Congreso Nacional de Control Automático 2013
Ensenada, Baja California, Mexico, Octubre 16-18, 2013

Output Integral Sliding Mode Observer for Linear Time Variant Systems

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Abstract— **This paper is devoted to the construction of a hierarchical observer for a linear time variant system. It presents an extension of the Output Integral Sliding Mode observer to the time variant case, and gives an algebraic procedure in order to reconstruct the state right after the first moment, if we assume that the sliding mode exist and the equivalent control is available**

Keywords: Output Integral Sliding Mode, Hierarchical Observer, Linear Time Variant System.

I. INTRODUCTION

The sliding modes control is a suitable way to control and estimate uncertain/perturbed systems (Utkin, 1992; Shtessel *et al.*, 2013). The only restriction for exact compensation of the uncertainties/perturbations is that they must be matched. The disadvantages of this kind of control are the presence of a reaching phase, making the system not robust to any disturbance/uncertainty, and the presence of chattering during this phase (Utkin, 1992). However, in the last years there have been developed different variants of sliding modes techniques that eliminate the chattering and the reaching phase. The Integral Sliding Modes (see (Matthews and DeCarlo, 1988; Utkin and Shi, 1996; Castaños and Fridman, 2006), for more details), were developed in order to eliminate the reaching phase from the controlled system, i.e. the system is invariant with respect to uncertainties/perturbations since the initial moment. One of the main issue of this theory is the design of the sliding surface. In (Castaños and Fridman, 2006) it was proved that in the presence of unmatched disturbances the best way to minimize the effects of the uncertainty is to choose the integral sliding surface gain matrix as the pseudoinverse of the input matrix. So far this design methodology had been developed for linear time invariant systems and non linear systems, considering an invariant time sliding matrix (Castaños and Fridman, 2006; Rubagotti *et al.*, 2011). *What can we do if the input matrix is time variant?* The main problem when it is trying to implement an integral sliding mode control is that we require full knowledge of the state vector, including the initial condition. In order to solve this problem it is necessary to design an observer. There are two possibilities for the design of an output sliding mode control. One is to use an output feedback control, i.e. to design a sliding surface using only output information, such that the dynamics of the system fulfils the given requirements (Shtessel *et al.*, 2013). Another way is to construct an observer in order to estimate the state, and use this estimation in a control law instead of the real ones. Following this approach there are two viewpoints. One is devoted to eliminate the tracking error between the observer state and the system state in order to reconstruct the state as fast as possible (Fridman *et al.*, 2009; Fridman *et al.*, 2011). The other is dedicated to the state estimation step by step (Hashimoto *et al.*, 1990; Floquet and Barbot, 2004). In (Bejarano *et al.*, 2007) the use of a hierarchical observer is proposed based on an output integral sliding mode technique for linear time invariant systems, allowing to reconstruct the state right after the initial time even in the presence of unknown inputs. The aim of this paper is to propose a methodology to design a hierarchical observer for linear time variant systems in order to reconstruct the state since the first moment using the time variant observability matrix. This paper is organized as follows: First we present the definition of an uncertain linear time variant system and the necessary assumptions follows by the main objective. Then in section III we present a formal definition of a time variant output integral sliding mode. In section IV we present the design procedure for a hierarchical observer, that would help us to reconstruct the state needed for the time variant output integral sliding mode. The applicability of the proposed approach is illustrated by simulation results. Section VI concludes the paper.

II. PROBLEM FORMULATION

Let us consider the following uncertain linear time variant system defined on the time interval $[t_0, t_f]$

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\left(u(t) + \phi(x,t)\right), \\ y(t) &= C(t)x(t), \qquad x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector and $y(t) \in \mathbb{R}^p$ is the output vector. The function $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ represents the uncertainties due to parameter variations, unmodelled dynamics and/or exogenous disturbances; and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R} \to \mathbb{R}^{p\times n}$ are continuously differentiable matrix functions. In order to eliminate the effects of the disturbances we need to define a nominal system for (1) assuming $\phi = 0$.

$$\dot{x}_n(t) = A(t)x_n(t) + B(t)u_n(t), \qquad x_n(t_0) = x_0, \quad (2)$$

where u_n be the nominal control designed to achieve any stable control objective.

