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Dynamical observers design for discrete-time descriptor linear systems

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Abstract: This paper addresses the problem of observer design for discrete-time descriptor systems by using a new concept of dynamical observer. The advantage of this new concept is the structure of this dynamical observer, which is more general than the Proportional-Integral observer and the Proportional observers. The sufficient and necessary conditions for the stability of the proposed dynamical observer are given in a set of linear matrix inequalities (LMIs). A numerical example is given to show the applicability of the present approach.

Keywords: Dynamical observer, Disrete-time descriptor systems, LMI.

1. INTRODUCTION

This paper concerns the dynamical observers design for discrete-time descriptor systems. An observer is a dynamical system which uses the available information on the inputs and outputs to reconstruct the unmeasured states of the system. A new structure of the observers, known as dynamical observers, was developed by Goodwin and Middleton (1989) and by Marquez (2003). This structure presents an alternative state estimation which can be considered as more general than Proportional Observers (PO) and Proportional-Integral Observers (PIO). Which can be only considered as particular cases of this structure. The idea of including additional dynamics in the observer was presented by Goodwin and Middleton (1989).
Observers design for discrete-time descriptor systems were presented by Wu et al. (2009) by using the PIO approach. In Darouach et al. (2010) and Wang et al. (2012) a PO design for discrete-time nonlinear descriptor systems was treated by using a linear matrix inequality (LMI) approach. In Lin (2012) a PO design for discrete-time rectangular descriptor systems with time varying delay was presented. In Liying and Zhaolin (2004) the authors present a PO design for discrete-time non linear descriptor systems with unknown input, where the simultaneous state and unknown input estimation is possible by extending the state vector with the unknown input vector.
Descriptor systems also known as singular or differential-algebraic systems are used to describe several real systems having complex interrelations between the state variables. These systems were introduced by Luenberger (1977) from a control theory point of view and since, great efforts have been made to investigate singular systems theory and its applications (see Müller and Hou (1993); Müller (2005); Liu et al. (2008); Boulkroune et al. (2009); Darouach (2009); Zhou and Lu (2009); Darouach (2012); Araujo et al. (2012)).
The main contribution of this paper is that this new observer structure is more general than those presented in Darouach et al. (2010) and Wu et al. (2009). This observer is composed by both a dynamical part and a statical part. A numerical example is given to illustrate our approach.

2. PRELIMINARIES

In this section we shall present some basic results which are used in the sequel of this paper. We shall use the following notations: The symbol A^T denotes the transpose of the matrix A. A^+ denotes any generalized inverse of the matrix A, i.e. it verifies $AA^+A = A$. I_n denotes an identity matrix with dimensions $n \times n$, I denotes an identity matrix with appropriate dimensions, 0 denotes a scalar or matrix zero with appropriate dimensions. The symbol $(*)$ denotes the transpose elements in the symmetric positions, $\mathsf{ones}_{n,m}$ denotes a matrix with dimensions $n \times m$ with all elements equal to one. The symbol $E^{\perp}$ denotes a maximal row rank matrix such that $E^{\perp}E = 0$.
The Schur complement Lemma is needed to deduce an LMI feasible problem

Lemma 1. (Schur complement) Given a symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, the following conditions are equivalent:

(1) $S < 0$;
(2) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
(3) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

In Section 5, we use the following theorem to solve LMIs

Theorem 1. (Skelton et al., 1998) Let matrices $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D} = \mathcal{D}^{\mathcal{T}}$ be given, then the following statements are equivalent:

(1) There exists a matrix $\mathcal{X}$ satisfying
$$\mathcal{BXC} + (\mathcal{BXC})^T + \mathcal{D} < 0.$$
(2) The following two conditions hold
$$\mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \text{ or } \mathcal{B}\mathcal{B}^T > 0$$
$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \text{ or } \mathcal{C}^T\mathcal{C} > 0.$$

Suppose that the statement 2 holds. Let r_b and r_c be the ranks of $\mathcal{B}$ and $\mathcal{C}$, respectively, and $(\mathcal{B}_l, \mathcal{B}_r)$ and $(\mathcal{C}_l, \mathcal{C}_r)$ be any full rank factors of $\mathcal{B}$ and $\mathcal{C}$, i.e. $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$, $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$. Then matrix $\mathcal{X}$ in statement 1 is given by
$$\mathcal{X} = \mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+$$
where $\mathcal{Z}$ is an arbitrary matrix and
$$\begin{aligned}\mathcal{K} =& -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2}\\ \mathcal{S} =& \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T\left[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta\right]\mathcal{B}_l\mathcal{R}^{-1}\end{aligned}$$
where $\mathcal{L}$ is an arbitrary matrix such that $\|\mathcal{L}\| < 1$ and $\mathcal{R}$ is an arbitrary positive definite matrix such that
$$\vartheta = (\mathcal{B}_r\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{D})^{-1} > 0.$$

