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PD + Twisting stabilization of the cart pole system in finite time ⋆

J. A. Mendoza-Mendoza * C. Aguilar-Ibáñez ** J. Davila-Montoya ***

* *CIC - IPN, Av. Juan de Dios Bátiz s/n, U.P.A.L.M, Col. San Pedro Zacatenco, A.P. 75476, México, D.F. 07738, Tel.: +(52-5) 729-6000 ext. 56568 Fax: +(52-5) 586-2936 (e-mail: jamendozam0300@alumno.ipn.mx).*
** *CIC - IPN, (e-mail: carlosaguilari@cic.ipn.mx)*
*** *ESIME Ticoman -IPN, Section of Graduate Studies and Research (e-mail: jadavila@ieee.org)*

Abstract: We present an hybrid **PD** plus Twisting-like algorithm control scheme, which stabilizes a damped cart pole in a robust way, provided that the pendulum is initially placed inside of the upper-half plane. To achieve this, the original system model is converted into a four-order chain of integrators, where the damping force is added through an additional nonlinear perturbation. Our technique consists of simultaneously bringing the velocity and position of the pendulum inside of a compact region by applying the **PD** controller. Meanwhile, the system state variables are brought to a small vicinity around to the origin by the Twisting-like algorithm in finite time. The convergence analysis is done with Lyapunov functions, also we show numerical simulations.

Keywords: Inverted cart pendulum, Sliding mode control, Lyapunov method, Super-Twisting, Finite-Time

1. INTRODUCTION

The inverted pendulum on a cart also known as cart pole system (**CPS**), is one of the classic systems extensively studied in control theory. This system was originally used as a benchmark for educational purposes, however, across the years this system has also attracted the attention as an important underactuated mechanical system, firstly due to the fact that pendulum angular acceleration cannot be directly controlled Fantoni and Lozano (2002); Sira-Ramirez and Agrawal (2004). Other reasons are because the **CPS** dynamics resembles that of many complex underactuated robot systems, it has been studied as a simplified model of underactuated robot systems (see Spong (1996); Olfati-Saber (1999); Saranlı et al. (2010); Martinez and Alvarez (2008); She et al. (2012); Almutairi and Zribi (2010)). The system is made up of a cart that moves, forward and backward, over a straight line and a pendulum hanging from it, which can move freely. The cart is moved by a horizontal force, which is the input of the system. Actually, the system is not feedback linearizable Jakubczyk and Respondek (1980); Sira-Ramirez and Agrawal (2004); also the system loses controllability when the pendulum passes through the horizontal plane Fantoni and Lozano (2002); Shiriaev et al. (2000). However, by applying the direct pole placement procedure, the system can be controlled when the pendulum is located close to the unstable equilibrium point Sira-Ramirez and Agrawal (2004); Lozano et al. (2000).

There are two important problems related to the control of the **CPS**. The first one is swinging up the pendulum from the hanging position to the upright position. In general, this problem has been tackled by using methods based on energy control and hybrid schemes Lozano et al. (2000); Spong et al. (1996); Astrom et al. (2008); Udhayakuma and Lakshmi (2007); Ordaz-Oliver et al. (2012); Gordillo and Aracil (2008); Aracil et al. (2007); Acosta et al. (2005, 2008). The second issue arises when the pendulum is located somewhere in the upper-half plane and the goal is bringing it to its unstable equilibrium point. Usually, this control challenge has been solved by applying nonlinear control tools. A full review of these tools is beyond the scope of this work; however it can be found at Bloch et al. (2001); Olfati-Saber (1999); Aguilar-Ibañez and Gutiérrez-Frias (2008b,a); Acosta et al. (2005); Udhayakuma and Lakshmi (2007); BenAbdallah and Mabrouk (2011); Gordillo and Aracil (2008).

