Memorias del XVI Congreso Latinoamericano
de Control Automático, CLCA 2014
Octubre 14-17, 2014. Cancún, Quintana Roo, México

LPV control methodology applied to LPV systems based on robust stabilizing controllers and mixed sensitivity

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Abstract: A Linear Parametric time-Varying (LPV) controller is designed for an LPV plant based on the parametrization of all stabilizing controllers. The state space realization of the LPV plant is a polytopic function of a time varying parameter. This parameter is measured in real time and lies between given bounds. Robust stable controllers are synthesized for each vertex of the convex hull of the plant based on the parametrization of all stabilizing controllers, and their free parameters are fixed solving a mixed sensitivity criterion. Then, an LPV controller is designed by interpolation of the robust controllers. Finally, convexity of the closed loop system is assured allowing Quadratic Stability (QS) analysis. In order to analyze QS, Horisberger's Theorem is applied solving a set of Linear Matrix Inequalities (LMI's). The results are illustrated by a simulation example of a two-degrees-of-freedom planar rotational robot.

Keywords: Linear Parametric Varying (LPV), Quadratic Stability, Linear Matrix Inequality (LMI), Stabilizing controllers, Mixed sensitivity.

1. INTRODUCTION

In real world applications the control must preserve stability and performance even in the presence of uncertainty conditions and disturbances, such as unmodelled dynamics, non-linearities, parametric uncertainties, load variations, vibrations, and aging. These heavily operating conditions deteriorate the performance of the control law. Robust control techniques deal with these situations, in particular LPV control preserves stability and performance under fast time-variations of the plant parameters. These systems can represent non-linear systems approximated by polynomials or non-linear systems linearized along a time varying trajectory, so, the LPV control has been applied for instance to coupled tanks in Abdullah and Zribi (2009), flexible robots in Apkarian and Adams (1998), and DC motors in Galindo et al. (2012). Also, the LPV plant is closer to the real non-linear plant and have more information than a Linear Time-Invariant (LTI) plant, holding many mathematical properties due to their parametric affinity. These LPV systems have parameters that belong to given known intervals. This work is focus on a state space realization that is a polytopic function of these time varying parameters. In this work it is assumed that this parameter is measured in real time and lies between given bounds. Modeling a plant as an LPV model allows designing a controller for each vertex of the convex hull of the plant, and the LPV control is gotten by interpolation. In this work, robust stable and stabilizing controllers are designed at each vertex of the convex hull of the plant, adding their robust properties to the overall LPV control. Also, convexity is assured in closed loop when the LPV control is applied to the LPV plant using the plant parameters measurement. For this LPV system, QS is analyzed solving Linear Matrix Inequalities (LMI's) at each vertex of the convex hull of the plant, these LMI's are based on Horisberger's Theorem (see Horisberger and Belanger (1976), Apkarian and Gahinet (1995), and Amato (2006)).

Also, it is considered that the LTI systems at each vertex of the convex hull of the plant, that is, $P_1(s)$, ..., $P_{2^q}(s)$ where q is the number of time varying parameters, are subject to admissible disturbances and uncertainties. Hence, robust controllers stabilizing $P_1(s), \ldots, P_{2^q}(s)$, are designed based on the parametrization of all stabilizing controllers as proposed by Youla et al. (1976), Kucera (1979), Desoer et al. (1980) and Vidyasagar (1985), and the performance problems can be solved by their free control parameters (see Vidyasagar (1985)). Moreover, computational efficient formulas for the stabilizing controllers and for their free parameters are given by Bonilla and Galindo (2011) and Galindo and Conejo (2012). Also, a stable controller is desired to minimize numerical errors or in case of sensor failure or loop break. A stable controller exists between the family of all stabilizing controllers if the plant satisfies the parity interlacing property (see Vidyasagar (1985)), that is, the number of real unstable poles between any pair of real unstable McMillan zeros is even. In Bonilla and Galindo (2011) and Galindo and

⋆ Under project PAICyT IT956-11.

Conejo (2012) the free parameters solve a mixed sensitivity criterion, that minimizes the $\mathcal{H}_\infty$-norm of the output sensitivity function at low frequencies, improving the regulation and the attenuation of output additive disturbances, and minimizes the $\mathcal{H}_\infty$-norm of the transfer function from the output to the input of the uncertainty at high frequencies, preserving stability under uncertainties. The method is based on which usually the disturbances are of low frequencies and on which the mathematical models are more exact and accurate in low frequencies, neglecting generally the high frequency dynamics. The results are illustrated by a simulation example of a two-degrees-of-freedom planar rotational robot.

Notation. $\theta_{(i)}$ denotes the $i-$vertex obtained from the 2^q combinations of the time-invariant bounds $\underline{\theta}_j$ and $\bar{\theta}_j$, $j = 1,\ldots, q$, of the hyperbox of a continuous function $\theta(t) \in \Re^q$, that remains between given time-invariant bounds, *i.e.*, the $j-$entry of $\theta(t)$ satisfies $\theta_j(t) \in [\underline{\theta}_j, \bar{\theta}_j]$ for $j = 1,\ldots, q$; $\Re(s)$ denotes the set of all rational functions of the complex variable s with real coefficients; $\Re\mathcal{H}_\infty$ the set of proper stable rational functions; $\Re$ the set of real numbers; diag$\{a_1,\ldots, a_p\}$ is a $p\times p$ diagonal matrix whose elements are $a_1,\ldots, a_p$; and I_p is a $p \times p$ identity matrix.