3. PROBLEM FORMULATION

Consider the linear discrete-time descriptor system of the form
$$\begin{aligned}Ex(k+1) =& Ax(k) + Bu(k)\\ y(k) =& Cx(k)\end{aligned} \tag{1}$$
where $x(k) \in \mathbb{R}^n$ is the semi state vector, $u(k) \in \mathbb{R}^m$ is the input, and $y(k) \in \mathbb{R}^p$ represents the measured output vector. Matrices $E \in \mathbb{R}^{n_1\times n}$, $A \in \mathbb{R}^{n_1\times n}$, $B \in \mathbb{R}^{n_1\times m}$, and $C \in \mathbb{R}^{p\times n}$. Let $\mathsf{rank}(E) = r \leq n$, i.e. matrix E may be singular, and let $E^{\perp} \in \mathbb{R}^{r_1\times n_1}$ be a full row rank matrix such that $E^{\perp}E = 0$, in this case $r_1 = n_1 - r$.

Assumption 1.
$$\mathsf{rank}\begin{bmatrix}E\\ E^{\perp}A\\ C\end{bmatrix} = n$$

Remark 1. Assumption 1 is equivalent to the impulse observability (Dai, 1988, 1989). i.e.
$$\mathsf{rank}\begin{bmatrix}E & A\\ 0 & C\\ 0 & E\end{bmatrix} = \mathsf{rank}(E) + n.$$

Now, let us consider the following dynamical observer for system (1)
$$\zeta(k+1) = N\zeta(k) + Hv(k) + F\begin{bmatrix}-E^{\perp}Bu(k)\\ y(k)\end{bmatrix} + Ju(k) \tag{2}$$
$$v(k+1) = S\zeta(k) + Lv(k) + M\begin{bmatrix}-E^{\perp}Bu(k)\\ y(k)\end{bmatrix} \tag{3}$$
$$\hat{x}(k) = P\zeta(k) + Q\begin{bmatrix}-E^{\perp}Bu(k)\\ y(k)\end{bmatrix} \tag{4}$$
where $\zeta(k) \in \mathbb{R}^q$ represents the state vector of the observer, $v(k) \in \mathbb{R}^v$ is an auxiliary vector, and $\hat{x}(k) \in \mathbb{R}^n$ is the estimate of $x(k)$.

Remark 2.

- The observer (2)-(4) is in a general form and generalizes the existing ones, In fact:
  - ✻ For $H = 0$, $S = 0$, $M = 0$, and L a stability matrix, let matrices F and Q be partitioned according to the partition of $\begin{bmatrix}-E^{\perp}Bu(k)\\ y(k)\end{bmatrix}$ as $F = [0\ F_a]$, and $Q = [0\ Q_a]$ respectively, then the following observer is obtained
  $$\begin{aligned}\dot{\zeta}(k) =& N\zeta(k) + F_a y(k) + Ju(k)\\ \hat{x}(k) =& P\zeta(k) + Q_a y(k)\end{aligned}$$
  which is the form used for the PO for descriptor systems (Darouach et al., 2010).
  - ✻ For $L = 0$ and let $S = -C$, and $M = -CQ + [0\ I]$, then the following observer is obtained
  $$\begin{aligned}\dot{\zeta}(t) =& N\zeta(t) + Hv(t) + F\begin{bmatrix}-E^{\perp}Bu(k)\\ y(t)\end{bmatrix} + Ju(k)\\ \dot{v}(k) =& y(k) - C\hat{x}(k)\\ \hat{x}(k) =& P\zeta(k) + Q\begin{bmatrix}-E^{\perp}Bu(k)\\ y(t)\end{bmatrix}\end{aligned}$$
  which is in the form used for the unknown input PIO for descriptor systems.
- The order of the observer is $q \leq n$, when $q = n - p$, the reduced order observer is obtained and for $q = n$ the full order one.

