c such that for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{v} \in \mathbb{R}^n$ we have

$$\|C(\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}\| \leq k_c\|\boldsymbol{y}\|\|\boldsymbol{v}\|.$$

Our main analysis result is presented in the following

Proposition 1. Consider the class of Lagrangian systems (11), such that (17) is fulfilled, with a desired potential energy function V_c satisfying (19) and (20) for some strict positive constants $\beta, \beta', \varepsilon, \varepsilon'$. Assume that there is a positive constant k_s such that $\|S_c\dot{\boldsymbol{q}}\| \leq k_s\|\dot{\boldsymbol{q}}\|^2$, where the S_c matrix is skew symmetric [4]. Then, local zero position error convergence is assured, i.e., $\lim_{t\to\infty}(\boldsymbol{q}(t) - \boldsymbol{q}^*) = \boldsymbol{0}$ at least for a sufficiently small initial position and velocity $\boldsymbol{q}(0)$ and $\dot{\boldsymbol{q}}(0)$, and a Lyapunov function to prove this is given by

$$V_L(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^TM_c\dot{\boldsymbol{q}} + V_c(\boldsymbol{q}^*,\tilde{\boldsymbol{q}}) - V_c(\boldsymbol{q}^*,\boldsymbol{0}) + \gamma\boldsymbol{h}(\tilde{\boldsymbol{q}})^TM_c\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{z}}^T\Sigma_o\tilde{\boldsymbol{z}} \quad (22)$$

[3] To simplify notation, from now on for all expressions, which are functions of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, we will write explicitly their dependence only the first time they are defined.

[4] There are several examples that satisfies this condition. One of them is shown in [14].

where $\boldsymbol{h}(\tilde{\boldsymbol{q}})$ is the tangent hiperbolic vectorial function (21) and γ is a constant such that

$$\min\left\{\sqrt{\frac{2\beta}{\lambda_{\rm Max}\{M_c\}}}, \frac{\lambda_{\min}\{F\}}{\lambda_{\rm Max}\{M_c\}+\alpha_2[k_c+k_s]},\right.$$
$$\left.\sqrt{\frac{\beta'\lambda_{\min}\{F\}}{s(\boldsymbol{q},\dot{\boldsymbol{q}})}}\right\} > \gamma > 0 \quad (23)$$

where F is the symmetric part of $M_c M^{-1} R_d$, that is

$$F = \frac{1}{2}[M_c M^{-1} R_d + R_d M^{-1} M_c] \quad (24)$$

and $s = \beta'[\lambda_{\rm Max}\{F\} + \alpha_2[k_c + k_s]] + \frac{1}{4}[\lambda_{\rm Max}\{F\}]^2$.

Proof. The Lyapunov function candidate (22) can be rewritten as

$$V_L = \frac{1}{2}[\dot{\boldsymbol{q}} + \gamma\boldsymbol{h}]^T M_c[\dot{\boldsymbol{q}} + \gamma\boldsymbol{h}] + V_c(\boldsymbol{q}^*, \tilde{\boldsymbol{q}}) - V_c(\boldsymbol{q}^*, \boldsymbol{0})$$
$$-\frac{1}{2}\gamma^2\boldsymbol{h}^T M_c\boldsymbol{h} + \frac{1}{2}\tilde{\boldsymbol{z}}^T\Sigma_o\tilde{\boldsymbol{z}}. \quad (25)$$

Thus, it will be a positive-definite function, at least locally, provided that

$$V_c(\boldsymbol{q}^*, \tilde{\boldsymbol{q}}) - V_c(\boldsymbol{q}^*, \boldsymbol{0}) - \frac{1}{2}\gamma^2\boldsymbol{h}^T M_c\boldsymbol{h} \quad (26)$$

is also a positive-definite function in $\tilde{\boldsymbol{q}}$. Since (26) vanishes at $\tilde{\boldsymbol{q}} = \boldsymbol{0}$, only remains to show that it is positive for all $\tilde{\boldsymbol{q}} \neq \boldsymbol{0}$ into a domain $\Gamma \subseteq {\rm I\!R}^n$. To this end, we use the following important result:

$$\frac{1}{2}\gamma^2\boldsymbol{h}^T M_c\boldsymbol{h} \leq \frac{1}{2}\gamma^2\lambda_{\rm Max}\{M_c\}\|\tilde{\boldsymbol{q}}\|^2 \;\text{ if }\; \|\tilde{\boldsymbol{q}}\| < \varepsilon \quad (27)$$

which holds for any ε. Next, by taking the same ε in (27) as in (19), we obtain

$$V_c(\boldsymbol{q}^*, \tilde{\boldsymbol{q}}) - V_c(\boldsymbol{q}^*, \boldsymbol{0}) - \frac{1}{2}\gamma^2\boldsymbol{h}^T M_c\boldsymbol{h}$$
$$\geq \left[\beta - \frac{1}{2}\gamma^2\lambda_{\rm Max}\{M_c\}\right]\|\tilde{\boldsymbol{q}}\|^2 \quad \text{if } \; \|\tilde{\boldsymbol{q}}\| < \varepsilon. \quad (28)$$

Obviously, since γ satisfies (23), we have ensured that (26) is a positive-definite function in $\tilde{\boldsymbol{q}}$. In summary, the Lyapunov function candidate (22) is a positive-definite function, at least locally. It's important note that the constant γ is only required for purposes of analysis, and we do not need to know its numerical value.

The time derivative of the Lyapunov function candidate (22) along the trajectories of the closed-loop equation (11) can be written as

$$\dot{V}_L = -\dot{\boldsymbol{q}}^T F\dot{\boldsymbol{q}} + \gamma\dot{\boldsymbol{h}}^T M_c\dot{\boldsymbol{q}} + \gamma\boldsymbol{h}^T C_c^T\dot{\boldsymbol{q}} + \gamma\boldsymbol{h}^T S_c\dot{\boldsymbol{q}}$$
$$-\gamma\boldsymbol{h}^T F\dot{\boldsymbol{q}} - \gamma\boldsymbol{h}^T\boldsymbol{g}_c - \tilde{\boldsymbol{z}}^T\Sigma_o G\Psi G^T\Sigma_o\tilde{\boldsymbol{z}} \quad (29)$$

where we used the Property 1. We now provide upper bounds on each terms of (29). Taking into account that (17) is fulfilled, the positivity and symmetry of the matrix F is ensured. Therefore, an upper bound of the first term is

$$-\dot{\boldsymbol{q}}^T F\dot{\boldsymbol{q}} \leq -\lambda_{\min}\{F\}\|\dot{\boldsymbol{q}}\|^2. \quad (30)$$

The second term can be bounded as follows:

$$\gamma\dot{\boldsymbol{h}}^T M_c\dot{\boldsymbol{q}} \leq \gamma\lambda_{\rm Max}\{M_c\}\|\dot{\boldsymbol{q}}\|^2, \quad (31)$$

where we have used the last property of the Definition 1 and the fact that $\dot{\tilde{\boldsymbol{q}}} = \dot{\boldsymbol{q}}$, because $\boldsymbol{q}^*$ is constant for all $t \geq 0$. Now, in according at the Property 2 and the second property of the Definition 1, the third term is bounded by

$$\gamma\boldsymbol{h}^T C_c^T\dot{\boldsymbol{q}} \leq \gamma\alpha_2 k_c\|\dot{\boldsymbol{q}}\|^2, \quad (32)$$

while the upper bound of the forth term is

$$\gamma\boldsymbol{h}^T S_c\dot{\boldsymbol{q}} \leq \gamma k_s\alpha_2\|\dot{\boldsymbol{q}}\|^2 \quad (33)$$

where we have considered the second property of the Definition 1 and the positive constant k_s assumed in the Proposition 1. Following similar arguments as in the first term and considering the first property of the Definition 1, the bound of the fifth term result

$$-\gamma\boldsymbol{h}^T F\dot{\boldsymbol{q}} \leq \gamma\lambda_{\rm Max}\{F\}\|\tilde{\boldsymbol{q}}\|\|\dot{\boldsymbol{q}}\|. \quad (34)$$

Finally, taking into account (20) and the first property of the Definition 1, the sixth term of (29) is bounded by

$$-\gamma\boldsymbol{h}^T\boldsymbol{g}_c \leq -\gamma\beta'\|\tilde{\boldsymbol{q}}\|^2. \quad (35)$$