Here we propose a control strategy to stabilize the damped **CPS** around to its unstable equilibrium point by assuming the pendulum starts to move from some position inside of the upper-half plane, and the damped coefficient is known. The damping force in the non-actuated coordinate makes controlling this system a challenge, because this force can easily destroy the system stability Woolsey et al. (2001); Ortega and Garcia-Canseco (2004); Gomez-Estern

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and Van der Schaft (2004). Additionally, an accurate estimation of the effects that this force brings to the system, is a very difficult task. Hence, several works have neglected the damping coefficient, also we must underscore that this strategy is highly robust against perturbations and peaking phenomenon such as is showed in Sepulchre (2000) .

To developed our strategy, we turn the original system into a four-order chain of integrators, with an additional nonlinear perturbation. Then, we propose to use an strategy that is a combination of a linear **PD** controller and a modified version of the Twisting algorithm Riachy et al. (2008); Santiesteban et al. (2008), such that he **PD** controller acts over the pendulum position and velocity; while the Twisting algorithm is in charge of bringing the whole system state to the origin, in finite time. The stability analysis was carried out by using Lyapunov functions. Finally, we must refer that this work was inspired in Riachy et al. (2008); Santiesteban et al. (2008). Nevertheless, our strategy guaranties finite time convergency and the absence of singularities in the control. That is, the domain of attraction is valid for any initial condition belonging to the upper-half plane; having, on the other hand, the disadvantage of being less robust in the presence of unmodeled perturbations. During the development of this study we use the following functions:

$$sgn[x] = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ if } x < 0, \\ \in [-1,1] \text{ if } x = 0. \end{cases} ; \ \frac{d}{dx}|x| = sgn[x] .$$

The following sections are organized as follows. The nonlinear model of the system is presented in Section 2. In Section 3, we develop the control strategy. The numerical simulations and the conclusions are given in Section 4 and 5, respectively.

2. PROBLEM STATEMENT

Consider the damped inverted pendulum mounted on a cart. This system can be described by the following set of normalized differential equations Sira-Ramirez and Agrawal (2004):

$$\begin{aligned} &(1+\mu)\,\ddot{x} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = f, \\ &\ddot{x}\cos\theta + \ddot{\theta} - \sin\theta = -d\dot{\theta}, \end{aligned} \tag{1}$$

where x is the normalized cart displacement, θ is the angle between the pendulum and the vertical, f is the normalized force applied to the cart, which is also the input to the system, and $\mu > 0$ is a scalar constant dependent on both the cart and pendulum masses. The pendulum viscous friction is considered as a linear function of the angular velocity, $d\dot{\theta}$, with $d \geq 0$.

In this work, the physical parameters μ and d are actually given by Sira-Ramirez and Agrawal (2004):

$$\mu = \frac{M}{m};\ d = \frac{\gamma}{mL^{1/2}g^{3/2}},$$

where M and m stand for the cart and pendulum masses, respectively, the pendulum length is L and g is the gravity constant, γ is the actual dissipation coefficient presented in the non-actuated coordinated θ. The damping force presented in the actuated coordinate is neglected in order to simplify the methodology presented here. It is important to remark that this force can be easily compensate by using any adaptive control algorithm (Moreno-Valenzuela and Kelly (2006); Garćıa-Alarćon et al. (2012)).

The control objective consists on bringing the pendulum to its unstable equilibrium point:

$$p = (\theta = 0, \dot{\theta} = 0, x = 0, \dot{x} = 0),$$

under the following important considerations:

C1) The system is initialized inside of the set:

$$U = \{(\theta, \dot{\theta}, x, \dot{x}) : (-\pi/2, \pi/2) \times R^3\}$$

C2) The state variables are available and the parameters are known.

It should be notice that **C1** is not very restrictive, because the pendulum is assumed to be somewhere inside of the upper-half plane; in fact, it can be easily accomplished by using some suitable controller like the ones proposed in Lozano et al. (2000).

Differential equations are understood in the Filippov sense Filippov (1988) in order to provide for the possibility to use discontinuous signals in controls. Filippov solutions coincide with the usual solutions, when the right-hand sides are Lipschitzian. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole semi-axis $t \geq 0$.

