Memorias del XVI Congreso Latinoamericano de Control Automático, CLCA 2014 Octubre 14-17, 2014. Cancún, Quintana Roo, México

Min-Max Robust Stabilization of Parametrically Perturbed Second Order Control Systems

Victor N. Zhermolenko[1] and Alex S. Poznyak[2]

***Abstract*— A parametrically perturbed second order oscillatory system is discussed. Such perturbations try to swing the system maximally, but a bounded control intends to provide the robust stabilization of this system in spite of the perturbation effect. So, here a min-max feedback design is tackled. By the obtained results an attraction set containing the origin in the phase plane is found.**

***Keywords:* Integral funnel, Worst Parametric Perturbation, Limit Cycles, Domain of Robust Stabilization.**

I. Introduction

A. Brief survey

The basic notion treated in this paper is an *Integral Funnel (IF)*. By the definition (see, for example, [1], [2], [3], [4]), *IF* of a point $P(t_0, x_0)$ for a differential equation $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$ is the set of all points lying on the integral curves passing through $P(t_0, x_0)$. If only one integral curve passes through this point, then the *IF* consists of this single curve. In the $n = 1$ case, that is, when x is scalar, the integral funnel consists of points (t, x) for which $x_*(t) \leq x(t) \leq x^*(t)$ where $x^*(t)$ and $x_*(t)$ are the upper and lower solutions, that is, the largest and smallest solutions passing through P.

If the function $f(t, x)$ is continuous (or satisfies the conditions of the Carathéodory existence theorem), then the integral funnel is a closed set. Furthermore, if all the solutions passing through $P(t_0, x_0)$ exist on the interval $t \in [a, b]$, then this segment of the funnel and the section of the integral funnel by any plane $t = t_1 \in [a, b]$ are connected compact sets. Any point on the boundary of *IF* can be joined to P by a piece of the integral curve lying on the boundary of IF. If the sequence of points P_k $(k = 1, 2, ...)$, converges to P, then the segments of the funnels of the points P_k converge to the segment of the funnel of P in the sense that for any $\varepsilon > 0$ they are contained in an ε-neighbourhood of the segment of the funnel of P if $k > k_1(\varepsilon)$. Analogous properties are possessed by integral funnels for differential inclusions $\dot{x} \in F(t, x)$ under specified hypotheses (see [*Filippov 1988*]) concerning the set $F(t, x)$.

The notion of the upper $x^*(t)$ and lower $x_*(t)$ solutions have been extended to the plain case $n = 2$ in [5], [6], [7], [8], [9] and [10].

Main properties of extremal boundary curves, characterizing *IF*, were extensively used in the, so-called, *Funnel Control Method* (see [11], [12], [13], [8], [14], [15], [16], [17], [18], [19] and [20]). So, the work [12] deals with tracking of a reference signal (assumed bounded with essentially bounded derivative) considered in the context of a class of nonlinear systems, with output, described by functional differential equations (a generalization of the class of linear minimum-phase systems of relative degree one with positive high-frequency gain). There the primary control objective is tracking with prescribed accuracy. In [16] tracking (by the system output) of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of linear input-output systems subject to input saturation where the system is assumed to have strict relative degree one with stable zero dynamics. There the prespecified notion is a performance funnel, within which the tracking error is required to evolve: transient and asymptotic behavior of the tracking error is influenced through choice of parameter values which define this funnel. In [17] the given overview presents a simple high-gain adaptive controller – the funnel controller and its possible applications in mechatronis. The funnel controller neither identifies nor estimates the system under control and is applicable for (nonlinear) systems being minimum-phase (or having stable zero dynamics in the nonlinear case), having relative degree one or two and known sign of the high-frequency gain. The paper [18] deals with the partial case of the same problem, but when the parametric perturbations are periodic. In [19] it is shown that PI-funnel control (i.e. funnel control in combination with a PI controller) in presence of actuator (input) saturation may lead to integrator windup deteriorating control performance.

B. Main contribution

Here we develop a new approach, called below as *Extremal Deviations Method (EDM),* providing an effective instrument for finding extremal actions of controls and external perturbations corresponding that extremal trajectories. Based on this *EDM* in the two-dimensional case the solution of the min-max robust stabilization problem with parameters perturbations is obtain. There the maximum is taken over the set of existing uncertainties and minimum is taken over the admissible control actions. Notice that we deal with parametric uncertainties such corresponding min-max problem was solved in [21].

The main contribution of this paper consists in the following issues:

- the *necessary and sufficient conditions* guarantying the robust stabilizability for a class parametrically perturbed systems in two-dimensional case are obtained and the behavior of the trajectories on the phase plane with *extremal min-max swing amplitude* is investigated;

[1] Russian State University of Petrol and Gaz, Moscow, Russia, e-mail: zhermol@yandex.ru

[2] Department of Automatic Control, CINVESTAV-IPN, Mexico DF, e-mail: apoznyak@ctrl.cinvestav.mx

- the *unstable limit cycle* for these extremal trajectories in the corresponding phase plane is found in analytical form; here is shown that if initial conditions of the system are inside of this limit cycle, then the suggested min-max controller obligatory guarantees the convergence of all trajectories within *IF* to the origin providing the property of the robust stabilization; if initial condition are outside of the cycle, then there exist instable trajectories (of a nonlinear measure) which tend to infinity (disconverge).

