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A Family of Continuous State Feedback Synthesis: Lyapunov Approach

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Abstract— **A family of continuous output feedback synthesis is analyzed using strict non smooth Lyapunov functions, such that compensation of growing perturbations together with state variables is shown. Indeed, from twisting algorithm to pd control law, a general continuous family of control algorithms are considered. A strict non-smooth Lyapunov function is proposed allowing to create tuning rules for the gains of a family of controllers such that global finite time stability of the origin is shown. The proposed methodology estimate an upper bound for convergence time of the closed loop system spite of growing perturbation with respect to the state. To illustrate performance and robustness properties a numerical experiment is presented, using one-link pendulum as a test bed.**

Keywords: Second-order sliding modes; Lyapunov function; Stability analysis.

I. Introduction

Second order sliding mode algorithms (SOSM) have become very popular, particularly with electromechanical systems, due to their finite-time convergence to the origin in the presence of bounded, persisting external disturbances and parametric uncertainties (Orlov *et al.*, 2011), added to these properties, some particular versions of SOSM algorithms, like the twisting algorithm, have the advantage of considering the Coulomb friction as part of the controller (see (Emelyanov *et al.*, 1986)). However, the "chattering" phenomenon is always present in SOSM algorithms, turning into one of the principal problems of sliding mode control techniques.

Several works on the literature have been devoted to modify SOSM algorithms to reduce chattering during last years (see (Orlov *et al.*, 2003), (Orlov, 2009) and references therein). An example of them is (Orlov *et al.*, 2011), where a smooth version of twisting algorithm has been developed applying the invariance principle to prove global asymptotic stability of the perturbed double integrator in spite of external growing perturbations. The authors design a family of controllers, nevertheless, they use a weak Lyapunov function design, and the upper bound for the growing perturbation is with respect to one state variable only. In (Santiesteban *et al.*, 2010), a strict non smooth Lyapunov function is proposed using a twisting algorithm, and the stability of the closed loop of a general mechanical system is shown. The strictness of this function allows to estimate the convergence time of the closed loop system to the origin, and therefore a deeply study of the robustness of the algorithm is allowed. Moreover, (Santiesteban, 2013) developed a strict Lyapunov function using the family of controllers of (Orlov *et al.*, 2011) as an extension of their gain restrictions. Note that in this work the upper bound of the growing perturbation is with respect to both state variables.

In this paper, a new strict Lyapunov function is proposed for perturbed systems, which results in the improvement of the convergence time estimation and an easier methodology to compute the gain constraints, compared with (Santiesteban, 2013). A continuous stabilizing feedback controller is designed to show the results using a one-link pendulum affected by Coulomb friction and growing perturbations with respect to the state as a test bed.

The structure of the paper is as follows: basic assumptions of the systems under interest and some mathematical background are given in Section II. The proposed Theorem to stabilize perturbed systems is developed in Section III. In section IV to support theoretical results, a numerical example is shown and in section V are the conclusions of this work.

II. Preliminaries

This section states definitions and the mathematical background, as well as the class of systems considered throughout the paper.

II-A. Considered systems

The general model of second-order mechanical systems will be considered. It is described by equations of the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x,y) + g(x,y)\tau + \delta(t,x,y) \end{aligned} \qquad (1)$$

with the smooth function $g(x,y) \neq 0$. The origin of (1) is an equilibrium point and x and $y \in \mathbb{R}$ are scalar state variables. The function $f(x,y)$ represents the nominal known part of the system dynamics, it can be discontinuous, and $\delta(t,x,y)$ the uncertainties such as growing perturbations with respect to the state variables (see (Orlov *et al.*, 2011)).

Since the right hand side of the equation (1) has discontinuous terms, their solutions are understood in the Filippov sense (see (Filippov, 1988)). It is assumed that the full vector state of the dynamic system (1) is available for measurement, note that this assumption is not restrictive because there are several works in the literature about observers and differentiators design (see (Angulo *et al.*, 2013), (Moreno, 2013), (Orlov *et al.*, 2011), (Dávila *et al.*, 2005) and references therein).