Remark 2.1: Note that the trajectories (1) and (2) could be in general quite different. Also the trajectory of (1) is not unique, while the trajectory of (2) is unique.

Assuming that the state vector of (1) is unmeasurable, it is necessary to design an observer to reconstruct this state vector and use it in a suitable control law. Our aim is to design an algebraic hierarchical observer (Bejarano *et al.*, 2007) that allow us to approximate step by step the states of the time variant system (1) based on the measured outputs since the first moment, i.e. $\hat{x}(t) = x(t)$, where $\hat{x}(t)$ is the approximate state. The approximate states $\hat{x}(t)$ will be used to design an output integral sliding mode control (Bejarano *et al.*, 2009) for an uncertain linear time variant system on the based of a nominal system (2), allowing the trajectory of (1) be equal to the one of (2), i.e. $x(t) = x_n(t)$ for all $t \in [t_0, t_f]$. Obviously the control objective is achieved only if the equivalent control is able to reconstruct the negative of the matched uncertainty/perturbation. For the proposed sliding mode approach it is necessary to assume that:

- A1 rank $B(t) = m, \quad \forall t \in [t_0, t_f]$
- A2 The uncertainty/disturbance $\phi(x,t)$ is bounded: $\|\phi(x,t)\| \leq \phi_{Max}$, for all $t \in [t_0, t_f]$, where $\phi_{Max} \in \mathbb{R}_+$ is given.
- A3 The matrices $A(t)$, $B(t)$ and $C(t)$ are $l-1$ times continuously differentiable matrix functions in the time interval $[t_0, t_f]$, and these matrices and their derivatives are bounded and known, and $l_c, l_o \in \mathbb{Z}_+$ and $l = \max\{l_c, l_o\}$.
- A4 The pair $(A(t), B(t))$ is controllable in $[t_0, t_f]$ (Rugh, 1993; Kratz and Liebscher, 1998) with controllability index l_c.
- A5 The pair $(A(t), C(t))$ is observable in $[t_0, t_f]$ (Rugh, 1993; Kratz and Liebscher, 1998), i.e. for some $t_o \in [t_0, t_f]$, the observability index l_o is the minimum integer such that rank$(\mathcal{O}_{l_o}(t_o)) = n$, where

$$\mathcal{O}_{l_o}(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ \vdots \\ N_{l_o-1}(t) \end{bmatrix} \in \mathbb{R}^{pc \times n}, \quad (3)$$

  where $N_0(t) = C(t)$ and $N_i(t) = N_{i-1}(t)A(t) + \frac{dN_{i-1}(t)}{dt}$ for $i = 1, \ldots, l_o$.
- A6 The initial condition is unknown but bounded, i.e. $\|x(t_0)\| \leq \mu$.

III. Output Integral Sliding Mode

In order to eliminate the effects of the uncertainties/perturbations from the system (1) we need to generalize the output integral sliding mode (Bejarano *et al.*, 2007; Bejarano *et al.*, 2009) to the time variant case. First, let $u(t) = u_n(t) + u_1(t)$, where u_n is any suitable nominal control, and $u_1(t)$ is the integral sliding mode control part guarantying the compensation of the uncertainty/perturbation $\phi(x,t)$, in the time interval $t \in [t_0, t_f]$. Consider the linear time variant system (1). Define the sliding surface

$$\begin{aligned} s(y,t) = G(t)y(t) - G(t_0)y(t_0) - \int_{t_0}^{t} \Big(\dot{G}(\tau)C(\tau)\hat{x}(t) \\ + G(\tau)C(\tau)\left(A(\tau)\hat{x}(\tau) + B(\tau)u_n(\tau)\right)\Big) d\tau, \end{aligned} \quad (4)$$

where $\hat{x}(t)$ is the observed state and $G(t) \in \mathbb{R}^{m \times n}$ is a continuously differentiable design matrix. In contrast with the integral sliding modes presented in (Castaños and Fridman, 2006; Dullerud and Paganini, 2000; Rubagotti *et al.*, 2011), we defined a time variant sliding mode manifold, where the matrix G is not assumed constant in t and $\dot{G}$ is known. Also note that the system is in the sliding mode at the first moment, i.e. $s(y(t_0), t_0) = 0$. Taking the derivative of the surface along the trajectories of (1)

$$\begin{aligned} \dot{s}(y,t) &= \Big(\dot{G}(t)C(t) + G(t)C(t)A(t)\Big)(x(t) - \hat{x}(t)) \\ &\quad + G(t)C(t)B(t)\left(u_1(t) + \phi(x,t)\right), \\ s(t_0) &= 0. \end{aligned} \quad (5)$$