II. Problem Statement

Consider differential equation

$$\begin{gathered}\frac{d^2}{d^2\tau}x(\tau) + \left[\omega^2 + \tilde{v}(\tau)\right]x(\tau) = \tilde{u}(\tau)\\ x(\tau=0)=0,\ \dot{x}(\tau=0)=0\end{gathered}$$

representing a parametrically perturbed second-order controlled system. Here all functions $\tilde{v}(\tau)$ and $\tilde{u}(\tau)$ are piecewise continuos scalar functions of $\tau \geq 0$. Applying the time-transformation $\tau = t/\omega$, $\omega > 0$ this differential equation can be written in the *Frobenius form*

$$\begin{gathered}\dot{x}_1 = x_2\\ \dot{x}_2 = -[1+v(t)]x_1 + u(t)\\ x_1(0) = x_2(0) = 0\end{gathered} \tag{1}$$

where $t = \tau\omega$ is a time variable without physical dimension and

$$x_1(t) = x\left(t/\omega\right),\ x_2(t) = \frac{d}{dt}x_1\left(t/\omega\right)$$

The control input

$$\begin{gathered}u(t) := u\left(x_1\left(t\right), x_2\left(t\right), t\right) =\\ \frac{1}{\omega^2}\tilde{u}\left(x_1\left(\frac{t}{\omega}\right), x_2\left(\frac{t}{\omega}\right), \frac{t}{\omega}\right) := \frac{1}{\omega^2}\tilde{u}(t/\omega)\end{gathered}$$

and the parametric perturbation $v(\tau) := \dfrac{1}{\omega^2}\tilde{v}(t/\omega)$ are assumed to be piecewise continuous and bounded by $\lambda > 0$ and $\mu \in (0,1)$ respectively:

$$\begin{gathered}u(t) \in U = \{\mid u(t) \mid \leq \lambda\}\\ v(t) \in V = \{\mid v(t) \mid \leq \mu\}\end{gathered} \tag{2}$$

DEFINITION 2.1. By the robust stabilizability of ODE (1) we mean that there is an admissible control action $u(t) \in U$ providing the existence in phase plane (x_1, x_2) an attraction region containing the origin which is invariant with respect to the set of perturbations $v(t) \in V$.

According to Definition 2.1, control $u(\cdot) \in U$, which we are interested in, is intended to provide the *robust stabilization* of system (2.1), that is, to guarantee that $\|x(t)\| \to 0$ ($x = (x_1, x_2)^{\intercal}$) when $t \to \infty$ for any admissible parametric perturbation from V.

REMARK 2.1. Certainly, the considered robust stabilizing law in the presence of parametric perturbations can be treated as one providing a practical stabilization. But, as it will be clear from the consideration below, the uniform local stabilization of the set Z (zone of sticking) never can be guaranteed.

Assuming that perturbations try to violate the desired stabilizing control, we suggest to treat the control and perturbations as two dynamic players, pursuing conflicting goals with on-line available information on the realized states $\{t, x(t)\}$ of the system (1).

III. Boundaries of IF and Criteria of Oscillation

A. Boundaries of IF

If the control is identically equal to zero, that is, $u(\cdot) \equiv 0$, then the system (1) is the absolute oscillatory, i.e., having all oscillating trajectories. In this case the coordinate $x_1(x^{(0)}, u, v, t)$ performs *simple oscillations* for any nonzero initial position $x^0 = (x_1^0, x_2^0)$ and any parametric perturbation $v(\cdot) \in V$. This means that between two sequential zeros of coordinate $x_1(x^{(0)}, u, v, t)$ there obligatory exists only one zero of its derivative. Therefore, without loss of generality, we may assume that the stabilization starts at the instant t_0 when

$$x_1(t_0) := -a_0 < 0,\ x_2(t_0) = \dot{x}_1(t_0) = 0$$

DEFINITION 3.1. Trajectories of type 1 (or 2) we call the collection of trajectories of system (1) between two sequential zeros of the derivative $\dot{x}_1 = x_2$, if at this interval there exists zero of coordinate $x_1(t)$ (or, for type 2, if there does not exist zero of coordinate $x_1(t)$).

Thus, a simple swinging of coordinate $x_1(x^{(0)}, u, v, t)$ of (1) consists of the trajectories of the 1-st type.

Consider the segment of *IF* $X_{uv}(x^{(0)})$ with vertex $x^{(0)} = (-a_0, 0)$ on the negative semi-axis OX_1^-.