Now, we can give the following lemma

Lemma 2. There exists an observer of the form (2)-(4) for the system (1) if and only if the following two statements hold

I. There exists a matrix T of appropriate dimensions such that the following conditions are satisfied
   (a) $NTE + F\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} - TA = 0$
   (b) $J = TB$
   (c) $M\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} + STE = 0$
   (d) $[P\ Q]\begin{bmatrix}TE\\ E^{\perp}A\\ C\end{bmatrix} = I_n$

II. The matrix $\begin{bmatrix}N & H\\ S & L\end{bmatrix}$ is a stability matrix.

Proof. Let $T \in \mathbb{R}^{q\times n_1}$ be a parameter matrix and define $\varepsilon(k) = \zeta(k) - TEx(k)$, then its dynamic is given by
$$\begin{aligned}\varepsilon(k+1) =& N\varepsilon(k) + \left(NTE - TA + F\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix}\right)x(k)\\ &+ (J - TB)u(k) + Hv(k)\end{aligned} \tag{5}$$
by using the definition of $\varepsilon(k)$, equations (3) and (4) can be written as
$$v(k+1) = S\varepsilon(k) + \left(M\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} + STE\right)x(k) + Lv(k) \tag{6}$$
$$\hat{x}(k) = P\varepsilon(k) + [P\ Q]\begin{bmatrix}TE\\ E^{\perp}A\\ C\end{bmatrix}x(k) \tag{7}$$

Now, if conditions (a)-(d) of Lemma 2 are satisfied the following observer error dynamics is obtained from (5) and (6)

$$\underbrace{\begin{bmatrix}\varepsilon(k+1)\\ v(k+1)\end{bmatrix}}_{\varphi(k+1)} = \underbrace{\begin{bmatrix}N & H\\ S & L\end{bmatrix}}_{\mathbb{A}}\underbrace{\begin{bmatrix}\varepsilon(k)\\ v(k)\end{bmatrix}}_{\varphi(k)} \tag{8}$$

and from (7)

$$\begin{aligned}\hat{x}(k) - x(k) =& P\varepsilon(k)\\ e(k) =& P\varepsilon(k)\end{aligned} \tag{9}$$

in this case if $\mathbb{A}$ is a stability matrix, then $\lim_{k\to\infty} e(k) = 0$.

The problem of the dynamical observer is reduced to determine the matrices N, F, J, H, L, M, S, P, Q, and T such that Lemma 2 is satisfied.

4. PARAMETERIZATION OF THE DYNAMICAL OBSERVER

In this section we shall present a parameterization of the dynamical observer matrices by solving the constrained Sylvester equations (a)-(d) of Lemma 2.

Define the following matrices: $\Omega = \begin{bmatrix}E\\ E^{\perp}A\\ C\end{bmatrix}$, $P_1 = \Sigma^+\begin{bmatrix}I_q\\ 0\end{bmatrix}$, $N_1 = T_1A\Sigma^+\begin{bmatrix}I_q\\ 0\end{bmatrix}$, $N_2 = T_2A\Sigma^+\begin{bmatrix}I_q\\ 0\end{bmatrix}$, $T_1 = R\Omega^+\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$, $N_3 = (I_{n_1+r_1+p} - \Sigma\Sigma^+)\begin{bmatrix}I_q\\ 0\end{bmatrix}$, $T_2 = (I_{n_1+r_1+p} - \Omega\Omega^+)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$, $K_1 = R\Omega^+\begin{bmatrix}0\\ I_{r_1+p}\end{bmatrix}$, $K_2 = (I_{n_1+r_1+p} - \Omega\Omega^+)\begin{bmatrix}0\\ I_{r_1+p}\end{bmatrix}$, $\tilde{K}_1 = T_1A\Sigma^+\begin{bmatrix}0\\ I_{r_1+p}\end{bmatrix}$, $\tilde{K}_2 = T_2A\Sigma^+\begin{bmatrix}0\\ I_{r_1+p}\end{bmatrix}$, $\tilde{K}_3 = (I_{q+r_1+p} - \Sigma\Sigma^+)\begin{bmatrix}0\\ I_{r_1+p}\end{bmatrix}$, $F_1 = T_1A\Sigma^+\begin{bmatrix}K\\ I_{r_1+p}\end{bmatrix}$, $F_2 = T_2A\Sigma^+\begin{bmatrix}K\\ I_{r_1+p}\end{bmatrix}$, $F_3 = (I_{q+r_1+p} - \Sigma\Sigma^+)\begin{bmatrix}K\\ I_{r_1+p}\end{bmatrix}$, $Q_1 = \Sigma^+\begin{bmatrix}K\\ I_{r_1+p}\end{bmatrix}$ and let $R \in \mathbb{R}^{q\times n}$ be a full rank matrix and define the matrix $\Sigma = \begin{bmatrix}R\\ E^{\perp}A\\ C\end{bmatrix}$ such that $\mathsf{rank}(\Sigma) = n$.