3. USEFUL TRANSFORMATIONS

After introduce the following feedback law:

$$f = u(\mu + \sin^2\theta) - d\cos\theta\dot{\theta} - \dot{\theta}^2\sin\theta + \cos\theta\sin\theta, \tag{2}$$

the system can be rewritten as:

$$\begin{aligned} \ddot{\theta} &= \sin\theta - \cos\theta u - d\dot{\theta}, \\ \ddot{x} &= u, \end{aligned} \tag{3}$$

Now, in order to represent the system (3) as a four-order chain of integrators plus an additional nonlinear perturbation, we define the following new change of coordinates:

$$\begin{aligned} &z_1 = x + 2\tanh^{-1}(\tan\frac{\theta}{2})\ ;\ z_3 = \tan\theta; \\ &z_2 = \dot{x} + \dot{\theta}\sec\theta \qquad\quad ;\ z_4 = \dot{\theta}\sec^2\theta. \end{aligned} \tag{4}$$

Then, the system (3) can be rewritten as:

$$\begin{aligned} \dot{z}_1 &= z_2; \\ \dot{z}_2 &= z_3 + \alpha(z_3)\, z_4^2 - dz_4\beta(z_3)\,; \\ \dot{z}_3 &= z_4; \\ \dot{z}_4 &= \upsilon; \end{aligned} \tag{5}$$

where

$$\alpha(z_3) = \frac{z_3}{(1+z_3^2)^{\frac{3}{2}}}\ ;\ \beta(z_3) = \frac{1}{\sqrt{1+z_3^2}},$$

and υ is the new control variable, defined as:

$$\upsilon = \sec^2\theta\left(\sin\theta - \cos\theta u - d\dot{\theta}\right) + 2\dot{\theta}^2\sec^2\theta\tan\theta \tag{6}$$

It is important to remark that:

$$|\alpha(x)| \leq \kappa_0 = \frac{2}{3^{3/2}}\ ;\ |\beta(x)| \leq 1. \tag{7}$$

Now, to dominate the undesirable term $d\beta(z_3)z_4$, found in the second equation of (5), we use the following change of coordinates and the following scale of time:

$$q_1 = \epsilon^2 z_1;\ q_2 = \epsilon z_2;\ q_3 = z_3;\ q_4 = z_4/\epsilon;\ \tau = \epsilon t, \tag{8}$$

where ϵ is a strictly positive free parameter. Hence, the system (5) can be written in the new coordinates as:

$$\begin{aligned}\dot{q}_1 &= q_2;\\ \dot{q}_2 &= q_3 + \epsilon q_4 \rho(q);\\ \dot{q}_3 &= q_4\\ \dot{q}_4 &= \frac{v}{\epsilon^2} = v_\epsilon,\end{aligned} \tag{9}$$

where $\rho(q)$ is a vanishing perturbation defined by:

$$\rho(q) = \epsilon\alpha(q_3)\, q_4 - d\beta(q_3) \tag{10}$$

Here the symbol "dot" stands for differentiation with respect to the dimensionless time τ. We must underscore that the free parameter $\epsilon > 0$ can be tuned as desired.

4. CONTROL DESIGN

The control law is proposed as:

$$v_\epsilon(q) = v_s(q) + v_e(q),$$

where $v_e(q)$ is a linear controller devoted to bring the states q_3 and q_4 close enough to the origin, and $v_s(q)$ is a bounded controller designed using the twisting sliding mode algorithm.

The linear control part of the controller v_ϵ, is selected as:

$$v_e = -\frac{k_1 q_3 + k_2 q_4}{k_3}; \tag{11}$$

Let us introduce the following auxiliary variables:

$$\begin{aligned}s_1 &= k_1 q_1 + (k_1+k_2) q_2 + (k_3+k_2)\, q_3 + k_3 q_4;\\ s_2 &= k_1 q_2 + k_2 q_3 + k_3 q_4;\end{aligned} \tag{12}$$

where the set of constants $k_i > 0$ should be selected such that they satisfy the following, this choice is explained along the text :

$$\begin{aligned}k_2 &> 1/2 + \lambda_1 \epsilon^2 (k_2 + k_1)\, \kappa_0 + \delta_1;\\ \lambda_1^2 &> \lambda_2^2 + 2\lambda_1 \epsilon d\, (k_2+k_1)\, \overline{q}_4 + \delta_2;\\ \lambda_2 &> k_1 \left(\kappa_0 \epsilon^2 \overline{q}_4^2 + \epsilon d \overline{q}_4\right) + \delta_3;\end{aligned} \tag{13}$$

where $\delta_i > 0$, $i = \{1,2,3\}$ [1]. The inequities in (13) are referred in the sequel as the assumption **A1**.