The maximum- and minimum-optimal boundary trajectories (branches) B_{max} *and* B_{min} of the trajectory funnel $X_{uv}(x^{(0)})$ can be found using the *Differential Geometry Method* [6] exploiting the extremal deviations analysis [9], [10]]. Maximal or minimal deviations correspond to the left or right boundary branches B_{max} and B_{min} of *IF* $X_{uv}(x^{(0)})$ generated by (1) where the right-hand side is associated with the extreme-left or extreme-right vector of the admissible phase velocities cone in the current phase point (x_1, x_2). These extreme vectors of the phase velocities cone are defined by the maximum or minimum of the "slope-tangent" coefficient

$$k(x,\ u,\ v,) = \frac{dx_2}{dx_1} = -\frac{x_1}{x_2} - \frac{vx_1}{x_2} + \frac{u}{x_2} \tag{3}$$

at each point of the corresponding extremal trajectory. For the maximization (or minimization) of $k(x,u,v)$ it is necessary and sufficient to maximize by u and v two last term in (3). The extremal controls u and v providing these optimal boundary branches of *IF* $X_{uv}(x^{(0)})$ have the form:

$$\begin{gathered}u^+ = \lambda\,\text{sign}\{x_2\},\ v^+ = -\mu\,\text{sign}\{x_1\,x_2\}\\ u^- = -\lambda\,\text{sign}\{x_2\},\ v^- = \mu\,\text{sign}\{x_1\,x_2\}\end{gathered}$$

Obviously, with these controls we have oscillations having maximum and minimum swing amplitudes. If one of the functions $u(\cdot) \in U$ or $v(\cdot) \in V$ is fixed then boundaries of *IF* $X_{uv}(x^{(0)})$ can be found by the same rule. For example, under the chosen control $u = \widehat{u}$ the boundary branches of the trajectory funnel $X_{\widehat{u}v}(x^{(0)})$ are as follows: $v^+ = -\mu\,\mathrm{sign}\{x_1\,x_2\}$, $v^- = \mu\,\mathrm{sign}\{x_1\,x_2\}$.

B. Criteria of Oscillation

At first, we analyze the behavior of a piece of minimum-optimal (right) boundary branch B_{min} of the full trajectory funnel $X_{uv}(x^{(0)})$ from vertex $x^{(0)} = (-a_0, 0)$ to its first intersection with the axis $x_2 = 0$. Then by requiring that it must be a trajectory of type 1, but can't be a trajectory of type 2 we obtain the following criterion of absolute oscillation.

LEMMA 3.1. All trajectores from *IF* $X_{uv}(x^{(0)})$, $x^{(0)} = (-a_0, 0)$ until its first intersection with axis $x_2 = 0$ are the trajectories of type 1 if and only if

$$a_0 > \frac{2\lambda}{1-\mu} \tag{4}$$

IV. Robust Stabilization as a Differential Game

Suppose now that the condition (4) holds. Then by Lemma 3.1 there exists the time-instant t_1 such that $x_2(t_1) = 0,\ x_2(t) \neq 0,\ t \in (t_0, t_1)$. In view of that the following extremal problem (with non-fixed time horizon)

$$\sup_{v(\cdot)\in V} \left|x_1(u,\, v,\, t_1,\, x^0)\right| \to \inf_{u(\cdot)\in U}$$
$$x^0 = (x_1(t_0),\, x_2(t_0)) = (-a_0, 0)$$

takes place. In fact, it consists in the minimization of the greatest possible amplitude of the oscillation of the type 1 between its two sequential extremal points. First, we consider the auxiliary differential game with non-fixed time horizon and the following individual aims of the participants:

$$\sup_{v(\cdot)\in V} \left|x_1(u,\, v,\, t_1,\, x^0)\right| \to \inf_{u(\cdot)\in U} \tag{5}$$

$$\inf_{u(\cdot)\in U} \left|x_1(u,\, v,\, t_1,\, x^0)\right| \to \sup_{v(\cdot)\in V} \tag{6}$$

with interval conditions $x^{(0)} = (-a_0, 0);\ \ t_1 : x_2(t_1) = 0,\ x_2(t) \neq 0,\ t \in (t_0, t_1)$. These extremal problems (5) and (6) with a non-fixed time horizon have a simple geometrical illustration. In the problem (5), first, we search the greatest possible value of amplitude $v(\cdot) \in V$ between two sequential zero of its derivative, and then, the lower bound of the found greatest possible amplitude of a simple oscillation is calculated by variation of $u(\cdot) \in U$. In the problem (6) the order of the decision making is opposite.

In general case we have

$$\gamma_0 := \sup_{v(\cdot)\in V} \inf_{u(\cdot)\in U} x_1(u,\, v,\, t_1,\, x^0) \leq$$
$$\gamma^0 := \inf_{u(\cdot)\in U} \sup_{v(\cdot)\in V} x_1(u,\, v,\, t_1,\, x^0)$$

If $\gamma_0 = \gamma^0$ then the differential game is said to have a saddle point. In our problem we have exactly this situation.

THEOREM 4.1 The differential game (5)-(6) has a saddle point u^*, v^*satisfying

$$\min_{u(\cdot)\in U} \max_{v(\cdot)\in V} x_1(x^{(0)},\, u,\, v,\, t_1) =$$
$$\max_{u(\cdot)\in U} \min_{v(\cdot)\in V} x_1(x^{(0)},\, u,\, v,\, t_1) = x_1(x^{(0)},\, u^*,\, v^*,\, t_1)$$

where

$$u^* = -\lambda\,\mathrm{sign}\{x_2\},\ v^* = -\mu\,\mathrm{sign}\{x_1 x_2\} \tag{7}$$