For system (1) the following controller design is proposed

$$\tau = \frac{1}{g(x,y)}\left(U - f(x,y)\right) \tag{2}$$

where $U \in \mathbb{R}$ is a new control input given by

$$U = -k_1|x|^{\frac{\alpha}{2-\alpha}}\mathrm{sgn}(x) - k_2|y|^{\alpha}\mathrm{sgn}(y) \tag{3}$$

with k_1, k_2 are positive constants and $0 \le \alpha \le 1$.

Let consider an external bounded perturbation $\delta(x,y)$ given by

$$|\delta(x,y)| \le \mu_y|y|^{\alpha} + \mu_x|x|^{\frac{\alpha}{2-\alpha}}, \tag{4}$$

where $\mu_y, \mu_x \in \mathbb{R}$ are positive constants and α is defined as 3. Therefore system 1 is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= U + \delta(x,y) \end{aligned} \tag{5}$$

Previous results for the considered systems defined above are shown in the following sections.

II-B. *Mathematical background*

A main contribution in (Santiesteban, 2013) is reported here by the following definitions and theorems. The notation of some theorems was modified for readability.

Let consider the positive definite Lyapunov function

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2}k_1^2|x|^{\frac{4}{2-\alpha}} + k_1|x|^{\frac{2}{2-\alpha}}y^2 \\ &+ |x|^{\frac{3}{2-\alpha}}\mathrm{sgn}(x)y + \frac{1}{2(2-\alpha)}y^4. \end{aligned} \tag{6}$$

Note that (6) is continuous everywhere but not differentiable for $x = 0$. Since (6) is a strict Lyapunov function of system (1-3), then finite time stability can be concluded (see (Bacciotti and Rosier, 2001)).

Theorem 1: (Santiesteban, 2013) Unperturbed System (5) has finite time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \le \frac{1}{\zeta_{min}}\gamma_{max}^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(x(0),y(0)) \tag{7}$$

as an estimation of the convergence time, with $V(x,y)$ defined as (6),

$$\begin{aligned} \zeta_{min} &= \min\Big\{k_1k_2 - \frac{\alpha}{1+\alpha},\ k_1 - \frac{1}{1+\alpha}k_2, \\ k_1k_2 &- \frac{3(1+\alpha)}{2(2-\alpha)},\ k_2 - \frac{3}{4}(1-\alpha)\Big\} \end{aligned} \tag{8}$$

and

$$\gamma_{max} = \max\left\{\lambda_{\max}\left(\begin{bmatrix} \frac{2-\alpha}{2}k_1^2 & \frac{1}{2} \\ \frac{1}{2} & k_1 \end{bmatrix}\right), \frac{1}{2(2-\alpha)}\right\} \tag{9}$$

if

$$\begin{aligned} k_1 &> \frac{1}{1+\alpha}\max\{\alpha, k_2\} \\ k_2 &> \frac{3}{4}(1-\alpha) \\ k_1k_2 &> \left(\frac{3}{2}\right)\frac{1+\alpha}{2-\alpha} \end{aligned} \tag{10}$$

for $0 \le \alpha < 1$ and asymptotic stability for $\alpha = 1$.

Remark 1: Note that the analyzed controllers in equation (3) are a continuous family of control algorithms that contain from twisting algorithm to pd control law acting in the closed-loop system (5). Considering $\alpha = 0$, the algorithm (3) is known as twisting algorithm, a well-known controller and considering $\alpha = 1$ it is not difficult to show that the algorithm (3) is known as PD control law, another well-known controller.

Now, consider a positive definite Lyapunov function of the form

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2}k_1\left(k_1 - \mu_x\mathrm{sgn}(x)\right)|x|^{\frac{4}{2-\alpha}} \\ &+ k_1|x|^{\frac{2}{2-\alpha}}y^2 + |x|^{\frac{3}{2-\alpha}}\mathrm{sgn}(x)y + \frac{1}{2(2-\alpha)}y^4 \end{aligned} \tag{11}$$

This function is continuous everywhere but not differentiable on $x = 0$.