2. LPV CONTROL

Let a linear time varying state space description of a given system, gotten by linearizing a non-linear system along a time varying trajectory; then, an LPV state space description of this system, $P(\theta(t))$, obtained by defining parameters $\theta_j(t)$, or in a less conservative way, by applying the polytopic covering technique (see Amato (2006)), is,

$$\left.\begin{array}{l}\dot{x}(t) = F(\theta(t))\,x(t) + G(\theta(t))\,u(t)\\ y(t) = H(\theta(t))\,x(t),\ t \in [0,\ +\infty)\end{array}\right\} \quad (1)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$ are the state, the input and regulated output of the system, $\theta(t) \in \Re^q$ is a measured or estimated function of the plant parameters and the matrices $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$ are polytopic functions of the parameter $\theta(t)$, that is,

$$F(\theta(t)) = \sum_{i=1}^{2^q} \alpha_i(\theta(t))\,F_i,\ \alpha_i(\theta(t)) > 0,\ \sum_{i=1}^{2^q} \alpha_i(\theta(t)) = 1 \quad (2)$$

and analogously for $G(\theta(t))$ and $H(\theta(t))$, being $F_1, \ldots, F_{2^q}$ time-invariant matrices. Hence, $F(\theta(t))$, $G(\theta(t))$ and $H(\theta(t))$ varies in the convex hull of the plant, *i.e.*, the convex envelope of a set of LTI models that become by evaluating $F(\theta(t))$, $G(\theta(t))$ and $H(\theta(t))$ in the $\theta_{(i)}$.

Suppose that at each vertex of the convex hull of the plant, *i.e.*, $P_1(s), \ldots, P_{2^q}(s)$ both the $\mathcal{H}_2$ norms of the disturbances and the $\mathcal{H}_\infty$ -norm of the transfer function from the output to the input of the uncertainty are bounded, and at these vertices design robust stabilizing controllers $K_1(s), \ldots, K_{2^q}(s)$. Hence, an LPV controller,

$$K(\theta(t)) = \sum_{i=1}^{2^q} \alpha_i(\theta(t))\,K_i,\ \alpha_i(\theta(t)) > 0,\ \sum_{i=1}^{2^q} \alpha_i(\theta(t)) = 1 \quad (3)$$

is designed by interpolation of the robust controllers, applying the interpolation algorithm proposed by Pellanda et al. (2001), where $K_1, \ldots, K_{2^q}$ are the input-output relations of $K_1(s), \ldots, K_{2^q}(s)$ and, $\alpha_1(\theta(t)) = \frac{1}{\Gamma}\prod_{j=1}^{q}(\bar{\theta}_j - \theta_j(t))$,
$\alpha_2(\theta(t)) = \frac{1}{\Gamma}\prod_{j=2}^{q}(\bar{\theta}_j - \theta_j(t))(\theta_1 - \underline{\theta}_1), \ldots, \alpha_{2^q}(\theta(t)) = \frac{1}{\Gamma}\prod_{k=1}^{q}(\theta_k(t) - \underline{\theta}_k)$, being $\Gamma := \prod_{j=1}^{q}(\bar{\theta}_j - \underline{\theta}_j)$.

The LPV controller given by (3) is composed by the parallel connection of the state space realizations of the stabilizing controllers $K_1(s), \ldots, K_{2^q}(s)$ that have at their input or output $\alpha_1, \ldots, \alpha_{2^q}$. So the whole connection admits two standard state space realizations, if $\alpha_1, \ldots, \alpha_{2^q}$ are at the output of $K_1(s), \ldots, K_{2^q}(s)$, then,

$$\begin{array}{l}A_k = \text{diag}\{A_{k1},\ldots, A_{k2^q}\},\\ B_k = \left[\, B_{k1}^T \ \cdots \ B_{k2^q}^T \,\right]^T,\\ C_k(\theta(t)) = [\,\alpha_1(\theta(t))\,C_{k1} \ \cdots \ \alpha_{2^q}(\theta(t))\,C_{k2^q}\,],\\ D_k(\theta(t)) = \displaystyle\sum_{i=1}^{2^q} \alpha_i(\theta(t))\,D_{ki}\end{array} \quad (4)$$

while if $\alpha_1, \ldots, \alpha_{2^q}$ are at the input of $K_1(s), \ldots, K_{2^q}(s)$, then, $B_k(\theta(t)) = \left[\,\alpha_1(\theta(t))\,B_{k1}^T \ \cdots \ \alpha_{2^q}(\theta(t))\,B_{k2^q}^T\,\right]^T$, $C_k = [\,C_{k1} \ \cdots \ C_{k2^q}\,]$ and A_k and D_k are given by (4).