Proof. Indeed a pair of functions $(u_0,\, v_0)$ is a solution of the min-max problem (6) iff

$$k(x,\, u_0,\, v_0) = \min_{|u|\leq\lambda} \max_{|v|\leq\mu} k(x,\, u,\, v) \tag{8}$$

for any $t \in (t_0,\, t_1)$. Similarly, a pair of functions $(u^0,\, v^0)$ gives a solution to the max-min problem (5) iff

$$k(x,\, u^0,\, v^0) = \max_{|v|\leq\mu} \min_{|u|\leq\lambda} k(x,\, u,\, v)$$

for any $t \in (t_0,\, t_1)$. It is obvious that the linear function $k(x,\, u,\, v)$ (3) of u and v has a unique saddle point, i.e., the pair of functions $(u_0,\, v_0)$ and $(u^0,\, v^0)$ coincide, namely,

$$u_0 = u^0 = u^*,\ \ v_0 = v^0 = v^*$$

So, by [22] the differential game (5) - (6) has a saddle point too. The functions realizing the optimal minimax (maximin) synthesis are given in (7). Theorem is proven.

V. Oscillations with Min-max Swing Amplitudes and Limit Cycle

A. Oscillations with Min-max Swing Amplitudes

Substitution of (7) in (1) with the initial conditions $x^{(0)} = (-a_0, 0)$ leads to the solution at the point $t = t_1$ as

$$x_1(x^{(0)},\, u^*,\, v^*,\, t_1) =$$
$$\tfrac{1}{1-\mu}\left(-\lambda + \sqrt{\lambda^2 + a_0(1-\mu)[(1+\mu)a_0 - 2\lambda]}\right) \tag{9}$$

where the end point of the interval is given by

$$t_1 = t_0 + \frac{1}{\sqrt{1+\mu}}\arccos\frac{\lambda}{\lambda-(1+\mu)a_0} +$$
$$\frac{1}{\sqrt{1-\mu}}\mathrm{arctg}\frac{1}{\lambda}\sqrt{a_0(1-\mu)[(1+\mu)a_0 - 2\lambda]}$$

The following corollary takes place.

COROLLARY 5.1. The oscillation of coordinate $x_1(x^{(0)},\, u^*,\, v^*,\, t)$ under the initial condition $x^{(0)} = (-a_0, 0)$ in (1) on the interval $[t_0,\, t_1]$ is simple (of the type 1) iff the condition

$$a_0 > \frac{2\lambda}{1+\mu}$$

holds.

REMARK 5.1. Notice that the inequality in the corollary above does not contradict to (3.2) since it means that the coordinate $x_1\,(u, v, t_1, x_0)$ of the integral curve with "min − max = max − min" amplitude is the trajectory of the 1-st type.

Denote $b_0 := \left|x_1(x^{(0)},\, u^*,\, v^*,\, t_1)\right|$ and assume that $b_0 > \dfrac{2\lambda}{1+\mu}$. Taking the starting point $x^{(0)} = (b_0, 0)$ implies that the pair of functions (7) gives the solution to the following extremal problem with a non-fixed horizon:

$$\inf_{u(\cdot)\in U} \left|x_1(x^{(0)},\, u,\, v,\, t_2)\right| \to \sup_{v(\cdot)\in V}$$

$$x^{(0)} = (x_1(t_0),\, x_2(t_0)) = (b_0, 0)$$
$$t_2 :\, x_2(t_2) = 0,\, x_2(t) \neq 0 \; t \in (t_1,\, t_2)$$

which has the same geometric illustration as in Fig.1. In

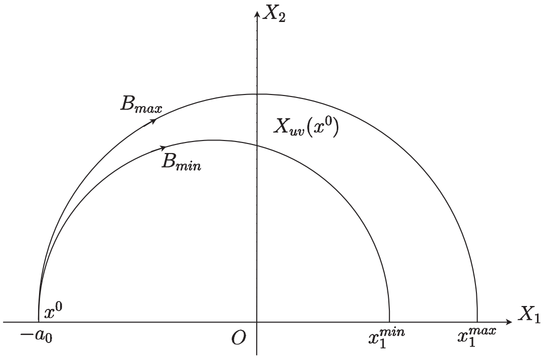

Fig. 1. Integral Funnel $X_{uv}(x^{(0)})$.

this case the maximum value of $\left|x_1(x^{(0)},\, u,\, v,\, t)\right|$ can be found from the expression $-a_1 = -x_1(x^0,\, u^*,\, v^*,\, t_2) < 0$. Here $x_1(x^{(0)},\, u^*,\, v^*,\, t_2)$ calculated by the formula (9) where instead of a_0 there should be taken b_0, and the time t_2 means the first moment of crossing the positive semi-axis $0X^+$ by the considered phase trajectory from a given IF.

CONCLUSION 5.1. Iterating this procedure we obtain the extremal trajectory $x(x^{(0)},\, u^*,\, v^*,\, t)$ with sequential min-max amplitudes $\{a_i\}$, $\{b_i\}$ $(i = 0, 1, 2, ...)$ which are related as

$$b_i = \varphi(a_i), a_{i+1} = \varphi(b_i)$$

$$\varphi(-z) = \tfrac{1}{1-\mu}\left(-\lambda + \sqrt{\lambda^2 + z(1-\mu)[(1+\mu)z - 2\lambda]}\right) \tag{10}$$

B. The Poincaré map and limit cycle

Recall (see [23]) that a *first recurrence map* (or the Poincaré map) is the intersection of a periodic orbit in the state space of a continuous dynamic system with a certain lower dimensional subspace, called the Poincaré section, transversal to the flow of the system. More precisely, one considers a periodic orbit with initial conditions within a section of the space, which leaves that section afterwards, and observes the point at which this orbit first returns to the section. One then creates a map to send the first point to the second (the first recurrence map). The transversality of the Poincaré section means that periodic orbits, starting on the subspace, flow through it and not parallel to it.