Theorem 2: (Santiesteban, 2013) System (5) has finite time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \le \frac{1}{\zeta_{minp}}\gamma_{max}^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(x(0),y(0)) \tag{12}$$

as an estimation of the convergence time with $V(x,y)$

defined as (11),

$$\zeta_{minp} = \min\Big\{ k_1(k_2-\mu_y) - \frac{6}{(1+\alpha)(2-\alpha)}, \; k_2-\mu_y, \; k_2-\mu_y-\frac{3}{2(1+\alpha)}, \; k_2-\mu_y-\frac{6}{3+\alpha}, \; \frac{k_1}{2}-4\frac{\alpha}{(3+\alpha)(2-\alpha)}\mu_x, \; k_1-\frac{\alpha}{1+\alpha}\mu_x, \; \frac{1}{2}(k_1-\mu_x)-\frac{1}{1+\alpha}(k_2-\mu_y)\Big\} \quad (13)$$

and

$$\gamma_{max} = \max\Big\{\lambda_{max}(P_1), \frac{1}{2(2-\alpha)}\Big\}, \quad (14)$$

with $P_1 = \begin{bmatrix} \frac{2-\alpha}{2}k_1(k_1-\mu_x\text{sgn}(x)) & \frac{1}{2} \\ \frac{1}{2} & \frac{2-\alpha}{2}k_1 \end{bmatrix}$. Notice that $\lambda_{max}(P_1)$ denotes the largest eigenvalue of matrix P_1.

if

$$\begin{aligned} k_1 &> \max\Big\{4\frac{\alpha}{(3+\alpha)(2-\alpha)}, \frac{\alpha}{1+\alpha}\Big\}\mu_x \\ k_2 &> \max\Big\{\mu_y, \; \mu_y+\frac{3}{2(1+\alpha)}\Big\} \\ \frac{1}{2}(k_1-\mu_x) &> \frac{1}{1+\alpha}(k_2-\mu_y) > \frac{6}{(3+\alpha)(1+\alpha)}\mu_x \\ k_1(k_2-\mu_y) &> \frac{6}{(1+\alpha)(2-\alpha)} \\ 1 &< k_1^2(k_1-\mu_x\text{sgn}(x)) \end{aligned} \quad (15)$$

for $0 \le \alpha < 1$ and asymptotic stability for $\alpha = 1$.

Now, in the following sections the main results of this paper will be developed.

III. Main results

In this section a new strict Lyapunov function is proposed for the perturbed system (5).

Let consider a positive definite Lyapunov function of the form

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2}(k_1-\mu_x\text{sgn}(x))^2|x|^{\frac{4}{2-\alpha}} \\ &+ (k_1-\mu_x\text{sgn}(x))|x|^{\frac{2}{2-\alpha}}y^2 \\ &+ |x|^{\frac{3}{2-\alpha}}\text{sgn}(x)y + \frac{1}{2(2-\alpha)}y^4 nlf) \end{aligned} \quad (16)$$

This function is continuous everywhere but not differentiable on $x = 0$.

Theorem 3: System (5) has finite time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \le \frac{1}{\zeta_{minp}}\gamma_{max}^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(x(0),y(0)) \quad (17)$$

as an estimation of the convergence time with $V(x,y)$ defined as (11),

$$\zeta_{minp} = \min\Big\{\eta_x\eta_y - \frac{3}{2}\frac{1+\alpha}{2-\alpha}, \; \frac{1}{2-\alpha}\Big(\eta_y-\frac{3}{2}(1+\alpha)\Big), \; \eta_x-\eta_y\frac{1}{1+\alpha}, \; \eta_x-\frac{\alpha}{1+\alpha}\Big\} \quad (18)$$

$$\gamma_{max} = \max\Big\{\lambda_{max}(P_2), \frac{1}{2(2-\alpha)}\Big\}, \quad (19)$$

with $P_2 = \begin{bmatrix} \frac{2-\alpha}{2}(k_1-\mu_x\text{sgn}(x))^2 & \frac{1}{2} \\ \frac{1}{2} & \frac{2-\alpha}{2}(k_1-\mu_x\text{sgn}(x)) \end{bmatrix}$.