An alternative is to interpolate the state space matrices of $K_i(s)$, $i = 1,\ldots, 2^q$, that is,

$$A_k(\theta(t)) = \sum_{i=1}^{2^q} \alpha_i(\theta(t))\,A_{ki} \quad (5)$$

and analogously for $B_k(\theta(t))$, $C_k(\theta(t))$ and $D_k(\theta(t))$, where $(A_k(\theta(t)), B_k(\theta(t)), C_k(\theta(t)), D_k(\theta(t)))$ and $(A_{ki}, B_{ki}, C_{ki}, D_{ki})$ are the state space realizations of $K(\theta(t))$ and $K_i(s)$, $i = 1,\ldots, 2^q$, respectively.

The control strategy must be such that the state matrix of the closed loop system has a multi-affine dependency on $\theta(t)$, so, assuring convexity. If convexity is assured it allows to apply the Theorem of Horisberger and Belanger (1976) (see also Amato (2006)), in order to analyze Quadratic Stability (QS) of $K(\theta(t))$ applied to $P(\theta(t))$ in a feedback configuration. It is well known that even if the system is stable at each vertex of the convex hull of the plant, instability can be induced by the time variations of the plant parameters. Stability is assured at each vertex of the convex hull of the plant by the designed robust stabilizing controllers, and the stability of the overall system is analyzed applying Horisberger's Theorem. Also, it is assumed that these robust properties are extended to the overall LPV system.

Let $K(\theta(t)) \in \Re^{m\times p}$ and $K_r(\theta(t)) \in \Re^{m\times p}$ be LPV state space descriptions of controllers designed for $P(\theta(t))$,

$$\left.\begin{array}{l}\dot{x}_k(t) = A_k(\theta(t))\,x_k(t) + B_k(\theta(t))\,u_2(t)\\ v_1(t) = C_k(\theta(t))\,x_k(t) + D_k(\theta(t))\,u_2(t)\end{array}\right\} \text{ and} \quad (6)$$

$$\left.\begin{array}{l}\dot{x}_{kr}(t) = A_{kr}(\theta(t))\,x_{kr}(t) + B_{kr}(\theta(t))\,y_d(t)\\ v_2(t) = C_{kr}(\theta(t))\,x_{kr}(t) + D_{kr}(\theta(t))\,y_d(t)\end{array}\right\} \quad (7)$$

respectively, in the one-parameter and two-parameter feedback configurations of Figures 1 and 2, respectively, where $y_d(t)$ is the reference input, $d_o(t)$ is an external disturbance at the output of the plant, and $v_1 = u$ in the configuration of Fig. 1. The role of $K(\theta(t))$ in Fig. 1 is to guarantee internal stability and to improve the performance, while in Fig. 2 the role of $K(\theta(t))$ is to guarantee internal stability while the one of $K_r(\theta(t))$ is

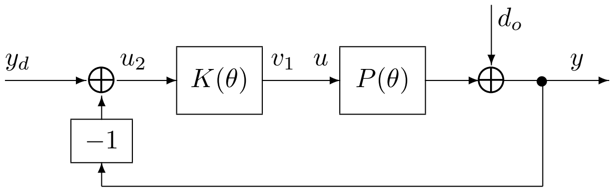

Fig. 1. Feedback system with one-parameter controller.

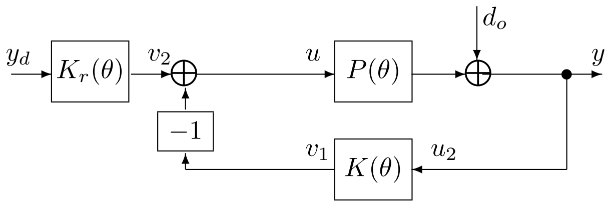

Fig. 2. Feedback system with two-parameter controller.

to improve the performance. In the feedback configuration of Fig. 1, the closed loop system is,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_k(t) \end{bmatrix} = A_{CL1}(\theta(t)) \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} + B_{CL1}(\theta(t))\, y_d(t)$$
$$y(t) = [\, H(\theta(t)) \;\; 0 \,] \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} \tag{8}$$

where,

$$A_{CL1}(\theta(t)) := \begin{bmatrix} \hat{F}(\theta(t)) & G(\theta(t))\, C_k(\theta(t)) \\ -B_k(\theta(t))\, H(\theta(t)) & A_k(\theta(t)) \end{bmatrix},$$
$$B_{CL1}(\theta(t)) := \begin{bmatrix} G(\theta(t))\, D_k(\theta(t)) \\ B_k(\theta(t)) \end{bmatrix}, \tag{9}$$

being $\hat{F}(\theta(t)) := F(\theta(t)) - G(\theta(t))\, D_k(\theta(t))\, H(\theta(t))$, while in the feedback configuration of Fig. 2, the closed loop system is,

$$\left.\begin{array}{ll} \dot{x}_a(t) & = A_{CL2}(\theta(t))\, x_a(t) + B_{CL2}(\theta(t))\, y_d(t) \\ y(t) & = [\, H(\theta(t)) \;\; 0 \;\; 0 \,]\, x_a(t) \end{array}\right\} \tag{10}$$

where $x_a(t) := \left[\, x^T(t) \;\; x_k^T(t) \;\; x_{kr}^T(t) \,\right]^T$, and,

$$A_{CL2}(\theta(t)) :=$$
$$\begin{bmatrix} \hat{F}(\theta(t)) & -G(\theta(t))\, C_k(\theta(t)) & G(\theta(t))\, C_{kr}(\theta(t)) \\ B_k(\theta(t))\, H(\theta(t)) & A_k(\theta(t)) & 0 \\ 0 & 0 & A_{kr}(\theta(t)) \end{bmatrix}$$
$$B_{CL2}(\theta(t)) := \begin{bmatrix} G(\theta(t))\, D_{kr}(\theta(t)) \\ 0 \\ B_{kr}(\theta(t)) \end{bmatrix}. \tag{11}$$