The following lemma gives the general form of the matrices T, S, M, P, Q, N and F.

Lemma 3. Under assumption 1, the general form of matrices T, S, M, P, Q, N and F are

$$T = T_1 - Z_1T_2 \tag{10}$$
$$S = -Y_1N_3 \tag{11}$$
$$M = -Y_1F_3 \tag{12}$$
$$P = P_1 - Y_2N_3 \tag{13}$$
$$Q = Q_1 - Y_2F_3 \tag{14}$$
$$N = N_1 - Z_1N_2 - Y_3N_3 \tag{15}$$
$$F = F_1 - Z_1F_2 - Y_3F_3 \tag{16}$$

where Z_1, Y_1, Y_2, and Y_3 are arbitrary matrices of appropriate dimensions.

Proof. Conditions (c) and (d) can be rewritten as

$$\begin{bmatrix}S & M\\ P & Q\end{bmatrix}\begin{bmatrix}TE\\ E^{\perp}A\\ C\end{bmatrix} = \begin{bmatrix}0\\ I_n\end{bmatrix} \tag{17}$$

the necessary and sufficient condition for (17) to have a solution is

$$\mathsf{rank}\begin{bmatrix}TE\\ E^{\perp}A\\ C\end{bmatrix} = \mathsf{rank}\begin{bmatrix}TE\\ E^{\perp}A\\ C\\ 0\\ I_n\end{bmatrix} = n \tag{18}$$

From the definition of matrices Σ and Ω, and by considering $\mathsf{rank}\begin{bmatrix}\Omega\\ \Sigma\end{bmatrix} = \mathsf{rank}(\Omega)$, condition (18) is equivalent to the existence of two matrices $T \in \mathbb{R}^{q\times n}$, and $K \in \mathbb{R}^{q\times(r_1+p)}$ such that

$$TE + K\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} = R \tag{19}$$

where R is an arbitrary matrix such that $\mathsf{rank}(\Sigma) = n$. Equation (19) can also be rewritten as

$$[T\ \ K]\,\Omega = R \tag{20}$$

since $\mathsf{rank}\begin{bmatrix}\Omega\\ R\end{bmatrix} = \mathsf{rank}(\Omega)$, the general solution to (20) is given by

$$T = T_1 - Z_1T_2 \tag{21}$$
$$K = K_1 - Z_1K_2 \tag{22}$$

where Z_1 is an arbitrary matrix of appropriate dimensions. By considering (19) we can get

$$\begin{bmatrix}TE\\ E^{\perp}A\\ C\end{bmatrix} = \begin{bmatrix}I_q & -K\\ 0 & I_p\end{bmatrix}\Sigma \tag{23}$$

replacing (23) in (17) we obtain

$$\begin{bmatrix}S & M\\ P & Q\end{bmatrix}\begin{bmatrix}I_q & -K\\ 0 & I_p\end{bmatrix}\Sigma = \begin{bmatrix}0\\ I_n\end{bmatrix} \tag{24}$$

Since Σ is of full column rank, the general solution of (24) is given by

$$S = -Y_1N_3 \tag{25}$$
$$M = -Y_1F_3 \tag{26}$$
$$P = P_1 - Y_2N_3 \tag{27}$$
$$Q = Q_1 - Y_2F_3 \tag{28}$$

where Y_1 and Y_2 are arbitrary matrices of appropriate dimensions.

Replacing TE from (19) in condition (a) we obtain

$$\begin{aligned}N\left(R - K\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix}\right) + F\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} =& TA\\ NR + \tilde{K}\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} =& TA\end{aligned} \tag{29}$$

where $\tilde{K} = F - NK$, equation (29) can be expressed also as

$$\left[N\ \ \tilde{K}\right]\Sigma = TA \tag{30}$$

the general solution to (30) is given by

$$N = N_1 - Z_1N_2 - Y_3N_3 \tag{31}$$
$$\tilde{K} = \tilde{K}_1 - Z_1\tilde{K}_2 - Y_3\tilde{K}_3 \tag{32}$$

where Z_1 and Y_3 are arbitrary matrices of appropriate dimensions.