if

$$\begin{aligned} k_2 &> \mu_y+\frac{3}{2}(1-\alpha) \\ k_1 &> \mu_x+\frac{1}{1+\alpha}\max\{\eta_y, \; \alpha\} \\ \eta_x\eta_y &> \frac{3}{2}\Big(\frac{1+\alpha}{2-\alpha}\Big) \end{aligned} \quad (20)$$

with

$$\eta_x = k_1-\mu_x; \;\; \eta_y = k_2-\mu_y \quad (21)$$

for $0 \le \alpha < 1$ and asymptotic stability for $\alpha = 1$.

An sketch of the proof of Theorem 3 is shown on Appendix I.

IV. Numerical Experiments

To illustrate the algorithm performance consider a tracking problem of the one-link pendulum system affected by Coulomb friction and external perturbations bounded by inequality (4). The state equation of a controlled one-link pendulum (see Fig. 1) is given by

$$(ml^2+J)\ddot{q} = mglsin(q) - F(\dot{q}) + \tau + \delta(t,q,\dot{q}) \quad (22)$$

where q is the angle made by the pendulum with the vertical, m is the mass of the pendulum, l is the distance to the center of mass, J is moment of inertia of the pendulum about the center of mass, g is the gravity acceleration, τ is the control torque. The friction force F is described by

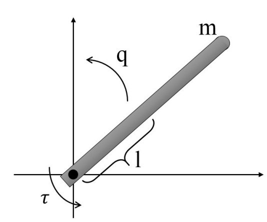

Fig. 1. The one-link pendulum system.

$$F(\dot{q}) = \rho_v\dot{q} + \rho_c sign(\dot{q}). \quad (23)$$

where $\rho_v > 0$ denotes the viscous friction coefficient and $\rho_c > 0$ denotes the Coulomb friction level. Suppose that the uncertainty term $\delta(t, q, \dot{q})$ is bounded by growing terms as in (4).

The control objective is to drive the one-link pendulum to a known trajectory in exact finite time, *i.e.*

$$q(t) - r(t) = 0 \tag{24}$$

where $r(t) = \frac{1}{8}sin(t)$ even in the presence of an admissible external disturbance (4). Let $x = q$ and $y = \dot{q}$, then equation (22) can be written in the state space form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{(ml^2 + J)}\Big(mglsin(x) - \rho_v y + \rho_c sign(y) \\ &+ \tau + \delta(t, x, y)\Big) \end{aligned} \tag{25}$$

Let the tracking error given by

$$e(t) = x(t) - r(t). \tag{26}$$

Using equations (25), the error dynamics are described by

$$\begin{aligned} (ml^2 + J)\ddot{e} &= mglsin(x) - \rho_v y + \rho_c sign(y) + \tau \\ &+ \delta(t, x, y) - (ml^2 + J)\ddot{r}. \end{aligned} \tag{27}$$

Let us choose the control in the form with $\alpha = \frac{1}{5}$

$$\begin{aligned} \tau &= \big(ml^2 + J\big)\,\ddot{r} - k_1|e|^{\frac{\alpha}{2-\alpha}}sign(e) - k_2|\dot{e}|^{\alpha}sign(\dot{e}) \\ &\quad -mglsin(x) + \rho_v y \end{aligned} \tag{28}$$

with $\alpha = 1/5$ and where (15) are satisfied. Parameters of a real laboratory one-link pendulum system are considered: the mass of the pendulum is $m = 0.5234kg$, the length of the link $l = 0.108m$, and the inertia about the center of the mass $j = 0.006kg \cdot m^2$. The Coulomb friction is given by $\rho_v = 0.00053N \cdot m \cdot s/rad$ and the viscous friction as $\rho_c = 0.05492N \cdot m$.