In these state space descriptions $A_{CL1}(\theta(t))$ and $A_{CL2}(\theta(t))$ in general have non-linearities in the parameters $\theta(t)$, due to the terms $G(\theta(t))\, D_k(\theta(t))\, H(\theta(t))$, $G(\theta(t))\, D_k(\theta(t))$, $G(\theta(t))\, C_k(\theta(t))$, $G(\theta(t))\, C_{kr}(\theta(t))$, and $B_k(\theta(t))\, H(\theta(t))$.

The closed loop systems of Figures 1 and 2 where the LPV plant is given by (1), are QS for all admissible $\theta(t)$ if, i) the input and the output matrices of the plant given by (1) are time invariant, ii) exist a positive definite matrix $P = P^T$ such that,

$$PA_{CL}\left(\theta_{(i)}\right) + A_{CL}\left(\theta_{(i)}\right) P < 0, i = 1, \ldots, 2^q, \tag{12}$$

where $A_{CL}(\theta(t))$ is either $A_{CL1}(\theta(t))$ or $A_{CL2}(\theta(t))$. Due to condition i) all the matrices of the state space realization of the controllers can be parameter dependent and convexity is not lost.

Regarding i), the input matrix of (1) is time invariant at least for linearized conservative Hamiltonian systems, that is, for a quadratic kinetic energy function, $T(q(t), \dot{q}(t)) = \frac{1}{2}\dot{q}^T(t)\, M(q(t))\, \dot{q}(t)$, where $q(t) \in \Re^m$ are the generalized coordinates and $M(q(t))$ is the inertia matrix, the Hamiltonian function is, $H_a(p(t), q(t), t) = T(q(t), p(t)) + V(q(t)) = \frac{1}{2}p^T(t)\, M^{-1}(q(t))p(t) + V(q(t))$, where $p(t)$ is the generalized momentum, and $V(q(t))$ the potential energy; hence, the non-linear Hamilton equations are, $\dot{q}(t) = \dfrac{\partial H_a(\cdot)}{\partial p(t)}$ and $\dot{p}(t) = \dfrac{-\partial H_a(\cdot)}{\partial q(t)} + \tau(t)$ that linearizing in the equilibrium point $(q_e, 0)$, arrives to the state space realization,

$$A = \begin{bmatrix} 0 & M^{-1}(q(t))\,|_{q(t)=q_e} \\ \dfrac{-\partial^2 V(q(t))}{\partial q^2(t)}|_{q(t)=q_e} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$
$$C = \left[\bar{C}_1 \;\; \bar{C}_2\right] \tag{13}$$

having a time-invariant input matrix, where the state is $\left[\, q^T(t) \;\; p^T(t) \,\right]^T$ and the input is $\tau(t)$.

If the input matrix of (1) is time invariant, a change of coordinates $\xi(t) = Tx(t)$ is realized such that $TB = \left[\, 0 \;\; B_m^T \,\right]^T$ and $A_{11}\left(\theta_{(i)}\right) = 0$, so, the LPV system at each vertex in new coordinates has the state space description,

$$\dot{\xi}(t) = \begin{bmatrix} 0 & A_{12}\left(\theta_{(i)}\right) \\ A_{21}\left(\theta_{(i)}\right) & A_{22}\left(\theta_{(i)}\right) \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ B_m \end{bmatrix} u(t)$$
$$y(t) = \left[\, C_1 \;\; C_2 \,\right] \xi(t), \; t \in [0, \, +\infty) \tag{14}$$

This form holds also for linearized Hamiltonian systems given by (13). It is assumed that the state space description given by (14) is causal LTI detectable and stabilizable, for $i = 1, \ldots, 2^q$. Suppose that n is even, $m = n/2$, and $p = m$, such that $A_{12}\left(\theta_{(i)}\right) \in \Re^{m\times m}$, $A_{21}\left(\theta_{(i)}\right) \in \Re^{m\times m}$, $A_{22}\left(\theta_{(i)}\right) \in \Re^{m\times m}$, and $B_m \in \Re^{m\times m}$. Since all the entries of $u(t)$ and $y(t)$ are linearly independent, without loss of generality, it is assumed that either B_m, and either C_2 or C_1, respectively, are non-singular matrices. Also, suppose that either $A_{12}\left(\theta_{(i)}\right)$ and $A_{21}\left(\theta_{(i)}\right)$ are non-singular matrices, as required by Bonilla and Galindo (2011) and Galindo and Conejo (2012). Then, computational efficient formulas of the stabilizing controllers and their free parameters are the ones of Galindo and Conejo (2012) to design a mixed sensitivity stabilizing controller at each vertex of $P(\theta(t))$. In particular in Bonilla and Galindo (2011) the cases of either $C_1 = 0$ or $C_2 = 0$ are considered. If $C_2 = 0$ coprime factorizations of $P\left(\theta_{(i)}\right)$ are,