As N, T, K, and $\tilde{K}$ are known matrices we can deduce the form of F as follows

$$F = F_1 - Z_1F_2 - Y_3F_3 \tag{33}$$

5. DYNAMICAL OBSERVER DESIGN

In this section a synthesis method for a dynamical observer given by (2)-(4) is presented. This method is obtained such that the matrix $\mathbb{A}$ given in (8) is a stability matrix.
By using (11) and (15) the observer error dynamics (8) can be rewritten as

$$\varphi(k+1) = (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(k) \quad (34)$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z_1N_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_3 & H \\ Y_1 & L \end{bmatrix}$, and $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_v \end{bmatrix}$.
The following theorem gives the LMI conditions that allow the determination of all dynamical observer matrices and which guarantee the stability of matrix $\mathbb{A}$.

Theorem 2. Under Assumption 1 there exist two parameter matrices $\mathbb{Y}$ and Z_1 such that the system (34) is asymptotically stable if there exist some symmetric positive definite matrices X_1 and X_2 and a matrix W_1 such that the following LMIs are satisfied

$$X_2 - X_1 > 0 \quad (35)$$

$$\begin{bmatrix} -N_3^{T\perp}X_1N_3^{T\perp T} & (*) & (*) \\ X_1N_1N_3^{T\perp T} - W_1N_2N_3^{T\perp T} & -X_1 & (*) \\ X_1N_1N_3^{T\perp T} - W_1N_2N_3^{T\perp T} & -X_1 & -X_2 \end{bmatrix} < 0 \quad (36)$$

In this case $Z_1 = X_1^{-1}W_1$ and matrix $\mathbb{Y}$ is parametrized as follows

$$\mathbb{Y} = X^{-1}(\mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+) \quad (37)$$

where

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}_l^T\vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1/2} \quad (38)$$

$$\vartheta = (\mathcal{B}_r\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{D})^{-1} > 0 \quad (39)$$

$$\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T\left[\vartheta - \vartheta\mathcal{C}_r^T(\mathcal{C}_r\vartheta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\vartheta\right]\mathcal{B}_l\mathcal{R}^{-1} \quad (40)$$

with $\mathcal{D} = \begin{bmatrix} -X_1 & (*) & (*) & (*) \\ -X_1 & -X_2 & 0 & 0 \\ X_1N_1 - W_1N_2 & 0 & -X_1 & (*) \\ X_1N_1 - W_1N_2 & 0 & -X_1 & -X_2 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ I_{q+v} \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} \begin{bmatrix} N_3 & 0 \\ 0 & -I_v \end{bmatrix} & 0 \end{bmatrix}$, and matrices $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\| < 1$. Matrices $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{B}_l$, and $\mathcal{B}_r$ are full rank matrices such that $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ and $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$, respectively.

Proof. Consider the following Lyapunov function

$$V(\varphi(k)) = \varphi(k)^TX\varphi(k) \quad (41)$$

where $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$, then by using the complement Schur lemma, since $X_1 = X_1^T > 0$, we obtain $X_2 - X_1 > 0$. Now, the time difference of $V(\varphi(k))$ along the solution of (2)-(4) is

$$\Delta V(\varphi(k)) = \varphi(k)^T(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\varphi(k) - \varphi(k)^TX\varphi(k) \quad (42)$$

The inequality $\Delta V(\varphi(k)) < 0$ holds for all $\varphi(k) \neq 0$ if and only if

$$(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) - X < 0. \quad (43)$$

By using Lemma 1 for (43) gives $X > 0$, and

$$\begin{bmatrix} -X & (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^TX \\ X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) & -X \end{bmatrix} < 0. \quad (44)$$

which can be rewritten as

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{D} < 0 \quad (45)$$

where $\mathcal{X} = X\mathbb{Y}$, $\mathcal{B} = \begin{bmatrix} 0 \\ -I_{q+v} \end{bmatrix}$, $\mathcal{C} = [\mathbb{A}_2 \; 0]$, and $\mathcal{D} = \begin{bmatrix} -X & \mathbb{A}_1^TX \\ X\mathbb{A}_1 & -X \end{bmatrix}$ which are the same as the ones defined in Theorem 2.
According to Theorem 1, the inequality (45) is equivalent to

$$\mathcal{B}^\perp\mathcal{D}\mathcal{B}^{\perp T} < 0 \quad (46)$$

$$\mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \quad (47)$$

with $\mathcal{B}^\perp = [I_{q+v} \; 0]$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} [N_3^{T\perp} \; 0] & 0 \\ 0 & I_{q+v} \end{bmatrix}$. By using the definition of matrices $\mathcal{B}^\perp$, and $\mathcal{D}$ the inequality (46) is equivalent to (35), and by using matrices $\mathcal{C}^{T\perp}$, $\mathcal{D}$ and W_1 (47) is equivalent to (36).