Considering (10) as the formulas for the Poincaré map of the semi-axis OX_1^- (OX_1^+) in itself, we find that the first back Poincare's function $\overline{a} = f(a)$, $\overline{b} = f(b)$ has the form: $\overline{a} = \varphi[\varphi(a)]$, $a > \dfrac{2\lambda}{1+\mu}$ and $\overline{b} = \varphi[\varphi(b)]$, $b > \dfrac{2\lambda}{1+\mu}$. It is easy to find that fixed points of the Poincare's mapping $\overline{a} = \varphi[\varphi(a)]$ and $\overline{b} = \varphi[\varphi(b)]$ on the axis OX_1 will be as follows: $(-a^*, 0)$, $(b^*, 0)$, $a^* = b^* = \dfrac{2\lambda}{\mu}$. Indeed,

$$\varphi\left(-z\right)|_{z=\frac{2\lambda}{\mu}} = \tfrac{1}{1-\mu}\left(-\lambda + \tfrac{4\lambda^2}{\mu^2}\left(1-\mu^2\right) - \tfrac{4\lambda^2}{\mu}\left(1-\mu\right)\right)$$

$$= \tfrac{\lambda}{1-\mu}\left[-1 + \sqrt{\left(\tfrac{2}{\mu} - 1\right)^2}\right] = \tfrac{2\lambda}{\mu}$$

Moreover, $\left(\frac{\partial \overline{a}}{\partial a}\right)_{a=a^*} = \frac{2+\mu}{2-\mu} > 1$. According to the Köenig's theorem (see, for example, [24] and the fixed-point theorems in [25]) the fixed points $(-a^*, 0)$, $(a^*, 0)$, as well as passing through them closed phase trajectory $x(x^{(0*)},\, u^*,\, v^*,\, t)$, $x^{(0*)} = (\pm a^*, 0)$ are unstable. Thus, the system (1) under action of $u = u^*$, $v = v^*$ has an unstable limit cycle C for the extremal oscillating trajectories $x(x^{(0)},\, u^*,\, v^*,\, t)$, $x^{(0*)} = (-a^*, 0)$ with the sequential min-max amplitudes $\{a_i\}$, $\{b_i\}$.

The *parametric equations of the limiting cycle* C can be found analytically by the following way. Using the shooting method on switching line $x_1 = 0$ to the first intersection of the phase trajectory $x(x^{(0*)},\, u^*,\, v^*,\, t)$ with the positive semi-axis $x_2 = 0$, the parametric equations of the upper half of limiting cycle C for $t \in [0, \arccos(-\frac{\mu}{2+\mu})]$ (before switching) are as follows:

$$x_1\left(t\right) = -\tfrac{\lambda(2+\mu)}{\mu(1+\mu)}\cos\left(t\sqrt{1+\mu}\right) - \tfrac{\lambda}{1+\mu}$$

$$x_2\left(t\right) = \tfrac{\lambda(2+\mu)}{\sqrt{1+\mu}}\sin\left(t\sqrt{1+\mu}\right)$$

and for $t \in [\arccos(-\frac{\mu}{2+\mu}), \arctan(\frac{2\sqrt{1-\mu}}{\mu})]$ (after switching) we have

$$x_1\left(t\right) = \tfrac{\lambda}{1-\mu}\cos\left(t\sqrt{1-\mu}\right) + \tfrac{2\lambda}{\mu\sqrt{1-\mu}}\sin\left(t\sqrt{1-\mu}\right) - \tfrac{\lambda}{1-\mu}$$

$$x_2\left(t\right) = \tfrac{2\lambda}{\mu}\cos\left(t\sqrt{1-\mu}\right) - \lambda\sqrt{1-\mu}\sin\left(t\sqrt{1-\mu}\right)$$

PROPOSITION 5.1. If the initial point x^0 is outside of the limit cycle C (or on it), then the sequential min-max amplitudes $\{a_i\}$, $\{b_i\}$ of the extremal oscillating trajectories $x(x^{(0)},\, u^*,\, v^*,\, t)$ increase monotonically (or do not decrease). In this case the min-max robust (for all possible trajectories) stabilization is impossible.

PROPOSITION 5.2. If the initial point x^0 is located inside of the domain $Q = \operatorname{int} C$ bounded by limit cycle C, then the sequential min-max amplitudes of the extremal trajectory $x(x^{(0)},\, u^*,\, v^*,\, t)$ decrease monotonically. Through a finite number of semi-coils of the extremal trajectory ends in the "zone of sticking" $Z := [-\frac{\lambda}{1+\mu},\, \frac{\lambda}{1+\mu}] \in OX_1$, i.e., on the switching line $x_2 = 0$. The number of semi-coils depends on the x^0. For the case $x^0 \in Q$ the possibility of the min-max robust stabilization will be discussed in the next section.