$$\tilde{N}_i(s) = \frac{1}{(s+a_i)^2} B_m, \; \tilde{D}_i(s) = \Gamma_i(s) A_{12}^{-1}\left(\theta_{(i)}\right) C_1^{-1},$$
$$N_i(s) = \frac{1}{(s+a_i)^2} C_1 A_{12}\left(\theta_{(i)}\right), \;\; D_i(s) = B_m^{-1}\Gamma_i(s) \tag{15}$$

where
$\Gamma_i(s) := \frac{1}{(s+a_i)^2}(s^2 I_m - sA_{22}\left(\theta_{(i)}\right) - A_{21}\left(\theta_{(i)}\right) A_{12}\left(\theta_{(i)}\right))$, and a solution of the Diophantine equation is,

$$X_i(s) = \frac{\left(X_{1i}s + Y_{0i}A_{21}\left(\theta_{(i)}\right) A_{12}\left(\theta_{(i)}\right) + a_i^3 I_m\right) A_{12}^{-1}\left(\theta_{(i)}\right) C_1^{-1}}{s+a_i},$$
$$Y_i(s) = \frac{1}{s+a_i}(Y_{0i} + sI_m)B_m, \tag{16}$$

where $X_{1i} = Y_{0i}A_{22}\left(\theta_{(i)}\right) + A_{21}\left(\theta_{(i)}\right)A_{12}\left(\theta_{(i)}\right) + 3a^2 I_m$ and $Y_{0i} = A_{22}\left(\theta_{(i)}\right) + 3a_i I_m$. Then, from the parametrization of all stabilizing controllers (see Desoer et al. (1980) and Vidyasagar (1985)), $K_{ri}(s) = \tilde{D}_{ki}^{-1}(s)Q_i(s)$ and $K_i(s) = \tilde{D}_{ki}^{-1}(s)\left(X_i(s) + R_i(s)\tilde{D}_i(s)\right)$ where $\tilde{D}_{ki}(s) := Y_i(s) - R_i(s)\tilde{N}_i(s)$, and $R_i(s) \in \Re\mathcal{H}_\infty$ and $Q_i(s) \in \Re\mathcal{H}_\infty$ are free control parameters. Let $R_i(s)$ and $Q_i(s)$ be R_i and Q_i,

$$R_i := a_i\left(r_i I_m + A_{22}\left(\theta_{(i)}\right)\right),\ Q_i := q_i a_i^2 A_{12}^{-1}\left(\theta_{(i)}\right)C_1^{-1} \tag{17}$$

where

$$\begin{aligned} r_i &= \frac{a_i\left(3\left\|C_1A_{12}\left(\theta_{(i)}\right)A_{21}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty - a_i^2 b_i\right)}{a_i^3 c_i + \left\|C_1A_{12}\left(\theta_{(i)}\right)A_{21}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty},\\ q_i &= \frac{w_{hi}^2}{a_i^2 + w_{hi}^2} \end{aligned} \tag{18}$$

being w_{hi} a frequency in the high frequency bandwidth of $P\left(\theta_{(i)}\right)$, and,

$$\begin{aligned} b_i &= \frac{\left\|C_1A_{12}\left(\theta_{(i)}\right)\left(X_{1i} + a_iA_{22}\left(\theta_{(i)}\right)\right)A_{12}^{-1}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty}{w_{hi}^2}\\ c_i &= \frac{\left\|C_1A_{12}\left(\theta_{(i)}\right)\left(X_{1i} + a_iY_{0i}\right)A_{12}^{-1}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty - w_{hi}^2 b_i}{3a_i w_{hi}^2} \end{aligned} \tag{19}$$

solving a mixed sensitivity problem (see Bonilla and Galindo (2011)), that is, minimize $\left\|\begin{bmatrix} S_{oli} \\ T_{ohi}\end{bmatrix}\right\|_\infty$, *i.e.*,

$$\begin{aligned} \|T_{ohi}\|_\infty &= \frac{\left\|C_1A_{12}\left(\theta_{(i)}\right)\left(X_{1i} + R_i\right)A_{12}^{-1}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty}{w_{hi}^2}\\ \|S_{oli}\|_\infty &= \frac{|3a_i - r_i|}{a_i^3}\left\|C_1A_{12}\left(\theta_{(i)}\right)A_{21}\left(\theta_{(i)}\right)C_1^{-1}\right\|_\infty \end{aligned} \tag{20}$$

where $S_{oli} := \lim_{s\to 0} S_{oi}(s)$ and $T_{ohi} := \lim_{s\to\infty} T_{oi}(s)$, being $S_{oi}(s) := \left(I + P\left(\theta_{(i)}\right)K_i(s)\right)^{-1}$ and $T_{oi}(s) := S_{oi}(s)P\left(\theta_{(i)}\right)K_i(s)$. Also, from, (15), (16) and (17) the characteristic polynomial of the controller, $\det\left(\tilde{D}_{ki}(s)\right)$, simplifies to,

$$\det\left(\frac{s^2 I_m + \left(A_{22}\left(\theta_{(i)}\right) + 4a_iI_m\right)s + a_i\left(3a_i - r_i\right)I_m}{(s+a_i)^2}B_m\right) \tag{21}$$

that is stable if

$$\det\left(s^2 I_m + \left(A_{22}\left(\theta_{(i)}\right) + 4a_iI_m\right)s + a_i\left(3a_i - r_i\right)I_m\right) \tag{22}$$

is a Hurwitz polynomial. Moreover, for linearized Hamiltonian systems given by (13), *i.e.*, $A_{22}\left(\theta_{(i)}\right) = 0$, the controller is stable if $r_i < 3a_i$.