The equivalent control (Utkin, 1992) that maintains the trajectories on the surface (5) is

$$\begin{aligned} &u_{1eq} = -\phi(x,t) \\ &- D^{-1}(t)\Big(\dot{G}(t)C(t) + G(t)C(t)A(t)\Big)(x(t) - \hat{x}(t)); \\ &D(t) = G(t)C(t)B(t), \quad t \in [t_0, t_f]. \end{aligned} \quad (6)$$

Substituting (6) in (1) yields the sliding mode dynamics

$$\begin{aligned} &\dot{x}(t) = \tilde{A}(t)x(t) + B(t)u_n(t) \\ &- B(t)D^{-1}(t)\Big(\dot{G}(t)C(t) + G(t)C(t)A(t)\Big)\hat{x}(t); \\ &y(t) = C(t)x(t); \quad x(t_0) = x_0, \quad t \in [t_0, t_f]. \end{aligned} \quad (7)$$

where the matrix

$$\begin{aligned} \tilde{A}(t) = \left[I - B(t)D^{-1}(t)G(t)C(t)\right]A(t) \\ - B(t)D^{-1}(t)\dot{G}(t)C(t). \end{aligned}$$

Remark 3.1: Notice that in the time invariant case the output integral sliding surface and the sliding mode dynamics are equal to the one proposed in (Bejarano *et al.*, 2007). Then, the proposed methodology constitute a generalization of the one in (Bejarano *et al.*, 2007) for the time variant case.

Remark 3.2: Note that the dynamics of the system under the sliding mode control are of the same order than the one of (1), but due to the equivalent dynamics it is not sure that the observability properties remains in the controlled system.

In order to assure that the observability properties remains in the pair $(\tilde{A}(t), C(t))$ we state the following lemma.

Lemma 3.1: When the number of outputs is less than or equal to the number of inputs and $\dot{G}(t)$ belongs to the null space of the matrix C, the matrix $\tilde{A}(t)$ always belongs to the null space of the matrix C, and, consequently, the pair $(\tilde{A}, C)$ is not observable.

Proof: Consider the system (1) with $p \leq m$ and $\text{rank}(C(t)B(t)) = p$ for all $t \in [t_0, t_f]$. Assume the control law $u(t)$ is designed as $u(t) = u_n(t) + u_1(t)$, where u_n is the nominal control used after the compensation of the matched disturbances/uncertainties and u_1 is designed to compensate the matched disturbances/uncertainties. Let us analyze the case when $p = m$ and later when $p < m$.

1) Consider the case when $p = m$. Define an output integral sliding surface as in (4). The matrix $G(t)$ must satisfy $\text{rank}\,(G(t)C(t)B(t)) = m$ for all $t \in [t_0, t_f]$, but this is only satisfied when $det(G(t)) \neq 0$. As we see in this section, the equivalent control takes the form (6). Substituting this equivalent control in the system (1) yields (7). Premultiplying $\tilde{A}(t)$ by $G(t)C(t)$ we get $G(t)C(t)\tilde{A}(t) = 0$. This means $\tilde{A}(t)$ belongs to the null space of $G(t)C(t)$ whenever $\dot{G}(t)$ belongs to the same space, and since $G(t)$ is a non singular matrix for all $t \in [t_0, t_f]$, then $\tilde{A}(t)$ belongs to the null space of C and it implies that $(\tilde{A}(t), C(t))$ is not observable.
2) Suppose that $p < m$. Let the output integral sliding surface be again as in (4), but since $\text{rank}(C(t)B(t)) = p$ and $p < m$, then there is not any matrix $G(t) \in \mathbb{R}^{m\times p}$ satisfying $\text{rank}(G(t)C(t)B(t)) = m$. That is why the sliding surface s can not be designed in a space of dimension greater than p. Let us define s in the space $\mathbb{R}^p$, with the matrix $G \in \mathbb{R}^{p\times p}$, and the sliding surface has the form (5). Redefine the control u_1 as $u_1(t) = \bar{F}(t)\bar{u}(t)$ where the matrix $\bar{F} \in \mathbb{R}^{m\times p}$ should satisfy $\text{rank}(G(t)C(t)B(t)\bar{F}(t)) = p$. Then $B(t)\bar{F}(t)$ can be considered as a new input matrix and $\bar{u}$ as a control input, and we go back to case 1.