Let us propose v_s, as a discontinuous injection based on the twisting control algorithm Riachy et al. (2008); Santiesteban et al. (2008). That is:

$$v_s = -\frac{1}{k_3}\left(\lambda_1 sgn[s_1] + \lambda_2 sgn[s_2]\right) = -\frac{\mathfrak{v}_s}{k_3}; \tag{14}$$

where $\lambda_1 > \lambda_2 > 0$.

Let us synthesize the main result of this work in the following theorem.

Consider the system (9) in closed-loop with:

$$v = -\frac{1}{k_3}\left(k_1 q_3 + k_2 q_4 + \lambda_1 sig[s_1] + \lambda_1 sig[s_2]\right),$$

where

$$\begin{aligned}s_1 &= k_1 q_1 + (k_1+k_2) q_2 + (k_3+k_2)\, q_3 + k_3 q_4;\\ s_2 &= k_1 q_2 + k_2 q_3 + k_3 q_4.\end{aligned}$$

Under the assumption that the control parameters $k_i > 0$; with $i = \{1,2,3\}$ and $\lambda_1 > \lambda_2 > 0$, satisfy the inequalities in (13), then the closed-loop system is asymptotically stable. In particular the variables s_1 and s_2 converge to zero in finite time. This will proof by following the application of the linear control, and after some simple algebra, the dynamics of s_1 and s_2 becomes:

$$\begin{aligned}\dot{s}_1 &= s_2 + (k_1+k_2)\epsilon q_4 \rho(q) + k_3 v_s;\\ \dot{s}_2 &= k_1 \epsilon q_4 \rho(q) + k_3 v_s.\end{aligned} \tag{15}$$

Then, the system composed by (15) and the last two equations of (5) reads as:

$$\begin{aligned}\dot{s}_1 &= s_2 + \epsilon q_4 \rho(q)(k_1+k_2) + k_3 v_s,\\ \dot{s}_2 &= \epsilon q_4 \rho(q) k_1 + k_3 v_s\\ \dot{q}_3 &= q_4,\\ \dot{q}_4 &= -\frac{1}{k_3}\left(k_1 q_3 + k_2 q_4\right) + v_s;\end{aligned} \tag{16}$$

System (16) has now the following configuration:

$$\begin{aligned}\dot{s}_1 &= s_2 + \Delta_1(q) + k v_s,\\ \dot{s}_2 &= k v_s + \Delta_2(q),\\ \dot{q}_3 &= q_4,\\ \dot{q}_4 &= v_e + v_s;\end{aligned} \tag{17}$$

where v_s is taken from (14), v_e is a linear control shown in (11) relying on q_3 and q_4; $\Delta_1(q)$ and $\Delta_2(q)$ are vanishing nonlinear perturbations, which can be made uniformly bounded after a small period of time and $k > 0$.

We must underscore that the above closed-loop configuration is inspired in the previous work of Sepulchre (2000).

To be able to carry out the convergence analysis we analyze the boundedness of the states q_3 and q_4, when the system (16) is feedback by the twisting controller, to assure the boundedness of the vanishing nonlinear perturbation $\rho(q)$. That is, the system (16), in closed-loop with (14), reads as

$$\begin{aligned}\dot{s}_1 &= s_2 + \epsilon q_4 \rho(q)(k_1+k_2) - \mathfrak{v}_s,\\ \dot{s}_2 &= \epsilon q_4 \rho(q) k_1 - \mathfrak{v}_s\\ \dot{q}_3 &= q_4,\\ \dot{q}_4 &= -\frac{1}{k_3}\left(k_1 q_3 + k_2 q_4\right) - \frac{1}{k_3}\mathfrak{v}_s\end{aligned}$$