Hence, the proposed methodology is:
Methodology.

(1) Linearize the non-linear Hamiltonian model getting a polytopic description of the form given by (14),
(2) Design stabilizing controllers at each vertex of the convex hull of the plant and their control parameters from (15), (16), and (17) solving a mixed sensitivity criterion,
(3) Analyze QS from (12), and

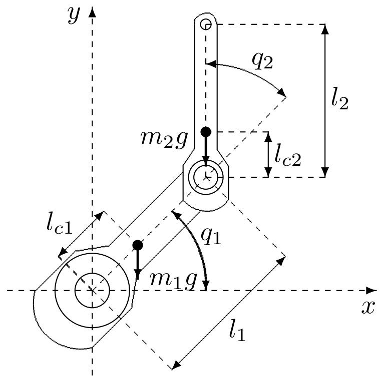

Fig. 3. Two DOF planar rotational robot.

(4) Get $K(\theta(t))$ by interpolation, for instance using (4) or (5).

These results are illustrated in the following section.

3. EXAMPLE

The results are applied to the non-linear and LPV models of a two-Degrees-Of-Freedom (DOF) planar rotational robot depicted in Fig. 3, where $q_1(t)$ and $q_2(t)$ are the angular positions. These models are obtained from the Hamilton equations. The inertia matrix is,

$$M(q(t)) = \begin{bmatrix} m_2\gamma_2 + \gamma_3 & m_2\gamma_1 + J_2 \\ m_2\gamma_1 + J_2 & m_2 l_{c2}^2 + J_2\end{bmatrix} \tag{23}$$

and the potential energy is

$$\begin{aligned} V(q(t)) &= m_1 g l_{c1}\sin(q_1(t)) + \\ &m_2 g\left[l_1\sin(q_1(t)) + l_{c2}\sin(q_1(t) + q_2(t))\right] \end{aligned} \tag{24}$$

where $q(t) := [\,q_1(t)\ q_2(t)\,]^T$, $\gamma_1 = l_{c2}^2 + l_1 l_{c2}\cos(q_2(t))$, $\gamma_2 = l_1^2 + l_{c2}^2 + 2l_1l_{c1}\cos(q_2(t))$ and $\gamma_3 = m_1 l_{c1}^2 + J_1 + J_2$ being J_k the inertia of the k-link. Linearizing the non-linear Hamiltonian model in the equilibrium point $(q_e, 0)$ where $q_e = [\,\pi/2\ 0\,]^T$, the linear model is given by (13), where,

$$\begin{aligned} &M^{-1}(q(t))\,|_{q(t)=q_e} = \\ &\begin{bmatrix} \theta_1(t)m_2(t)l_{c2}^2 + \theta_1(t)J_2 & -\theta_1(t)m_2(t)\gamma_{1e} - \theta_1(t)J_2 \\ -\theta_1(t)m_2(t)\gamma_{1e} - \theta_1 J_2 & \theta_1(t)m_2(t)\gamma_{2e} + \theta_1(t)\gamma_3 \end{bmatrix},\\ &\frac{-\partial^2 V(q(t))}{\partial q^2}\,|_{q(t)=q_e} = \\ &\begin{bmatrix} m_2(t)g(l_1 + l_{c2}) + m_1 l_{c1} g & m_2(t) g l_{c2} \\ m_2(t) g l_{c2} & m_2(t) g l_{c2}\end{bmatrix} \end{aligned} \tag{25}$$

being $\gamma_{1e} = l_{c2}^2 + l_1 l_{c2}$, $\gamma_{2e} = l_1^2 + l_{c2}^2 + 2l_1 l_{c1}$, and the considered time varying plant parameters are $\theta_1(t) := 1/\left[m_2(l_1^2 J_2 + l_{c2}^2 J_1 + m_1 l_{c1}^2 l_{c2}^2) + m_1 l_{c1}^2 + J_1 J_2\right] \in [3.71, 5.53]$ and $m_2(t) \in [2, 6]$, arriving to an state space LPV representation of the form (14).

We use the plant data $l_1 = 0.45\ m$, $l_{c1} = 0.091\ m$, $l_{c2} = 0.048\ m$, $m_1 = 23.902\ Kg$, $J_1 = 1.266\ Kg - m^2$, $J_2 = 0.093 Kg - m^2$ y $g = 9.81\ m/s^2$, borrowed from Bonilla and Galindo (2011). So, following the procedure of Bonilla and Galindo (2011), from (15), (16), and (17), the control parameters and norms of the sensitivity functions shown in Table 1, are obtained. At each vertex of the convex hull of the plant, the parity interlacing property is satisfied and $r_i < 3a_i$ of Table 1 assure a stable controller.