■

In order to prove that once the system enter to the sliding surface it will remains on it, let us state the following theorem.

Theorem 3.1: Under the assumptions A1-A6, the proposed sliding dynamics (5) are uniformly finite time stable (Rugh, 1993; Khalil, 2002) on the time interval $[t_0, t_f]$.

Proof: Let select

$$V = \frac{1}{2}\|s(\cdot)\|^2, \quad t \in [t_0, t_f],$$

as the candidate time variant Lyapunov function. Taking the derivative of V along the trajectories of the sliding surface (4)

$$\begin{aligned}\dot{V}(t) &= s^T(t)\dot{s}(t)\\ &= s^T\left(\left(\dot{G}(t)C(t) + G(t)C(t)A(t)\right)(x(t) - \hat{x}(t))\right.\\ &\quad \left. + D(t)\left(u_1(t) + \phi(x,t)\right)\right).\end{aligned}$$

Assume a first order sliding mode control (Utkin, 1992), i.e. $u_1 = -\beta D^{-1}(t)\text{sign}(s(t))$, for all $t \in [t_0, t_f]$, where β is a scalar. Then, the derivative of the candidate Lyapunov function along the trajectories of the sliding surface can be bounded as

$$\dot{V}(t) \leq -\|s\|\left(-\Xi\|x(t) - \hat{x}(t)\| + \beta - \Delta\phi_{Max}\right).$$

where $\max\limits_{t\in[t_0,t_f]} \|\dot{G}(t)C(t) + G(t)C(t)A(t)\| = \Xi$ and $\max\limits_{t\in[t_0,t_f]} \|D\| = \Delta$. Then, the proposed sliding mode control assures uniform stability (Rugh, 1993; Khalil, 2002) of the sliding surface (5) if the scalar β satisfies the inequality

$$-\Xi\|x(t) - \hat{x}(t)\| + \beta - \Delta\phi_{Max} \geq \lambda > 0,$$

where λ is a constant. By the comparison principle (Khalil, 2002) and knowing that for construction $V(s(t_0)) = 0$ since $s(y(t_0), t_0) = 0$ the convergence time is given by $t = t_0$ and the proposed sliding mode dynamics (5) are uniformly finite time stable and converge to the surface since the first moment. ■

Since the proposed sliding surface s is uniformly finite time stable, then $\|s(t)\| \leq \|s(t_0)\| = 0$, for all $t \in [t_0, t_f]$ and $s(t) = \dot{s}(t) = 0$ for all $t \in [t_0, t_f]$. Note that the proposed sliding control can be represented using a unitary vector form (Utkin, 1992)

$$u_1 = \begin{cases} -\beta D^{-1}(t)\frac{s(t)}{\|s(t)\|}, & if \quad s(t) \neq 0 \\ 0 & if \quad s(t) = 0. \end{cases} \quad \forall t \in [t_0, t_f]$$

IV. Hierarchical Observer for Linear Time Variant Systems

Now, we are able to state the main contribution of this paper, i.e., the generalization of the algebraic hierarchical observer to the time variant case. First, let us fix, without loss of generality, the value of the design matrix $G(t)$.

$$\begin{aligned}G(t) &= (C(t)B(t))^+\\ &:= \left[(C(t)B(t))^T(C(t)B(t))\right]^{-1}(C(t)B(t))^T.\end{aligned}$$