The initial conditions for the pendulum, are fixed as $\theta(0) = \pi\ rad$ and $\dot{\theta}(0) = 0\ rad/seg$ for the position and velocity, respectively . In Figure 2 shows the dynamics of one-link pendulum system in closed loop affected by the bounded external perturbations. The simulations of the double integrator use fixed gains as in (18-21),*i.e.* $k_1 = 7$, $k_2 = 3{,}2$. In this numeric experiment, the double integrator is affected by $\delta(x, y) = \sin(100t)\left(3|x|^{\frac{\alpha}{2-\alpha}} + |y|^{\alpha}\right)\ N \cdot m$ denotes a high frequency uncertainty term bounded by (4). This kind of tern is commonly used in control theory.

As Fig. 2 shows, the closed loop system with the studied design achieves the control objective and the one-link pendulum follows the desired trajectory in spite of bounded external uncertainties. Moreover, The strict Lyapunov function gives an estimation of convergence time. The estimation of convergence time using equation (17) gives

$$t_{reach} \leq 107{,}91\ \ sec \tag{29}$$

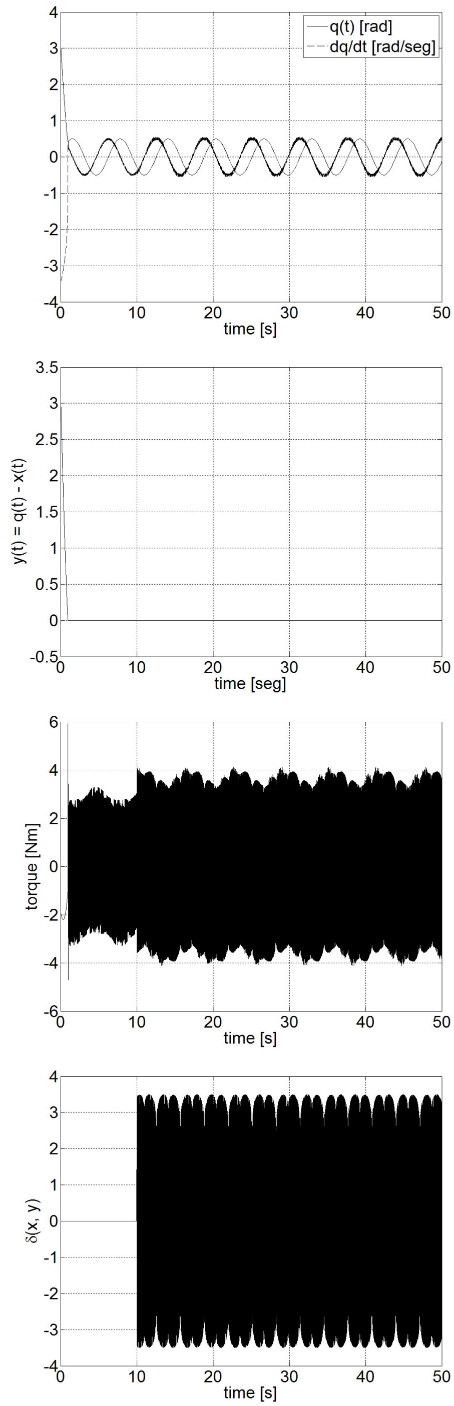

Fig. 2. Tracking stabilization of a one-link pendulum.

while in numerical simulations the convergence time is less than 10 seconds. In comparison, the estimation of convergence time given in [(Santiesteban, 2013)] is

$$t_{reach} \leq 251{,}5877\ \ sec \tag{30}$$

V. CONCLUSIONS

A family of continuous output feedback control is tuned such as global finite time convergence with respect to the growing perturbations is shown. With this aim, a non-smooth strict Lyapunov function is proposed allowing an estimation of the upper bound of the convergence time. The performance of the proposed algorithm was shown by solving the tracking control problem of a one-link pendulum in spite of bounded external and parametric perturbations. The closed loop mechanical system showed to be robust and provide nice performance in spite of unknown but bounded uncertainties. For future work, this result can be easily generalized for multidimensional case. Moreover, it can be extended when a state variable is not available for measurement, then a finite time observer can be applied such as super-twisting algorithm.