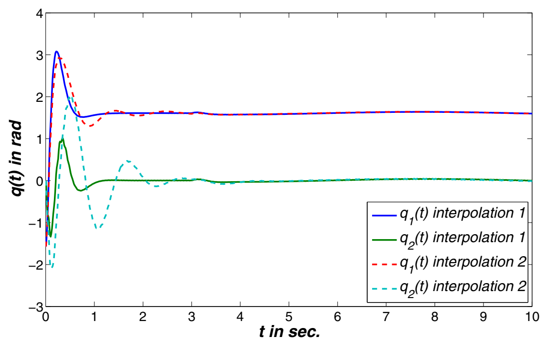

Fig. 4. Positions $q(t)$ for the one-parameter feedback configuration of $K(\theta(t))$ applied to $P(\theta(t))$

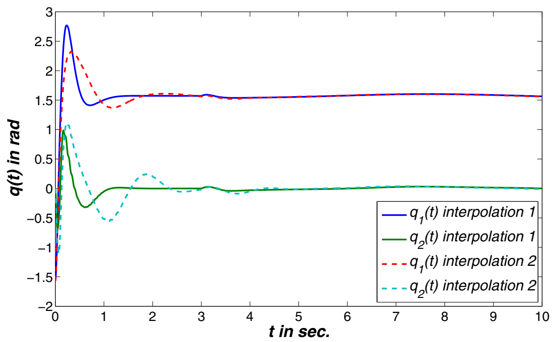

Fig. 5. Positions $q(t)$ for the one-parameter feedback configuration of $K(\theta(t))$ applied to the non-linear model

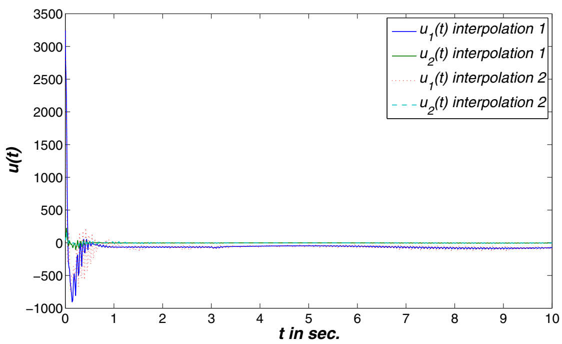

Fig. 6. Control law $u(t)$ for the one-parameter feedback configuration of $K(\theta(t))$ applied to $P(\theta(t))$

From the LMI toolbox of MatLab, the feasibility problem

Table 1. Control parameters at each vertex

$i-$vertex	a_i	w_{hi}	r_i	$\|T_{ohi}\|_\infty$ and $\|S_{oli}\|_\infty$
$\underline{\theta}_1, \underline{\theta}_2$	8	100	22.982	0.039
$\underline{\theta}_1, \bar{\theta}_2$	9	100	25.761	0.0496
$\bar{\theta}_1, \underline{\theta}_2$	9	100	26.284	0.052
$\bar{\theta}_1, \bar{\theta}_2$	10	100	29.296	0.0652

given by (12) has a solution, so, QS of the LPV control applied to the LPV system in the feedback configurations of Figures 1 and 2, is assured. In Figures 4 to 11, $K(\theta(t))$ is gotten by the interpolations 1 and 2 that are given by (4) and (5), respectively, the initial condition is $q(0) = [\frac{-\pi}{2},\ 0]^T$, the reference input is $y_d = [\frac{\pi}{2},\ 0]^T$

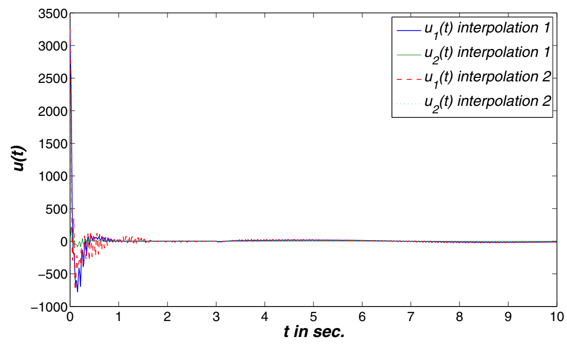

Fig. 7. Control law $u(t)$ for the one-parameter feedback configuration of $K(\theta(t))$ applied to the non-linear model

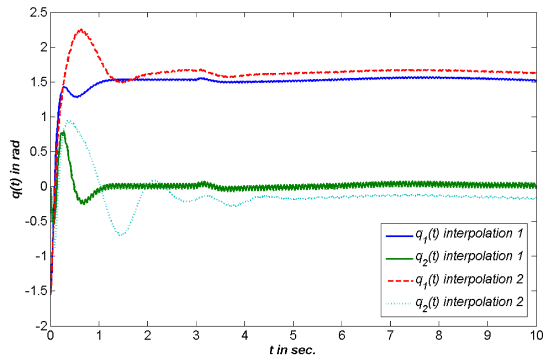

Fig. 8. Positions $q(t)$ for the two-parameter feedback configuration of $K(\theta(t))$ applied to $P(\theta(t))$