The following algorithm allows the determination of all the observer matrices, to summarize the above results.

Algorithm.

step 1. Choose the observer order q and a matrix $R \in \mathbb{R}^{q\times n}$ such that $\mathsf{rank}(\Sigma) = n$.
step 2. Compute matrices N_1, N_2, N_3, T_1, T_2, K_1, K_2, $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{K}_3$ and P_1 defined in Section 4.
step 3. Solve the LMI (35) and (36) to find X and Z_1.
step 4. Find $\mathcal{R} > 0$ such that (39) be positive definite.
step 5. Find $\mathcal{L}$ and $\mathcal{Z}$ such that $\|\mathcal{L}\| < 1$ to solve equations (38) and (40), then obtain matrix $\mathbb{Y}$ as in (37).
step 6. Compute the matrices of the dynamical observer (2)-(4): N, H, F, J, S, L, M, P, and Q, by using (15) to compute N, (37) to compute H and L, (11)-(14) to compute S, M, P, and Q by taking $Y_2 = 0$, F is defined in (16) and J is defined in Lemma 2

6. NUMERICAL EXAMPLE

In order to illustrate our results, let us consider the following descriptor system described by (1) where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0.7 & -0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0.3 & 0 \\ -0.1 & 0.2 & 0 & 0.5 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$

and $C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
By following the algorithm of Section 5 we have
step 1. An observer with order $q = 2$ was chosen with a matrix $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, such that $\mathsf{rank}(\Sigma) = 4$.
step 2. The following matrices were calculated

$$N_1 = \begin{bmatrix} 0.683 & -0.372 \\ 0.008 & 0.091 \end{bmatrix}, \; T_1 = \begin{bmatrix} 0.992 & 0.010 & -0.005 & 0 \\ 0.010 & 0.653 & -0.326 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.005 & -0.004 \\ -0.001 & 0.394 \\ 0.010 & 0.385 \\ -0.077 & 0.154 \\ 0.053 & -0.044 \\ -0.010 & -0.385 \\ -0.027 & 0.022 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0.008 & -0.010 & 0.005 & 0 \\ -0.010 & 0.347 & 0.326 & 0 \\ 0.005 & 0.326 & 0.337 & 0 \\ 0 & 0 & 0 & 1 \\ 0.078 & -0.103 & 0.052 & 0 \\ -0.005 & -0.326 & -0.337 & 0 \\ -0.039 & 0.052 & -0.026 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.008 & -0.015 \\ -0.015 & 0.031 \\ 0.077 & -0.154 \\ 0 & 0 \\ -0.038 & 0.077 \end{bmatrix}, \; P_1 = \begin{bmatrix} 0.992 & 0.015 \\ 0.015 & 0.969 \\ -0.015 & -0.969 \\ 0.038 & -0.077 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -0.078 & 0.005 & 0.039 \\ 0.104 & 0.326 & -0.052 \end{bmatrix}, \; \tilde{K}_1 = \begin{bmatrix} -0.114 & -0.002 & 0.057 \\ 0.014 & -0.098 & -0.007 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.078 & -0.005 & -0.039 \\ -0.104 & -0.326 & 0.052 \\ 0.052 & -0.337 & -0.026 \\ 0 & 0 & 0 \\ 0.778 & -0.052 & -0.389 \\ -0.052 & 0.337 & 0.026 \\ -0.389 & 0.026 & 0 \end{bmatrix}, \; \tilde{K}_2 = \begin{bmatrix} -0.001 & 0.002 & 0.001 \\ 0.063 & 0.098 & -0.031 \\ 0.061 & 0.101 & -0.030 \\ 0.231 & 0 & 0.385 \\ -0.011 & 0.015 & 0.006 \\ -0.061 & -0.101 & 0.030 \\ 0.006 & -0.008 & -0.003 \end{bmatrix},$$

and $\tilde{K}_3 = \begin{bmatrix} 0.077 & 0 & -0.038 \\ -0.154 & 0 & 0.077 \\ 0.769 & 0 & -0.385 \\ 0 & 0 & 0 \\ -0.385 & 0 & 0.192 \end{bmatrix}$.