VI. Min-max Robust Feedback Control Design and Stabilizability Domain

As it is mentioned above, the stabilization process starts at each instant t_i $(i = 0, 1, 2, ...)$ of the intersection of the phase trajectory of (1) with the axis OX_1 $(x_2 = 0)$, i.e., at the point $x^{(i)} = (x_1(t_i),\, 0)$. Consider now several possible situations.

A. Case $0 < \mu < 1/2$

Consider the case $0 < \mu < 1/2$. Then the inequalities $\frac{2\lambda}{1+\mu} < \frac{2\lambda}{1-\mu} < \frac{2\lambda}{\mu}$ are satisfied. Recall that the idea of the min-max robust stabilization is based on the following fact: if $u = u^*$ is fixed and at the instant t_i (when $x_2(t_i) = 0$), the following relations

$$\frac{2\lambda}{1-\mu} < |\, x_1(t_i)| < \frac{2\lambda}{\mu} \tag{11}$$

hold, then the first segment of *IF* $X_v(x^{(i)}) = \{x(x^{(i)}, u^*, v, t), u = u^*, v(\cdot) \in V\}$ from its vertex on the axis $x_2 = 0$ to the first intersection with the same axis $x_2 = 0$ consists of a simple oscillations which swing amplitudes in the interval $[t_i, t_{i+1}]$ between two sequential extremal points of the coordinate x_1 decreases, that is, $|\, x_1(t_{i+1})| < |\, x_1(t_i)|$. So, the inequality (11) is a sufficient condition for damping of the simple oscillations. As we noted above, *IF* $X_v(x^{(i)})$ is obtained at $u = u^*$, $v(\cdot) \in V$. The branches of its border are determined by the same way as above: the maximal (left) and minimal (right) boundary branches are defined by the system (1) at $u = u^*$, $v = v^*$ and $v = -v^*$ given by (7). Here we consider the situation that the functions v^* and v_*, being on the lines of their discontinuity $x_1 = 0$, $x_2 = 0$, can take any value from the admissible segment $[-\mu, \mu]$. Each of the branches border of *IF* $X_v(x^{(i)})$ is twisting, and through a finite number of semi-coils it ends in its "sticking zone" on the switching line OX_1 ($x_2 = 0$). More exactly, the maximal (left) branch ends in the interval $[-\dfrac{\lambda}{1+\mu}, \dfrac{\lambda}{1+\mu}]$ and minimal (right) branch ends within the interval $[-\frac{\lambda}{1-\mu}, \frac{\lambda}{1-\mu}]$.

The main aim of the min-max stabilizing control is to eliminate "sticking" at least one trajectory from $X_v(x^{(i)})$ including its boundary branches. To avoid this sticking at the instants t_i (when the path crosses the axis $x_2 = 0$) we have to use the control $u^* = -\lambda_i \mathrm{sign}\{x_2\}$. Similar to (11), in this case the damping condition of the simple oscillations is given by

$$\frac{2\lambda_i}{1-\mu} < |\, x_1(t_i)| < \frac{2\lambda_i}{\mu} \tag{12}$$

which can be rewritten in the following equivalent form: $\frac{\mu|\, x_1(t_i)|}{2} < \lambda_i < \frac{(1-\mu)|x_1(t_i)|}{2}$. Notice that the swing damping condition (12) is satisfied if the "adjusted" control resource λ_i takes value equal to the half-sum of its "adjusted" borders $\frac{\mu|\, x_1(t_i)|}{2}$ and $\frac{(1-\mu)|x_1(t_i)|}{2}$, i.e., when $\lambda_i = \frac{|\, x_1(t_i)|}{4}$. Thus, in fulfilling these conditions, the min-max robust stabilization is possible. The position-adjustable law of such min-max stabilization has the form

$$\tilde{u}^*(x) = -\frac{|\, x_1(t_i)|}{4}\,\mathrm{sign}\{x_2\},\ t_i : x_2(t_i) = 0 \tag{13}$$

Notice that the control (13) does not use the exact information about the parametric perturbation v. Only the resource μ is available, but it also does not participates in (13), that is, the control $\tilde{u}^*(x)$ is robust with respect to the perturbation $v(\cdot) \in V$

SUMMARY 6.1. In the case $0 < \mu < 1/2$ at $x^{(0)} \in Q$ both boundary branches of *IF* $X_v(x^{(0)})$, with $u = \tilde{u}^*(x)$, $v(\cdot) \in V$ are the spiral phase curves twisted to the asymptotic point at the origin. In the phase plane such behavior corresponds to simple oscillations (of the type 1) in the phase plane with monotonically decreasing and asymptotically tending to zero swing amplitudes. In other words, the domain $Q = \mathrm{int}\, C$ is a domain of the min-max robust stabilizability. This takes place because of the existing advanced contraction leads to the origin of the "zones of sticking"

$$Z_i^L := [-\tfrac{|\, x_1(t_i)|}{4(1+\mu)},\ \tfrac{|\, x_1(t_i)|}{4(1+\mu)}]$$

$$Z_i^R := [-\tfrac{|\, x_1(t_i)|}{4(1-\mu)},\ \tfrac{|\, x_1(t_i)|}{4(1-\mu)}]$$

corresponding to the left and right boundary branches of the trajectory funnel $X_v(x^{(0)})$, $u = \tilde{u}^*(x)$, $v(\cdot) \in V$.