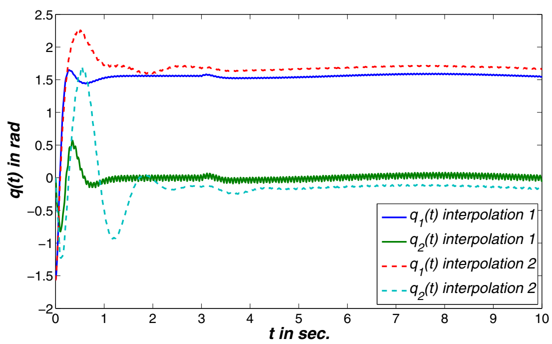

Fig. 9. Positions $q(t)$ for the two-parameter feedback configuration of $K(\theta(t))$ applied to the non-linear model

and the time variations of the parameters are $m_2(t) = 2\sin(100t) + 4$ and $p(t) = 0.89\sin(100t + \frac{\pi}{2}) + 4.63$. Also, the robot is tested with an output additive disturbance $d_o(t) = 0.4\sin(0.8t)$, $t > 3$ sec..

The plant output position $q(t)$, of $K(\theta(t))$ applied to $P(\theta(t))$ and $K(\theta(t))$ applied to the non-linear model in the one-parameter feedback configuration are shown in Figures 4 and 5, respectively, while in the two-parameter feedback configuration are shown in Figures 8 and 9, respectively. The plant input $u(t)$ of $K(\theta(t))$ applied to $P(\theta(t))$ and $K(\theta(t))$ applied to the non-linear model in the one-parameter feedback configuration are shown in

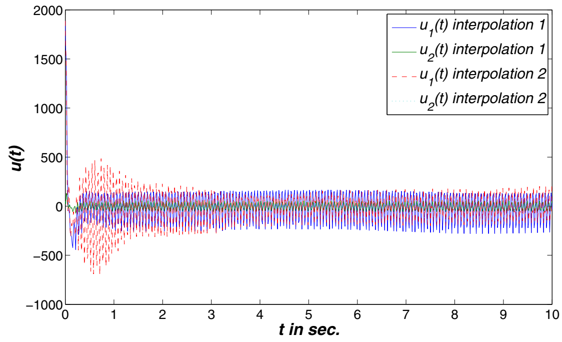

Fig. 10. Control law $u(t)$ for the two-parameter feedback configuration of $K(\theta(t))$ applied to $P(\theta(t))$

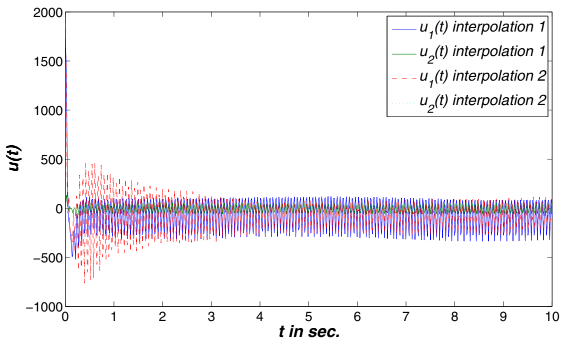

Fig. 11. Control law $u(t)$ for the two-parameter feedback configuration of $K(\theta(t))$ applied to the non-linear model

Figures 6 and 7, respectively, while in the two-parameter feedback configuration are shown in Figures 10 and 11, respectively.

As expected, due to the controller complexity, there are more oscillations at the output $q(t)$ using interpolation 1 than using interpolation 2. The time response of Figures 4 and 5 is less than the one of Figures 8 and 9, while the price to pay is that the overshoot of Figures 4 and 5 is higher than the one of Figures 8 and 9 and the one-parameter feedback configuration needs more plant input energy than the two-parameter feedback configuration, as shown in Figures 6, 7, 10 and 11. As expected stability is guaranteed in spite of the time variations of the parameters, the output additive disturbance and of the unmodelled dynamics present for the non-linear model. In Figures 8 and 9 is shown that there are small oscillations in $q_1(t)$ with frequency 0.8 rad/sec. due to the output additive disturbance, while it is not appreciable in Figures 4 and 5. In all the cases, the stationary state error is small as shown in Figures 4, 5, 8 and 9. Also, the overshoot of the plant input is bigger in Figures 6 and 7 than in Figures 10 and 11, and the oscillations of the plant input are better attenuated at the plant input in the one-parameter feedback configuration than in the two-parameter feedback configuration.

4. CONCLUSIONS

An LPV control methodology for LPV systems is proposed. An LPV controller is designed for an LPV plant, interpolating robust stable and stabilizing controllers. Convexity is assured for the feedback configurations of one or two parameters, so, Quadratic stability can be analyzed by Horisberger's Theorem. A comparison of the LPV controller gotten by two types of interpolation, applied to the LPV plant and to the non-linear plant in one and two parameter feedback configurations, is presented. This results show that, in all the cases, quadratic stability is achieved in spite of the fast time variation of the parameters, the output additive disturbance, and unmodelled dynamics. Also, the output is smooth and with a small stationary state error. The time response has less oscillations using the interpolation of the less complexity controller. The problem that the LPV controller guaranties closed stability in advance remains as a future work.

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