In virtue of that the last term of (29) is nonpositive for all $\tilde{\boldsymbol{z}}$, it now follows from the inequalities (30)-(35) that the time derivative $\dot{V}_L$ in (29) can be written as

$$\dot{V}_L \leq -\begin{bmatrix}\|\tilde{\boldsymbol{q}}\|\\ \|\dot{\boldsymbol{q}}\|\end{bmatrix}^T \underbrace{\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & Q_{22}\end{bmatrix}}_{Q(\boldsymbol{q},\dot{\boldsymbol{q}})}\begin{bmatrix}\|\tilde{\boldsymbol{q}}\|\\ \|\dot{\boldsymbol{q}}\|\end{bmatrix} \quad (36)$$

where

$$Q_{11} = \gamma\beta', Q_{12} = \frac{1}{2}\gamma\lambda_{\rm Max}\{F\},$$
$$Q_{22} = \lambda_{\min}\{F\} - \gamma[\lambda_{\rm Max}\{M_c\} + \alpha_2[k_c + k_s]].$$

For that Q be a positive-definite matrix is sufficient and neccesary that the elements Q_{11}, Q_{22} and the determinant $\det\{Q\}$ be strictly positive. To this end, the first condition is trivially fulfilled because γ and β' are strictly positive constants. Moreover, we have that Q_{22} is strictly positive in according at (23). Finally, the determinant of Q is given by

$$\det\{Q\} = \gamma\beta'[\lambda_{\min}\{F\} - \gamma[\lambda_{\rm Max}\{M_c\} + \alpha_2[k_c + k_s]]]$$
$$-\left[\frac{1}{2}\gamma\lambda_{\rm Max}\{F\}\right]^2$$

which also is strictly positive in accordance at (23). Because Q is a positive-definite matrix we have that $\dot{V}_L$ is a nonpositive function. It follows that V_L is bounded; hence $\tilde{\boldsymbol{z}}, \boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are in turn bounded vectors. Using Lyapunov's direct method we conclude stability of the equilibrium point $[\boldsymbol{q}^T \;\; \dot{\boldsymbol{q}}^T \;\; \tilde{\boldsymbol{z}}^T]^T = [\boldsymbol{q}^{*T} \;\; \boldsymbol{0}^T \;\; \boldsymbol{0}^T]^T$. To prove that the control objective is attained, i.e., $\lim_{t\to\infty}(\boldsymbol{q}(t) - \boldsymbol{q}^*) = \boldsymbol{0}$, we invoke functional analysis arguments. To this end, by integrating both sides of inequality (36), we conclude that $\tilde{\boldsymbol{q}}$ is a square integrable function, and because his derivative ($\dot{\tilde{\boldsymbol{q}}} = \dot{\boldsymbol{q}}$) is bounded, the error position $\tilde{\boldsymbol{q}}$ must tend to zero; hence $\lim_{t\to\infty}(\boldsymbol{q}(t) - \boldsymbol{q}^*) = \boldsymbol{0}$ as desired.

3. APPLICATION: THE BALL AND BEAM

The IDA-PBC control system of the ball and beam system has been reported in [12]. In that paper, the friction is absent all system joints and k_v is an arbitrary positive constant. A Lagrangian model of a ball and beam torque-driven system shown in Figure 1, where $n = 2$, $m = 1$ and $k = n - m = 1$ (joint 2 is the unique active joint), can be described by (3) with the following terms

$$M(q_1) = \begin{bmatrix} 1 & 0 \\ 0 & (L^2 + q_1^2) \end{bmatrix}, R = \begin{bmatrix} f_{v_1} & 0 \\ 0 & f_{v_2} \end{bmatrix}, G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\Psi = \frac{1}{f_{c_2}}|\dot{q}_2|, \Sigma_o = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{o_2} \end{bmatrix}, R_u = \begin{bmatrix} f_{v_1} & 0 \\ 0 & 0 \end{bmatrix}, F_v = f_{v_2}, \quad (37)$$

and the potential energy function $\mathcal{U}(q_1, q_2) = gq_1 \sin(q_2)$, where q_1 is the position of the ball, q_2 is the angle of the bar of angle, L is the length of the bar, the viscous friction coefficients of each coordinates are given by f_{v_1}, f_{v_2}, and g is the acceleration due to gravity. The control objective