So, we introduce the following auxiliary lemma

Lemma1: *Consider the following second order system:*

$$\dot{x} = y \;;\; \dot{y} = -k_p x - k_d y + v;$$

where the set of constants $k_i > 0$; for $i = \{p, d\}$, with $|v| \le \overline{v}$. Then, there exits a finite time $t_0 > 0$, such that:

$$|x| \le \frac{\overline{v} + \delta_0}{k_p};\; |y| \le \frac{\overline{v} + \delta_0}{k_d};\; \forall t \ge t_0,$$

where $\delta_0 > 0$, small enough. The proof of this lemma is omitted due to its obviousness Khalil (1996).

According to this lemma, the last two equations of (16) satisfy the following inequality:

$$|q_3| \le \frac{\lambda_1 + \lambda_2 + \delta_0}{k_1};\; |q_4| \le \overline{q}_4 = \frac{\lambda_1 + \lambda_2 + \delta_0}{k_2};\; \forall t \ge t_0, \tag{18}$$

where t_0 is a finite period of time and δ_0 is a small positive constant. That is, q_3 and q_4 are bounded after $t \ge t_0$. This fact allows to assure that proposed closed-loop system is Lipschitz, implying that the states s_1 and s_2 remain bounded during a finite time. Hence, the finite time of scape does not exist see Khalil (1996). On the other hand, from the relations (7) and (10), the following inequality:

$$|\rho(q)| \le \kappa_0 \epsilon \left(\frac{\lambda_1 + \lambda_2}{k_2}\right) + d; \forall t \ge t_0, \tag{19}$$

is fulfilled. Having shown that q_3 and q_4 are uniformly bounded after some finite time, we are in conditions to

[1] with $\overline{q}_4 = \frac{\lambda_1+\lambda_2}{k_2}$.and δ_i can be fixed as needed inside of a suitable range.

finally perform the convergence analysis of the whole system, based on the traditional Lyapunov method.

Discontinuous Lyapunov functions have been introduced since late nineties to prove the stability of discontinuous systems and systems with solutions intended in Filippov's sense, see for example, Bacciotti and Ceragioli (1999), Bacciotti and Rosier (2001) and Bacciotti and Ceragioli (2006). Let us propose the following discontinuous Lyapunov function:

$$V_T(p) = \frac{k_1}{2k_3}q_3^2 + \frac{1}{2}q_4^2 + \frac{\lambda_1}{k_3}|s_1| + \frac{1}{2k_3}s_2^2; \tag{20}$$

whose time derivative around the trajectories of the system (16) is almost everywhere given by:

$$\dot{V}_T(p) = -\frac{k_2}{k_3}q_4^2 + \frac{1}{k_3}q_4\mathfrak{v}_s + \overbrace{\frac{\lambda_1}{k_3}sgn[s_1]\dot{s}_1 + \frac{1}{k_3}s_2\dot{s}_2}^{W(p)}. \tag{21}$$

Notice that the derivative of the Lyapunov function (20) exists for all s_1 except the set of measure zero given by $s_1 = 0$. Notice that $W(p)$ can be expressed after using (15), as follows:

$$\begin{aligned} W(p) = &\frac{\lambda_1}{k_3}(k_2 + k_1)\, sgn[s_1]\rho(q)\epsilon + \frac{k_1}{k_3}s_2\rho(q)\epsilon \\ &-\frac{\lambda_1^2}{k_3} - \frac{\lambda_1\lambda_2 sgn[s_1]sgn[s_2]}{k_3} - \frac{\lambda_2}{k_3}|s_2|. \end{aligned}$$

By using the inequality $|q_4\mathfrak{v}_s| \leq (q_4^2 + \mathfrak{v}_s^2)/2$, we have that (21) can be upperbounded, as :

$$\dot{V}_T(p) \leq -\left(\frac{k_2}{k_3} - \frac{1}{2k_3}\right)q_4^2 - \frac{\mathfrak{v}_s^2}{2k_3} + W(p). \tag{22}$$