Substituting $G(t)$ into the matrix $D(\cdot)$ and $\tilde{A}(\cdot)$ lead the system to the following closed loop form

$$\begin{aligned}\dot{x}(t) &= \tilde{A}(t)x(t) + B(t)u_n(t)\\ &\quad + \left(B(t)\dot{G}(t)C(t) + B(t)G(t)C(t)A(t)\right)\hat{x}(t), \quad (8)\\ y(t) &= C(t)x(t), \quad x(t_0) = x_0.\end{aligned}$$

Now, let us recall assumption A3 and assume the pair $(\tilde{A}(t), C(t))$ is observable(Rugh, 1993; Kratz and Liebscher, 1998), then the observability matrix (3) with $A(t) = \tilde{A}(t)$ is well defined with observability index $l = \max\{c, o\}$

and has rank equal to n. The main idea of the observer is to recover step by step the vectors $f(t) = \mathcal{O}_l(t)x(t)$ for all $t \in [t_0, t_f]$. Note that in order to reconstruct the state $x(t)$ we only need to solve the set of algebraic equations $f(t) = \mathcal{O}_l(t)x(t)$. Also observe that the time variant observability matrix under consideration contains several derivatives. We assume that all this derivatives can be computed analytically. If that was not the case, then it is possible to use a suitable differentiator but that would affect the convergence properties of the proposed observer. Before design the hierarchical observer we need a start point that would help us to find some useful bounds and dynamics. Let us design an auxiliary observer that do not converge since the initial time. This observer should be design so that the dynamical error between the original system (7) and this auxiliary observer is stable. Let assume the following dynamical system

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}(t)\tilde{x}(t) + Bu_0 \\ &\quad + \left(B(t)\dot{G}(t)C(t) + B(t)G(t)C(t)A(t)\right)\hat{x}(t) \\ &\quad + K(t)\left(y(t) - C(t)\tilde{x}(t)\right), \quad \tilde{x}(t_0) = C^{+}(t_0)y(t_0)\end{aligned}$$

where $K(t)$ must be designed such that the dynamical observation error

$$\dot{r}(t) = \left(\tilde{A}(t) - K(t)C(t)\right)r(t) = \hat{A}(t)r(t), \quad r(t_0) = 0,$$

with $r(t) = x(t) - \tilde{x}(t)$; is exponentially stable.

Remark 4.1: Note that this dynamic system is any conventional observer. In the literature there are many methodologies that allow us to design the gain K assuring exponential stability. We proposed the use of a Kalman–Bucy Filter (Kalman and Bucy, 1961; Kwakernaak and Sivan, 1972; Chen and Kao, 1997).

If $K(t)$ is designed such that the error is exponentially stable, there exist constants ψ, η such that

$$\|r(t)\| \le \psi e^{-\eta(t-t_0)}\|r(t_0)\| \le \psi e^{-\eta(t-t_0)}\left(\|\mu + \tilde{x}(t_0)\|\right).$$

A. Algebraic Hierarchical Observer Form

The main idea in the step by step observers is the reconstruction of the output and its derivatives, so this allow us to reconstruct the state. To fulfil this requirement we need to recover the vectors $N_i(t)x(t)$, $i = 1, \ldots, l-1$ using an algebraic hierarchical observer for the time variant case. The observer design is given by the following theorem.

Theorem 4.1: If the auxiliary state vector x_{ak}, for all $k = 1, \ldots, l-1$ are designed as

$$\begin{aligned}\dot{x}_{ak} &= \tilde{A}(t)\tilde{x}(t) + B(t)\left(u_n(t)\right. \\ &\quad \left. + \left(G(t)C(t)A(t) + \dot{G}(t)C(t)\right)\hat{x}(t)\right) \\ &\quad + L_k(t)\left(N_{k-1}(t)L_k(t)\right)^{-1}\left(v_k(t)\right. \\ &\quad \left. + \dot{N}_{k-1}(t)(\tilde{x}(t) - x_{ak})\right);\end{aligned} \tag{9}$$

where $L_i(t) \in \mathbb{R}^{n\times p}$ is a design matrix such that $\det\left(N_{i-1}(t)L_i(t)\right) \neq 0$, and the initial conditions should satisfy