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VI. Appendix I

Proof of theorem 3: In order to show that function $V(x,y)$ is positive definite, let us describe it as follows

$$V(x,y)=|x|^{\frac{2}{2-\alpha}}\left(\rho^T P\rho\right)+\frac{1}{2(2-\alpha)}y^4 \tag{31}$$

where $\rho^T=[|x|^{\frac{1}{2-\alpha}} \quad y]$ and

$$P=\begin{pmatrix} \frac{2-\alpha}{2}\left(k_1-\mu_x \mathrm{sgn}(x)\right)^2 & \frac{1}{2} \\ \frac{1}{2} & \frac{2-\alpha}{2}\left(k_1-\mu_x \mathrm{sgn}(x)\right) \end{pmatrix} \tag{32}$$

Now, if $det(P)>0$ then function (31) is definite positive, then the following inequalities must hold at all time:

$$(2-\alpha)^2\cdot\frac{\left(k_1-\mu_x \mathrm{sgn}(x)\right)^3}{4}-\frac{1}{4}>0 \tag{33}$$

since $0\le\alpha\le 1$, if

$$\left(k_1-\mu_x \mathrm{sgn}(x)\right)^3>1\ge\frac{1}{(2-\alpha)^2} \tag{34}$$

A lower bound for function (31) can be written as follows

$$\gamma_{min}\left(|x|^{\frac{2}{2-\alpha}}\|\rho\|^2+y^4\right)\le V(x,y) \tag{35}$$

where $\gamma_{min} = \min\left\{\lambda_{min}(P),\frac{1}{2(2-\alpha)}\right\}$. Notice that $\lambda_{min}(P)$ denotes the minimum eigenvalue of matrix P. Now, an upper bound for the Lyapunov function (31) can be written as follows

$$V(x,y)\le\gamma_{max}\left(|x|^{\frac{2}{2-\alpha}}\|\rho\|^2+y^4\right) \tag{36}$$

where $\gamma_{max}=\max\left\{\lambda_{max}(P),\frac{1}{2(2-\alpha)}\right\}$. The time derivative of (31) along the trajectories of the perturbed system (5) is given by (after some algebraic simplifications)

$$\begin{aligned}\dot{V}(x,y) \le\ & -2\left(k_1-\mu_x\right)\left(k_2-\mu_y\right)|x|^{\frac{2}{2-\alpha}}|y|^{\alpha+1}\\ &+\frac{3}{2-\alpha}|x|^{\frac{1+\alpha}{2-\alpha}}y^2-\left(k_1-\mu_x\right)|x|^{\frac{3+\alpha}{2-\alpha}}\\ &-\left(k_2-\mu_y\right)|x|^{\frac{3}{2-\alpha}}|y|^{\alpha}\mathrm{sgn}(xy)\\ &-\frac{2}{2-\alpha}\left(k_2-\mu_y\right)|y|^{\alpha+3}\end{aligned} \tag{37}$$

Using equation (21), then the equation above can be written as follows

$$\begin{aligned}\dot{V}(x,y) \le\ & -|y|^{\alpha+1}\Bigg(\eta_x\eta_y|x|^{\frac{2}{2-\alpha}}\\ &-\frac{3}{2-\alpha}|x|^{\frac{1+\alpha}{2-\alpha}}|y|^{1-\alpha}+\frac{1}{2-\alpha}\eta_y|y|^2\Bigg)\\ &-|x|^{\frac{2}{2-\alpha}}\Bigg(\eta_x|x|^{\frac{1+\alpha}{2-\alpha}}-\eta_y|x|^{\frac{1}{2-\alpha}}|y|^{\alpha}\\ &+\eta_x\eta_y|y|^{\alpha+1}\Bigg)\end{aligned} \tag{38}$$