step 3. By using the LMI toolbox of MATLAB, we solved the inequalities (35) and (36) to get

$$Z_1 = \begin{bmatrix} -2.291 & 252.246 & -120.289 & 0 & 39.862 & 126.545 & -8.987 \\ -0.957 & 54.264 & -16.011 & 0 & 4.946 & 36.485 & -0.163 \end{bmatrix},$$ and

$$X = \begin{bmatrix} 0.400 & -0.105 & 0.400 & -0.105 \\ -0.105 & 0.488 & -0.105 & 0.488 \\ 0.400 & -0.105 & 0.976 & -0.125 \\ -0.105 & 0.488 & 0.061 & 1.267 \end{bmatrix}.$$

step 4. By considering $\mathcal{R} = I_4 \times 0.01$ to solve (39) as

$$\vartheta = \begin{bmatrix} 4.398 & 0.547 & -1.742 & -0.018 & 0.010 & 0.001 & 0.010 & 0.001 \\ 0.102 & 3.685 & 0.439 & -1.369 & -0.036 & -0.005 & -0.036 & -0.005 \\ -1.739 & -0.004 & 1.734 & 0.029 & -0.006 & -0.001 & -0.006 & -0.001 \\ 0.408 & -1.375 & -0.397 & 1.314 & 0.015 & 0.002 & 0.015 & 0.002 \\ 0.014 & -0.035 & -0.010 & 0.014 & 0.010 & 0 & 0. & 0 \\ 0.002 & -0.004 & -0.001 & 0.002 & 0 & 0.010 & 0 & 0 \\ 0.014 & -0.035 & -0.010 & 0.013 & 0 & 0 & 0.010 & 0 \\ 0.002 & -0.004 & -0.001 & 0.002 & 0 & 0 & 0 & 0.010 \end{bmatrix}.$$

step 5. By considering $\mathcal{L} = \mathsf{ones}_{4,3} \times 0.1$, and

$$\mathcal{Z} = \begin{bmatrix} 8 & 2 & 9 & 7 & 3 & 9 & 1 \\ 9 & 3 & 7 & 5 & 9 & 3 & 8 \\ 9 & 2 & 8 & 5 & 8 & 3 & 0 \\ 9 & 3 & 8 & 5 & 0 & 1 & 8 \end{bmatrix}$$ to solve (38) and (40), we obtain

$$\mathbb{Y} = \begin{bmatrix} 23.746 & 8.681 & 21.107 & 24.827 & 9.048 & -0.460 & -0.249 \\ 24.706 & 7.135 & 21.233 & 14.099 & 33.269 & -0.231 & -0.160 \\ 2.163 & -0.882 & 2.732 & -3.444 & 6.007 & -0.065 & -0.059 \\ -0.992 & 1.250 & -4.609 & 0.734 & -10.187 & -0.051 & -0.047 \end{bmatrix}.$$

step 6. Finally, we compute the matrices of the dynamical observer by considering $Y_2 = 0$, then we obtain the following dynamical observer

$$\zeta(k+1) = \begin{bmatrix} -0.357 & -0.086 \\ -0.127 & 0.016 \end{bmatrix} \zeta(k) + \begin{bmatrix} -0.460 & -0.249 \\ -0.231 & -0.160 \end{bmatrix} v(k) + \begin{bmatrix} -15.787 & -1.039 & 7.893 \\ -5.004 & -0.365 & 2.502 \end{bmatrix} \begin{bmatrix} -E^{\perp}Bu(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} -2.412 \\ -0.538 \end{bmatrix} u(k)$$

$$v(k+1) = \begin{bmatrix} -0.009 & 0.019 \\ -0.010 & 0.021 \end{bmatrix} \zeta(k) + \begin{bmatrix} -0.065 & -0.059 \\ -0.051 & -0.047 \end{bmatrix} v(k) + \begin{bmatrix} -0.050 & -0.002 & 0.025 \\ -0.055 & -0.003 & 0.028 \end{bmatrix} \begin{bmatrix} -E^{\perp}Bu(k) \\ y(k) \end{bmatrix}$$

$$\hat{x}(k+1) = \begin{bmatrix} 0.992 & 0.015 \\ 0.015 & 0.969 \\ -0.015 & -0.969 \\ 0.038 & -0.077 \end{bmatrix} \zeta(k) + \begin{bmatrix} 4.515 & 1.498 & -2.257 \\ 4.696 & 0.619 & -2.348 \\ -4.696 & 0.381 & 2.348 \\ 0.205 & 0.010 & 0.897 \end{bmatrix} \begin{bmatrix} -E^{\perp}Bu(k) \\ y(k) \end{bmatrix}$$