$$C(t_0)x_{a1}(t_0) = y(t_0),$$

and

$$N_{k-1}(t_0)\tilde{x}(t_0) + v_{k-1_{eq}}(t_0) = N_{k-1}(t_0)x_{ak}(t_0).$$

Moreover the variable s^k are designed as

$$s_1(y(t), x_{a1}(t)) = y(t) - C(t)x_{a1}(t) \tag{10}$$

and

$$\begin{aligned}s_k(y(t), x_{ak}(t)) &= N_{k-1}(t)\tilde{x}(t) + v_{k-1_{eq}}(t) \\ &\quad - N_{k-1}(t)x_{ak}(t),\end{aligned} \tag{11}$$

for $1 < k < l-1$. Then for all $t \in [t_0, t_f]$

$$v_{k_{eq}}(t) = N_k(t)\left(x(t) - \tilde{x}(t)\right) \quad \text{and } k = 1 \ldots l-1$$

and it is possible to reconstruct completely all the vector functions $N_i(t)x(t)$, $i = 1, \ldots, l-1$.

Proof: In order to recover the first vector $N_1x(t)$, we need the auxiliary state vector x_{a1} governed by (9) with $k = 1$, where $x_{a1}(t_0)$ satisfies $C(t_0)x_{a1}(t_0) = y(t_0)$. Using a sliding variable $s_1 \in \mathbb{R}^p$ defined by (10) we have that the dynamics of the sliding surface are ruled by

$$\begin{aligned}\dot{s}_1(y(t), x_{a1}(t)) &= \left(C(t)\tilde{A}(t) + \dot{C}(t)\right)\left(x(t) - \tilde{x}(t)\right) \\ &\quad - v_1(t) \\ &= N_1(t)\left(x(t) - \tilde{x}(t)\right) - v_1(t),\end{aligned}$$

with $v_1(t)$ defined as

$$v_1(t) = \begin{cases} M_1(t)\frac{s_1(t)}{\|s_1(t)\|} & if \quad s_1(t) \neq 0 \\ 0 & if \quad s_1(t) = 0 \end{cases}, \quad \forall t \in [t_0, t_f].$$

Here the scalar gain M_1 should satisfy the condition

$$\|N_1(t)\|\|x(t) - \tilde{x}(t)\| \le \|\Omega\|\|x(t) - \tilde{x}(t)\| < M_1(t);$$

where $\Omega_1 = \max_{t\in[t_0,t_f]}\{N_1(t)\}$. In order to assure exponential stability, this gain can be choose as

$$M_1(t) = \|\Omega_1(t)\|\psi e^{-\eta(t-t_0)}\left(\|\mu + \tilde{x}(t_0)\|\right).$$

Repeating the same stability proof as in section 4, we get $s_1(\cdot) = \dot{s}_1(\cdot) = 0, \ \forall t \ge t_0$. Thus, in view of (10), we have

$$C(t)x(t) = C(t)x_{a1}(t)$$

and the equivalent output injection is

$$v_{1_{eq}}(t) = N_1(t)\left(x(t) - \tilde{x}(t)\right), \quad \forall t \in [t_0, t_f].$$

Thus, $N_1(t)x(t)$ is recovered by means of the following representation

$$N_1(t)x(t) = N_1(t)\tilde{x}(t) + v_{1_{eq}}(t), \quad \forall t \in [t_0, t_f].$$

Thus, following this same procedure we have that the dynamics of the auxiliary state $x_{ak}(t)$ at the k-th level is governed by (9) where $x_{ak}(t_0)$ satisfies $N_{k-1}(t_0)\tilde{x}(t_0) +$

$v_{k-1_{eq}}(t_0) = N_{k-1}(t_0)x_{ak}(t_0)$. For the variable $s_k \in \mathbb{R}^p$ defined by (11) we have

$$\dot{s}_k(y(t), x_{ak}(t)) = N_k(t)\left(x(t) - \tilde{x}(t)\right) - v_k(t),$$

with $v_k(t)$ defined as

$$v_k(t) = \begin{cases} M_k(t)\frac{s_k(t)}{\|s_k(t)\|} & if \quad s_k(t) \neq 0 \\ 0 & if \quad s_k(t) = 0 \end{cases}, \quad \forall t \in [t_0, t_f]$$