In order to show that $\dot{V}(x,y)\le 0$, consider the following inequalities

$$\begin{aligned}|x|^{\frac{1+\alpha}{2-\alpha}}|y|^{1-\alpha} &\le \frac{1}{r_c}\gamma_c^{r_c}|x|^{\frac{2}{2-\alpha}}+\frac{1}{s_c}\gamma_c^{-s_c}|y|^2,\\ & for\ \ r_c=\frac{2}{1+\alpha}, s_c=\frac{2}{1-\alpha}\\ |x|^{\frac{1}{2-\alpha}}|y|^{\alpha} &\le \frac{1}{r_d}\gamma_d^{r_d}|x|^{\frac{1+\alpha}{2-\alpha}}+\frac{1}{s_d}\gamma_d^{-s_d}|y|^{1+\alpha},\\ & for\ \ r_d=1+\alpha, s_d=\frac{1+\alpha}{\alpha}\end{aligned} \tag{39}$$

Let $\gamma_c = \gamma_d = 1$, then equation (38) simplifies as follows

$$\begin{aligned} \dot{V}(x,y) \quad \leq \quad & -|y|^{\alpha+1}\left(|x|^{\frac{2}{2-\alpha}}\left(\eta_x\eta_y - \frac{3}{2}\frac{1+\alpha}{2-\alpha}\right)\right. \\ & \left. +\frac{1}{2-\alpha}|y|^2\left(\eta_y - \frac{3}{2}(1-\alpha)\right)\right) \\ - \quad & |x|^{\frac{2}{2-\alpha}}\left(|x|^{\frac{1+\alpha}{2-\alpha}}\left(\eta_x - \frac{1}{1+\alpha}\eta_y\right)\right. \\ & \left. +\eta_y|y|^{\alpha+1}\left(\eta_x - \frac{\alpha}{1+\alpha}\right)\right) \end{aligned} \tag{40}$$

if

$$\begin{aligned} (k_1 - \mu_x)(k_2 - \mu_y) \quad & > \quad \frac{3}{2}\left(\frac{1+\alpha}{2-\alpha}\right) \\ k_2 \quad & > \quad \mu_y + \frac{3}{2}(1-\alpha) \\ k_1 \quad & > \quad \mu_x + \frac{1}{1+\alpha}\max\{\eta_y, \ \ \alpha\} \end{aligned} \tag{41}$$

hold, the time derivative of the Lyapunov function $V(x,y)$ is negative definite. Since the inequalities (41) are in terms of the positive constants γ_c and γ_d, a solution can be found, always. In order to show the stability of the system (5), let us write the equation (36) as follows

$$V(x,y) \leq \gamma_{max}\left(|x|^{\frac{1}{2-\alpha}} + |y|\right)^4 \tag{42}$$

$$\left(\frac{V(x,y)}{\gamma_{max}}\right)^{\frac{1}{4}} \leq \left(|x|^{\frac{1}{2-\alpha}} + |y|\right) \tag{43}$$

and the let us write the equation (37) as follows

$$\dot{V}(x,y) \leq -\zeta_{minp}\left(|x|^{\frac{1}{2-\alpha}} + |y|\right)^{3+\alpha} \tag{44}$$

where ζ_{minp} is given by (18). Then equation (42-44) can be written as

$$\dot{V}(x,y) \leq -\zeta_{minp}\left(\frac{V(x,y)}{\gamma_{max}}\right)^{\frac{3+\alpha}{4}} \tag{45}$$

Then finite time stability of system (5)can be concluded. To estimate an upper bound for time convergence, let us consider the following comparison system

$$\dot{\omega} = -a\omega^{\frac{3+\alpha}{4}} \tag{46}$$

The solution of this system is $\omega(t) = (\omega^{\frac{1-\alpha}{4}}(0) - at)^4$, and thus the estimation for reaching time is $t_{reach} = \frac{1}{a}\omega^{\frac{1-\alpha}{4}}(0)$. Summing up, an estimation of an upper bound for the reaching time of the system (5) can be calculated as equation (12).

When $\alpha = 1$, equation 45 is as follows

$$\dot{V}(x,y) \leq -\frac{\zeta_{minp}}{\gamma_{max}}V(x,y) \tag{47}$$

then asymptotic stability of the closed loop system (5) is shown.

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