B. Case $1/2 \le \mu < 2/3$

In the case $1/2 \le \mu < 2/3$ in the domain $Q = \mathrm{int}\, C$ the min-max robust stabilization is also possible. Here the following inequality is valid: $\frac{2\lambda}{1+\mu} < \frac{\lambda}{1-\mu} < \frac{2\lambda}{\mu} \le \frac{2\lambda}{1-\mu}$. It is easy to show that if at the instant t_i for the state $x^{(i)} = (x_1(t_i), 0)$ of system (1) the following conditions

$$\begin{gathered}\tfrac{\theta\lambda}{1-\mu} < |\, x_1(t_i)| < \tfrac{2\lambda}{\mu} \\ \theta = 4/(\sqrt{1+3\mu}+1-\mu) > 1\end{gathered} \tag{14}$$

are satisfied, then the behavior of the maximal (left) boundary branch of the trajectory funnel $X_v(x^{(i)})$, $u = u^*$, $v(\cdot) \in V$ is the same as above in case $0 < \mu < 1/2$. But the minimal (right) boundary branch $x(x^{(i)}, u^*, v_*, t)$, corresponding to the segment of the trajectory funnel (*IF*) starting from the vertex $x^{(i)}$ to the next arrival point on the axis $x_2 = 0$, does not intersect axis $x_1 = 0$, i.e., it is not a simple oscillation (of type 1). It constitutes one half of the ellipse (semi-ellipse) with the center at the corresponding extreme point within its "zone of sticking" $[-\frac{\lambda}{1-\mu}, \frac{\lambda}{1-\mu}]$. Note that in the following vertex $x^{(i)}$ the next arrival point on the axis $x_2 = 0$ the module of the coordinate x_1 of the minimal boundary branch $x(x^{(i)}, u^*, v_*, t)$ is less than one of the maximal boundary branch $x(x^{(i)}, u^*, v^*, t)$. In other words, for the elements of *IF* $X_v(x^{(i)})$ at the interval $[t_i, t_{i+1}]$ between two sequential extremal points of the coordinate x_1 the swing damping condition is satisfied, namely, $|\, x_1(t_{i+1})| < |\, x_1(t_i)|$. Therefore, (14) is the condition of swing damping for the considered case of $1/2 \le \mu < 2/3$. By the same way as before, we obtain the expression for the positionally correcting control $(i = 0, 1, 2, ..., \infty)$

$$\tilde{u}^*(x) = -\frac{|\, x_1(t_i)|}{3}\,\mathrm{sign}\{x_2\}, t_i : x_2(t_i) = 0 \tag{15}$$

ensuring the min-max robust stabilization of system (1). The behavior of the maximal (left) boundary branch of the trajectory funnel $X_v(x^{(0)})$ will be the same as in the case of $0 < \mu < 1/2$, i.e., it represents a spiral curve twisted toward the asymptotic point at the origin. The corresponding minimal boundary branch in this case consists of the adjoined semi - ellipses, with joint points on the semi-axis OX_1, as well as the vertex $x^{(0)}$ of trajectory funnel $X_v(x^{(0)})$. Their centers are located in the corresponding extreme points of the "zones of sticking" $Z_i^R := [-\frac{|x_1(t_i)|}{3(1-\mu)}, \frac{|x_1(t_i)|}{3(1-\mu)}]$. Moreover, this sequence of adjoining semi-ellipses asymptotically tends to the origin.

SUMMARY 6.2. The contraction of these semi-ellipses to the origin at $t_i \to \infty$ is ensured according to the obtained law of the min-max robust stabilization which forces both sticking zones (for the left and right boundaries of the funnel) to converge to the origin making the "*current correction*" using (13) and (15) as the control law.

C. Case $\mu \geq 2/3$

If $\mu \geq 2/3$, then $\frac{2\lambda}{\mu} \leq \frac{\lambda}{1-\mu}$, i.e., the interval $(-\frac{2\lambda}{\mu}, \frac{2\lambda}{\mu})$ of the initial states $x^{(0)} = (x_1(t_0), 0)$, admissible at $\mu < 2/3$ and admitting the min-max stabilization, belongs to "zone of sticking" $[-\frac{\lambda}{1-\mu}, \frac{\lambda}{1-\mu}]$ for the minimal (right) boundary branch of the trajectory funnel $X_v(x^{(0)})$.

SUMMARY 6.3. So, at $\mu \geq 2/3$ the minimal boundary branch degenerates at the initial point, and therefore, in this case **the min-max robust stabilization is impossible**.