Here the scalar gain M_k should satisfy the condition

$$\|N_k(t)\|\|x(t) - \tilde{x}(t)\| \leq \|\Omega_k(t)\|\|x(t) - \tilde{x}(t)\| < M_k(t)$$

where $\Omega_k = \max_{t\in[t_0,t_f]} \{N_k(t)\}$. In order to assure exponential stability, the gain matrix M_k can be chosen as

$$M_k(t) = \|\Omega_k(t)\|\psi e^{-\eta(t-t_0)}\left(\|\mu + \tilde{x}(t_0)\|\right),$$

as in the former cases and using a Lyapunov stability test (Section 4), we get $s_k(\cdot) = \dot{s}_k(\cdot) = 0, \quad \forall t \geq t_0$, and the equivalent output injection is

$$v_{k_{eq}}(t) = N_k(t)\left(x(t) - \tilde{x}(t)\right), \quad \forall t \in [t_0, t_f]$$

and the vector $N_k(t)x(t)$ can be recovered by means of the equality:

$$N_k(t)x(t) = N_k(t)\tilde{x}(t) + v_{k_{eq}}(t), \quad \forall t \in [t_0, t_f].$$

■

Following the procedure presented in theorem 1, we can reconstruct the vector

$$\mathcal{O}_l(t)x(t) = \mathcal{O}_l(t)\tilde{x}(t) + v_{eq}(t), \quad \forall t \in [t_0, t_f],$$

where

$$v_{eq}(t) = \begin{bmatrix} C(t)x_{a1}(t) - C(t)\tilde{x}(t) \\ v_{1_{eq}}(t) \\ v_{2_{eq}}(t) \\ \vdots \\ v_{l-1_{eq}}(t) \end{bmatrix} \in \mathbb{R}^p.$$

Then, we have reconstructed the output and its l time-derivatives. Since the pair $(\tilde{A}, C)$ is observable. The pseudoinverse of $\mathcal{O}_l$ is well defined and the state can be recovered by means of the equation

$$x(t) = \tilde{x}(t) + \mathcal{O}_l^+(t)v_{eq}(t), \quad \forall t \in [t_0, t_f]. \tag{12}$$

Then the Hierarchical ISM observer is suggested as

$$\hat{x}(t) = \tilde{x}(t) + \mathcal{O}_l^+(t)v_{eq}(t), \quad \forall t \in [t_0, t_f]. \tag{13}$$

Remark 4.2: Note that in the time invariant case the proposed algebraic hierarchical observer is equivalent to the one presented in (Bejarano *et al.*, 2007)
Under the assumptions of this paper, and assuming the ideal output integral sliding mode exists, the following identity holds: $\hat{x}(t) \equiv x(t)$, for all $t \in [t_0, t_f]$. The proposed sliding mode observer needs a low-pass filter in order to reconstruct the equivalent output injection v_{eq} from the high frequency signal $v_k(t)$. Similar to (Utkin, 1992), we consider a first order low-pass filter where the filtered signal $v_{k_{av}}(t)$ would approximate $v_{k_{eq}}(t)$, i.e.

$$\frac{v_{k_{av}}(s)}{v_k(s)} = \frac{K}{\tau s + 1}, \tag{14}$$

where τ is the time constant of the filter and K is the filter gain. (see (Utkin, 1992) for details). Then, we can follow, the algorithm proposed in (Bejarano *et al.*, 2007) and the realization of the observer (13) takes the form

$$\hat{x}(t) = \tilde{x}(t) + \mathcal{O}_l(t)^+(t)v_{av}(t), \quad \forall t \in [t_0, t_f],$$
$$v_{av} = \begin{bmatrix} C(t)x_{a1}(t) - C(t)\tilde{x}(t) \\ v_{1_{av}}(t) \\ v_{2_{av}}(t) \\ \vdots \\ v_{l-1_{av}}(t) \end{bmatrix} \in \mathbb{R}^p. \tag{15}$$