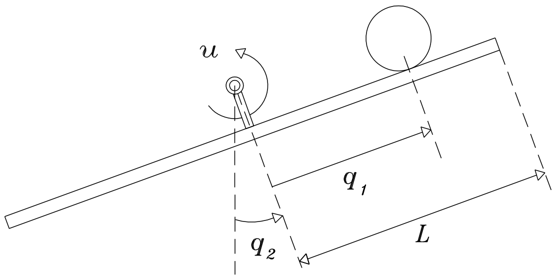

Fig. 1. Ball and beam system.

is to stabilize the ball and beam in rest position with $q_1^* = q_2^* = 0$.

3.1 A new IDA-PBC and analysis

By taking into account $G^\perp = [1 \;\; 0]$, the M matrix from (37) and the $\mathcal{U}$ potential energy function, solutions M_c, V_c and S_c of (12) and (13) are

$$M_c = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{L^2 + q_1^2}} & -1 \\ -1 & \sqrt{2[L^2 + q_1^2]} \end{bmatrix}, \quad (38)$$

$$V_c = g[1 - \cos(q_2)] + \frac{k_p}{2}\left[q_2 + \frac{1}{\sqrt{2}}\text{arcsinh}\left(\frac{q_1}{L}\right)\right]^2 \quad (39)$$

and

$$S_c = \begin{bmatrix} 0 & \frac{q_1\dot{q}_1}{L^2 + q_1^2} \\ -\frac{q_1\dot{q}_1}{L^2 + q_1^2} & 0 \end{bmatrix} \quad (40)$$

where S_c satisfies the condition $||S_c\dot{\boldsymbol{q}}|| \le k_s||\dot{\boldsymbol{q}}||^2$ given in Proposition 1, that is, $\frac{q_1\dot{q}_1}{L^2+q_1^2}\sqrt{\dot{q}_1^2 + \dot{q}_2^2} \le k_s[\dot{q}_1^2 + \dot{q}_2^2]$ with $k_s = 1$. Now, after some algebraics manipulations we verify that to accomplish the condition (17), the term k_v must be chosen as [16]:

$$k_v = \frac{2f_{v_2}^2[L^2 + q_1^2]^2 + f_{v_1}^2[L^2 + q_1^2]}{4f_{v_1}\sqrt{2[L^2 + q_1^2]}} \quad (41)$$

such that the positivity and symmetry of the matrix F given by (24) are ensured. Thus, considering (37)–(41) in (14), the new IDA-PBC control law of the ball and beam can be achieved directly.

Remark 2. It should noted that M_c and V_c have been obtained from the equivalences $M_c = MM_d(\boldsymbol{q})^{-1}M$ and $V_c = V_d$, where M_d and V_d are given in [12], being

$$M_d = [L^2 + q_1^2]\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{L^2 + q_1^2}} & 1 \\ 1 & \sqrt{2[L^2 + q_1^2]} \end{bmatrix}.$$

On the other hand, straightforward calculus lead us to find

$$\beta = \min\left\{\frac{k_p}{4L^2}, \frac{gk_p}{2k_p + 4L^2[g + k_p]}\right\}, \quad (42)$$

$$\beta' = \min\left\{\frac{k_p[1 + \sqrt{2}]}{4L^2}, \frac{k_p[2\sqrt{2} + 2][g + k_p]}{k_p[2\sqrt{2} + 2] + 8L^2[g + k_p]}\right\}, \quad (43)$$

with $\varepsilon = \varepsilon' = \pi$ in accordance at the conditions (19) and (20). Moreover, based in the properties of the Coriolis's matrix [10], the constant k_c can be obtained as

$$k_c = \max\left\{\frac{4\sqrt{6}}{9L^2}, 2\sqrt{2}\right\}.$$

Therefore, from Proposition 1, we conclude that a Lyapunov function is given by (22), with (38) and (39), where γ must satisfy (23), with β and β' given by (42) and (43) respectively.