In order to evaluate the performance of our observer, a measurement noise $w(k)$ was considered in the output, then the disturbed outputs become $y_1(k) = x_2(k) + x_3(k) + w(k)$, and $y_2(k) = x_4(k) + w(k)$.
The results of the simulation are depicted in Fig. 1-6. Fig. 1 shows the input $u(k)$ behavior. Fig. 2 shows the measurement noise behavior. Figs. 3-6 show the system states and their estimations. It can be seen that the observer have an acceptable performance despite the measurement noise.

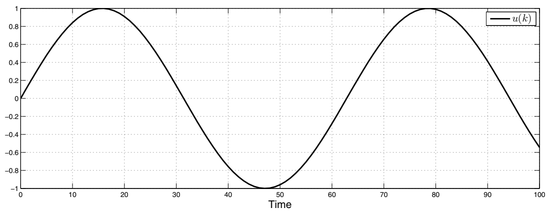

Fig. 1. Input $u(k)$ behavior.

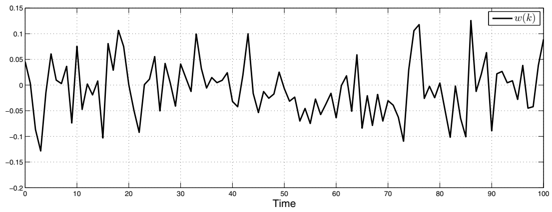

Fig. 2. Measurement noise $w(k)$ behavior.

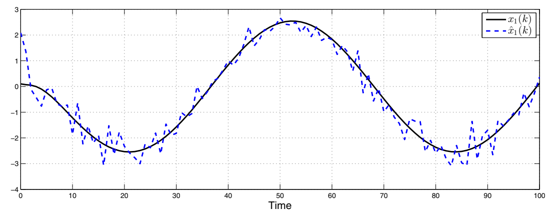

Fig. 3. Estimation $x_1(k)$ (black line: original state; blue dashed line: Dynamical observer)

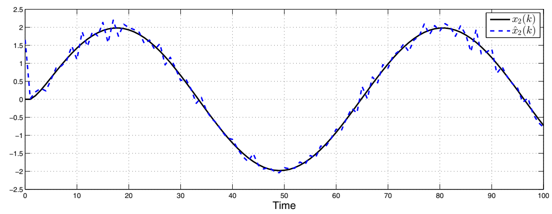

Fig. 4. Estimation $x_2(k)$ (black line: original state; blue dashed line: Dynamical observer)

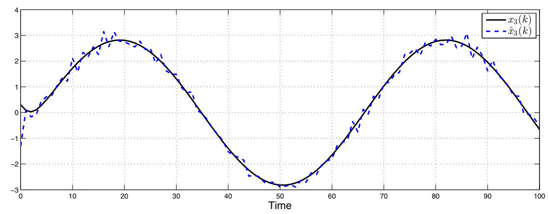

Fig. 5. Estimation $x_3(k)$ (black line: original state; blue dashed line: Dynamical observer)

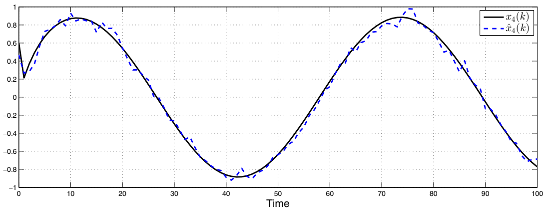

Fig. 6. Estimation $x_4(k)$ (black line: original state; blue dashed line: Dynamical observer)

7. CONCLUSION

In this paper, a dynamical observers design for disrete-time descriptor linear systems have been presented. Since regularity assumption is not required, the design procedure can be applied to square systems, but also to rectangular systems, which is a more general representation of linear systems, and square systems can be viewed as a particular case. The stability of the observer was proved by the Lyapunov method and it is guaranteed by solving a set of LMIs. An algorithm is described in order to facilitate the observer matrices computation. A numerical example was presented to show the performances of our approach.

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