VII. CONCLUSIONS

- Here we propose the new technique for designing of the feedback min-max robust stabilizing control for the class of two-dimensional parametrically perturbed oscillating systems.
- The corresponding analytical expression for worst parametric perturbation also is obtained. The behavior of the trajectories with extremal min-max swing amplitudes is analyzed: first, we find the maximal possible (on $v \in V$) amplitude value of the oscillation $x_1(u, v, t, x_0)$ between two sequential zeros of its derivative (in fact, at semi-period) and then we find the low (on $u \in U$) bound of the maximal possible value of the oscillation amplitude found before; in the max-min problem the order of the extremal search is inverse.
- As the result, one may formulate these procedures as an auxiliary differential game where the strategy set is selected (in view of the oscillatory character of the dynamics) $\{x : x_2 = 0\}$ and the value $x_1(u, v, t_1, x_0)$ is defined as the cost function (t_1 is the first moment when the trajectory reaches the set $\{x : x_2 = 0\}$); this differential game is shown to have a saddle-point.
- The unstable limit cycle for these trajectories in the phase plane is found. It is proved that its interiority is the attractive neighborhood of the origin representing the domain of min-max robust stabilizability.

REFERENCES

[1] A. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer, Dordrecht, 1988.

[2] A. I. Panasyuk, "Equations of attainable set dynamics, part 1: Integral funnel equations," *Journal of Optimization Theory and Applications*, vol. 64 (2), pp. 349–366, 1990.

[3] I. I. Turko and V. V. Filippov, "On the acyclicity of integral funnels," *Differential Equations*, vol. 45 (2), pp. 279–281, 2009.

[4] V. N. Ushakov, A. R. Matviichuk, and A. V. Ushakov, "Approximations of attainability sets and of integral funnels of differential inclusions," *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, vol. 4, p. 23?39, 2011.

[5] I. A. Sultanov, "Control processes described by equations with incompletely determined functional parameters," *Automation and Remote Control*, vol. 41 (10), pp. 1356–1364, 1980.

[6] A. Butkovskiy, *Phase Portrait of Control Dynamical Systems*. Kluwer, 1991.

[7] V. Alexandrov and V. Zhermolenko, "Minimax stabilization of parametrically perturbed oscilatory system," *Moscow University Mechanics Bulletin*, vol. 53 (6), pp. 1–6, 1998.

[8] V. N. Zhermolenko, "Trajectory funnels of two-dimensional control systems," *Journal of Computer and Systems Sciences International*, vol. 45 (2), pp. 191–203, 2006.

[9] ——, "Maximum deviation of oscillating system of the second order with external and parametric disturbances," *Journal of Computer and Systems Sciences International*, vol. 46 (3), pp. 407–412, 2007.

[10] A. Formalskii, "The design of optimal feedback control for systems of second order," *Applied Mathematics*, vol. 1, pp. 301–30, 2010.

[11] A. Kurzhanski and I. Valyi, *Ellipsoidal Calculus for Estimation and Control*. Boston, MA: Birkhauser, 1997.

[12] A. Ilchmann and S. Trenn, "Input constrained funnel control with applications to chemical reactor models," *Systems & Control Letters*, vol. 53 (5), p. 361?375, 2004.

[13] A. Ilchmann, E. Ryan, and S. Trenn, "Tracking control: Performance funnels and prescribed transient behaviour," *Systems & Control Letters*, vol. 54, p. 655?670, 2005.

[14] F. Blanchini and S. Miami, *Set Theoretic Methods in Control. Systems &Control: Foundations & Applications*. Boston, MA: Birkhauser, 2008.

[15] A. Ilchmann and H. Schuster, "Pi-funnel control for two mas systems," *IEEE Transactions on Automatic Control*, vol. 54 (4), p. 918?923, 2009.

[16] N. Hopfe, A. Ilchmann, and E. Ryan, "Funnel control with saturation: Linear mimo systems," *IEEE Transactions on Automatic Control*, vol. 55 (2), p. 532?538, 2010.

[17] C. Hackl, A. Hofmann, and R. Kennel, "Funnel control in mechatronics: An overview," in *Proceedings of the 50-th IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, 2011, p. 8000?8007.

[18] C. Vazquez, L. Fridman, and J. Collado, "Generalized twisting controller of a parametrically excited crane," in *Variable Structure Systems (VSS), 2012 12th International Workshop on Digital Object Identifier*, 2012, pp. 111 – 116.

[19] C. Hackl, "Pi-funnel control with anti-windup and its application to speed control of electrical drives," in *Conference on Decision and Control (CDC), Florence, Italy*, 2013.

[20] C. Hackl, N. Hopfe, A. Ilchmann, M. Mueller, and S. Trenn, "Funnel control for systems with relative degree two," *SIAM Journal on Control and Optimization (accepted)*, vol. 51 (2), pp. 965–995, 2013.

[21] V. Boltyanski and A. Poznyak, *Robust Maximum Principle: Theory and Applications*. Birkhauser, Boston, 2012.

[22] T. Basar and G. G.J. Olsted, *Dynamic Noncooperative Game Theory*. SIAM, Philadelphia, 1999.

[23] R. Devaney, *Introduction to Chaotic Dynamical Systems*. Benjamin-Cummings Publishing Co., Subs. of Addison Wesley Longman, US, 1986.

[24] A. Andronov, A. Vitt, and S. E. Khaikin, *Theory of oscillators. Dover, 1966*. Dover, 1966.

[25] A. Poznyak, *Advanced Mathematical Tools for Automatic Control Engineers: Deterministic Techniques*. Elsevier, 2008.