It is easy to check after some simple algebra that

$$\begin{aligned} -\frac{\mathfrak{v}_s^2}{2k_3} + W(p) \leq &-\frac{\lambda_1^2}{2k_3} - \frac{\lambda_2}{k_3}|s_2| \\ &+\frac{1}{2k_3}\lambda_2^2 + \frac{k_1}{k_3}|s_2|\left(\kappa_0\epsilon^2\bar{q}_4^2 + \epsilon d\bar{q}_4\right) \\ &+\frac{\lambda_1}{k_3}(k_2 + k_1)\left(\kappa_0\epsilon^2 q_4^2 + \epsilon d\bar{q}_4\right) \end{aligned} \tag{23}$$

After substituting (23) into the relation (22), we obtain the following inequality:

$$\begin{aligned} \dot{V}_T(q) \leq &-\left(\frac{k_2}{k_3} - \frac{1}{2k_3} - \frac{\lambda_1}{k_3}\epsilon^2(k_2 + k_1)\kappa_0\right)q_4^2 \\ &-\frac{1}{2k_3}\lambda_1^2 + \frac{1}{2k_3}\lambda_2^2 + \frac{\lambda_1 d}{k_3}\epsilon(k_2 + k_1)\bar{q}_4\,. \\ &-|s_2|\left(\frac{\lambda_2}{k_3} - \frac{k_1}{k_3}\left(\kappa_0\epsilon^2\bar{q}_4^2 + \epsilon d\bar{q}_4\right)\right) \end{aligned}$$

Then, according to the conditions in assumption **A1**, we have that after a finite time $t \geq t_0$, the following inequality is fulfilled:

$$\dot{V}_T(p) \leq -\frac{\delta_1}{k_3}q_4^2 - \frac{\delta_2}{2k_3} - \frac{\delta_3}{k_3}|s_2|.$$

Since V_T is strictly positive definite with its corresponding time derivative (21) satisfying $\dot{V}_T \leq -\delta_2/2k_3 < 0$,[2] then we conclude that $p = (s_1, s_2, q_3, q_4) \to 0$ in a finite period of time. ■

Remark 1: *The function V_T is continuous but not locally Lipschitz. Therefore, the usual version of the traditional Lyapunov theorem can not be applied Moreno (2012). However, it can be shown that function $V_T(p)$ is absolutely continuous along the trajectories of the closed-loop equation (16); implying that $V_T(p)$ is differentiable almost everywhere, monotone decreasing and converges to zero. These are the conditions needed by the theorem of Zubov Poznyak (2008).*

It is important to note that the proposed controller has a very simple structure and does not presents singularities, if the system is initialized inside of the upper half plane.

Tuning parameters: The correct performance of the control strategy requires a control parameters tuning according to the restriction (22). To illustrate this tuning, we fix the pendulum length, mass and damping as $L = 0.35[m]$, $m = 0.250[Kg]$ and $\gamma = 4[kgm^2/s]$. Then, according to the expression given in the comment **C1**, the normalized damping coefficient is $d = 0.9$. Now, fixing the control gains as $k_1 = 0.9$, $k_2 = 3.5$, $k_3 = 4$, $\lambda_1 = 6$ and $\lambda_2 = 0.8$ and setting the re-scale parameter as $0 < \epsilon < 0.404$, is easy to see in a plot that the inequalities in (13) hold.

Summarizing: Given $d > 0$ and $\delta_i \approx 0.1$, we need to find an admissible parameter vector $Q = (k_1, k_2, k_3, \lambda_1, \lambda_2, \epsilon) \in R_+^6$, fulfilling the restrictions given in (22). This problem can be solved using any numerical optimization program.