V. Academic Example

In order to show the effectiveness of the proposed approach, let us consider a matrix uncertain linear time variant system of the form

$$\dot{x}(t) = \begin{bmatrix} -1 - e^{-t} & 0 \\ 1 + e^{-t} & -1 - e^{-t} \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(u(t) + \phi(x,t)\right),$$
$$y(t) = \begin{bmatrix} 1 + e^{-2t} & 1 \\ 1 & 1 + e^{-2t} \end{bmatrix} x(t).$$

in the time interval $t \in [0, 5]$. Assume that the state is not available and that the nominal control input is an unitary step. The unknown disturbance/uncertainty has the form $\phi(x,t) = \frac{\sin^2(t)}{\cos(4\pi t)+2}$. Note that the observability index of this example system is 2. Applying the proposed approach we obtain the behavior show in Fig 1. In Fig.2 we show a comparative behavior of the Observation Mean Square Error (OMSE) for different values of τ. Note that the error convergence rate is inversely proportional to the value of τ. Moreover, the proposed observer is able to reconstruct the state since the first moment.

VI. Concluding Remarks

In this paper a framework to design an algebraic hierarchical observer based on integral sliding mode control for linear time variant systems is presented. This framework represent a generalization of the algebraic hierarchical observer proposed by (Bejarano *et al.*, 2007) to the time variant case. The time variant output integral sliding mode does not have reaching phase, ensuring complete compensation of the dynamics affecting the system since initial moment, and it is equivalent to the output integral sliding mode when the matrices are invariant in time. The advantage of the proposed algebraic hierarchical observer is that it assures full reconstruction of the time variant state since the initial time, but the accuracy of the observed state depends on the filter's sampling step on the level of the noise, i.e., if the sampling step goes to zero, the convergence time going to zero; as in the invariant case. The applicability of the proposed approach was proved using an illustrative simulation.

ACKNOWLEDGMENT

The authors gratefully acknowledges the financial support from PAPIIT, UNAM, grant 113613, and from CONACyT Ciencias Básicas 132125.

The Ph.D. Rosalba Galván Guerra is thankful to UNAM for the fellowship received during her postdoctoral stage at Facultad de Ingeniería, UNAM.

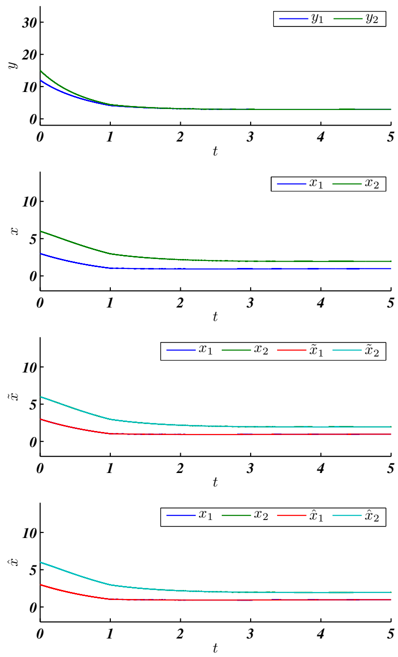

Figure 1. Output Integral Sliding Mode

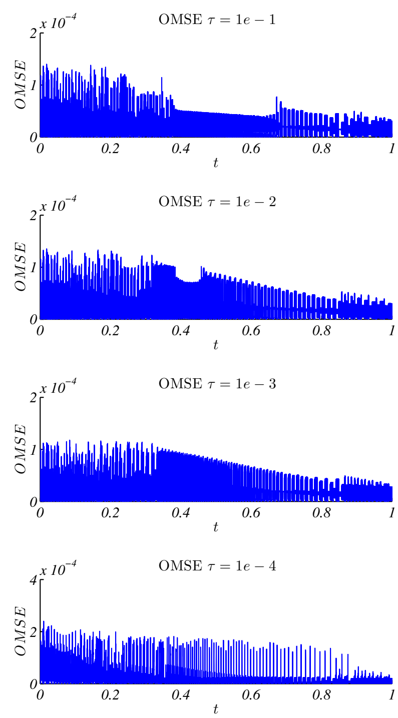

Figure 2. Observation Mean Square Error for different values of τ and $K = 1e-4$

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