3.2 Simulations

We have carried out simulations on a ball and beam system to verify the performance of the IDA-PBC with dynamic friction control. We use MATLAB software for simulations with ODE45 solver, which is based on an explicit Runge-Kutta formula, the Dormand-Prince pair, where we have used a relative error tolerance of 1×10^{-3}. The parameters of the unique actuator considered in the simulations correspond to the motor DM1004-C model from Parker Compumotor: $\sigma_{o_2} = 2764\,\text{Nm/rad}$, $f_{c_2} = 0.745\,\text{Nm}$, $f_{v_2} = 0.144\,\text{Nm s/rad}$. which they are obtained following the procedures shown in [8] and references therein.

We carried out two differents simulations to compare the performance of the IDA–PBC control system when friction compensation is absent and when it is present. We consider in both simulations the initial configuration: $\boldsymbol{x}(0) = [q_1(0) \;\; q_2(0) \;\; \dot{q}_1(0) \;\; \dot{q}_2(0) \;\; z_1(0) \;\; z_2(0)]^T$ and $\boldsymbol{x}(0) = [0.15 \;\; -5.7^o \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T$, with $L = 1$ [m], $f_{v_1} = 0.15$, and the gains $k_p = 10$ and k_v given in (41). We are interested in leading the ball and the beam to a zero position, as from a configuration of the ball and beam system where the efect of the friction be evident. An example of this latter occurs when the initial position error is small and the friction of the actuator is huge. The plots depicted from Figure 2 show the time evolution of positions when friction compensation is not present. The plot depicted in Figure 2 shown as besides of starting the ball close at the origin, the high torque due to the friction is such that constrains at the ball leave alone the bar. On the other hand, notice that Figure 3 shown as oscillatory behavior in the positions goes vanishing toward the desired position as desired, when we incorporate friction compensation in the control law.

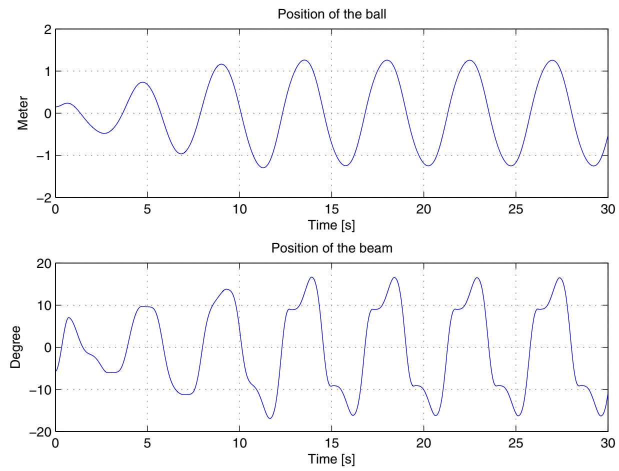

Fig. 2. Time evolution of positions without friction compensation.

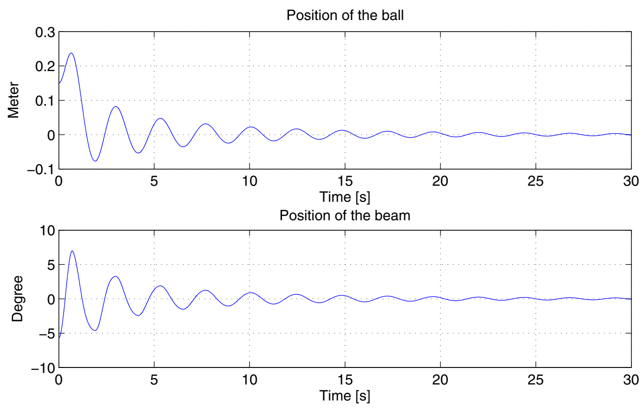

Fig. 3. Time evolution of positions with friction compensation.

4. CONCLUSIONS

In this paper, we have presented an extension of the IDA-PBC methodology to control a class of underactuated mechanical systems with friction using the Lagrangian formulation, where we have modeled friction through the Dahl model. We have carried out simulations to verify the performance of the IDA-PBC control system on a ball and beam system when friction compensation was absent and when it was present.

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