5. NUMERICAL SIMULATIONS

To verify the proposed control performance, we carried out some numerical simulations where the above proposed control gains were used; using $\epsilon = 0.4$. To make this experiment more interesting, we assume that the knowledge of the damping force has an accuracy of 85%. We ran two experiments with their own different initial conditions. The obtained closed-loop responses for $p_1 = (\theta(0) = 1.2[\text{rad}]$, $\dot{\theta}(0) = -0.1[\text{rad/s}]$, $x = 0, \dot{x} = 0)$and $p_2 = (\theta(0) = -0.7[\text{rad}]$, $\dot{\theta}(0) = 0$, $x = 0.2[m/s]$,$\dot{x} = 0)$ are shown in Figure 1. As we can see, the control strategy is able to render the system to the origin after $7[s]$ elapsed, even when the value of the damping coefficient d is partially known.

To provide an idea of how good is our control strategy **OC**, we compared it with the control technique proposed by Riachy *et al* in Riachy et al. (2008), here referred as **RC**. The control parameters of **RC** were tuning heuristically, but to be fair we tried to find the values that enable the best transient response. The simulation results are shown in Figure 2, when the system is initialized as $p = (\theta = 0.9, 0, 0, 0)$. The results so obtained are shown in Figure 2, where we can see that the closed-loop response of our control strategy is as good as the responses achieved by **RC**. Also, it can be seen that our strategy yields a better behavior in the angular variable, when compared with the **RF** strategy. However, the cart displacements in our strategy are larger than the ones in **RC**. Please keep in mind that this is a numeric comparison, doing it formally is beyond the scope of this work, as is a comparative study between our control strategy and others found in the literature. Finally, we stress that all the simulations were carried out using the actual coordinates of the pendulum system.

[2] We must remember that δ_i are strictly positive, as stated in the assumption **A1.**

6. CONCLUSIONS

We proposed a control scheme, based on a **PD** controller in combination with a Twisting-like algorithm, to solve the stabilization of the inverted pendulum on a cart, assuming that the pendulum is initially placed inside of the upper-half plane. The strategy was developed by means of a four-order chain of integrators transformation with an additional nonlinear perturbation associated to the damping force. The **PD** controller was designed to bring simultaneously the pendulum position and its velocity inside of a compact region. At the same time, the Twisting-like algorithm brings the whole system state to the origin. In order to carry out the convergence analysis, we used several Lyapunov functions, assuring that our strategy converges in finite time. The convergence time was not estimated because it would be needed a stronger Lyapunov function, which in our opinion cannot be constructed for such a fourth order nonlinear system. Finally, to test the effectiveness of this technique, we ran some numerical simulations.

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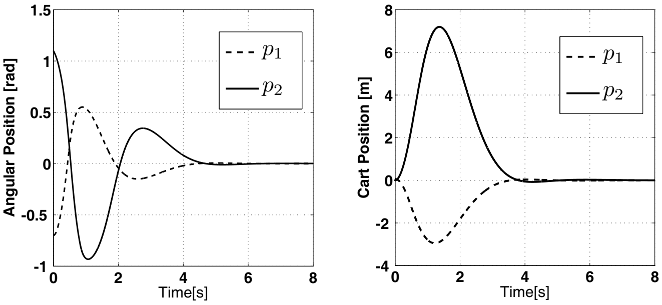

Fig. 1. Closed-loop responses for two initial conditions (p_1, p_2), and a partial knowledge of the 85% of the damping force.

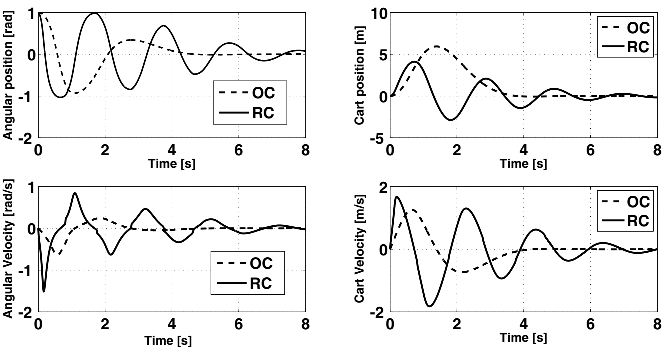

Fig. 2. Performance comparison between the closed-loop responses of the **OC** and the **RC** strategies, just for showing similar results.