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Multiple Stable Attractors In PWL Chaotic Systems

J. G. Barajas-Ramírez
IPICyT, División de Matemáticas Aplicadas
Camino a la Presa de San José 2055
Lomas 4a., CP. 78216, SLP, SLP.
jgbarajas@ipicyt.edu.mx
Phone: (52)-444-834 2000

***Resumen*— We extend previous results on chaotification to the design of piecewise-linear systems with multiple isolated attractors. Our proposed method allows one to assigned the location and orientation of the attractors in state space. We illustrate the effectiveness of the proposed chaotification method with numerical simulations.**

Keywords: Chaotic dynamics, Anticontrol of chaos, Linear time invariant systems.

I. Introduction

Chaotic behavior is now known to be useful and even under certain conditions a desirable feature of dynamical systems (Kapitaniak, 2000; Stark y Hardy, 2003). Due to its dynamical richness, it maybe beneficial to intensionally induce chaotic behaviors into a give system in order to, using chaos control and synchronization techniques, achieve a variety alternative goals (Chen y Dong, 1998). Over the past few years, the scientific community has place increasing attention on the design of chaotic systems and circuits (Chen y Ueta, 2002; Fortuna et al., 2009), and their potential applications to real-world problems, including secure communications, persistent excitation of systems, information processing and encryption, to mention but a few (Strogatz, 2001; Ott, 2002; Tam et al., 2007). In particular, significant research efforts have been devoted to the design, using simple electronics, of systems with multiscroll chaotic attractors. Different authors have shown that piecewise linear (PWL) systems, i. e., nonlinear systems with a PWL function in their mathematical description, can generate multiscroll chaotic attractors. In 1991, Suykens and Vandewalle were the first to described a family of multiscroll attractors, they realized their multiscroll attractor as a multitude of double-scroll attractor of Chua's circuit merge into a single attractor (Suykens and Vandewalle, 1991; Suykens and Vandewalle, 1993). Latter these results were extended to described a family of attractors with an arbitrary, even or odd, number of scrolls derived from a generalized version of Chua's circuit (Suykens and Vandewalle, 1997). Alternative methods have been proposed that generate multiscroll attractors using many different special nonlinear functions, including step functions, saturate functions, time-delayed functions, and many others (Yalcin et al., 2000; Yalcin et al., 2001; Yalcin et al., 2005). A switching manifold approach to generated multiple merged basins of attraction was proposed by Lü et al. in (Lü et al., 2003). The generation of n-scroll hyperchaotic attractors has also been considered, and even attractors with scrolls ordered on directed grids of one, two, and three dimensions (Lü and Chen, 2006).

Given the dynamical richness associated with chaotic behavior, sometimes it is intensionally introduce into a non-chaotic system, this technique is called chaotification [(Chen and Dong, 1998; Wang and Chen, 2000)]. In particular, when chaos is introduced to the system using a control law, the chaotification procedure is also known as anticontrol of chaos [(Chen and Lai, 1998; Wang and Chen, 1999)]. Recently, the use of switching controllers for linear time invariant systems was investigated as a form to generate chaotic multi-scroll attractors (Campos-Cantón et al., 2010; Barajas-Ramírez, 2012). In this contribution, we extend the use of chaotification of linear systems to the case where multiple stable attractors coexist on the same state space.

The remainder of the paper is organized as follows: In Section II, we describe the anticontrol of chaos procedure for a linear time invariant system. In Section III, we provide a procedure to design multiple stable attractors within the same state space and some general guideline rules to impose its location and orientation. To illustrate the effectiveness of our proposed design, in Section IV, we present some numerical simulations. Finally, in Section V, we close this contribution with conclusions.

II. Piecewise Linear Chaotic Systems

In the pursuit of finding systems that can be simple to build and capable of chaotic behavior we propose to chaotify a linear time invariant system:

$$\dot{x}(t) = Ax(t) + u(t) \tag{1}$$

with $u(t) = f(x(t))$ an anticontrol feedback to be designed. We restrict our attention to the specific case of a third dimensional dynamical system. Then, $x(t) = [x_1(t), x_2(t), x_3(t)]^\top \in \mathbf{R}^3$ are the state variables of the system; and $A = \{a_{ij}\} \in \mathbf{R}^{3\times 3}$ is the real-valued system matrix. In particular, we let the controller by given by $f(x_\sigma(t)) : \mathbf{R} \to \mathbf{R}$ a piecewise-linear function described by:

$$u(t) = f(x_\sigma(t)) = \begin{cases} \Lambda_1 x + B_1, & \text{for } h_1(x(t)) \\ \Lambda_2 x + B_2, & \text{for } h_2(x(t)) \\ \Lambda_3 x + B_3, & \text{for } h_3(x(t)) \end{cases} \quad (2)$$

where $\Lambda_i \in \mathbf{R}^{3\times 3}$ and $B_i\mathbf{R}^3$; with $h_i(x(t))$ switching conditions in terms of the state variables of the system, which describe the region of state space where that controller is applied. In particular, we consider that these sections cover the entire state space without overlaps. That is,

$$\bigcup_{i=1}^{N} h_i(x(t)) = \mathbf{R}^3, \text{ and } \bigcap_{i=1}^{N} h_i = \emptyset \quad (3)$$

The closed-loop system (1)-(2) becomes:

$$\dot{x}(t) = \begin{cases} A_1 x + B_1, & \text{for } h_1(x(t)) \\ A_2 x + B_2, & \text{for } h_2(x(t)) \\ A_3 x + B_3, & \text{for } h_3(x(t)) \end{cases} \quad (4)$$

with $A_i = A + \Lambda_i$.

Notwithstanding the simplicity of (4), for a wide range of parameter values produces chaotic trajectories. Form the point of view of chaotification (4) can be interpreted in two different ways. From (2) we can see the chaotification problem as the design of a switching controller for the linear system (1), similar to the ones considered in (Lü and Chen, 2006; Campos-Cantón et al., 2010; Barajas-Ramírez, 2012). Alternatively, the closed-loop system can be seen as a piecewise linear system with a structure that is very similar to some benchmark chaotic systems like Sprott, or Chua's circuits (Chua et al., 1993; Sprott, 2000; Barajas-Ramírez et al., 2003), then chaotification can be achieved by making the selection of A_i and b_i accordingly. For illustration purposes, consider the chaotic Sprott's circuit given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -0.6x_3 - x_2 - 1.2x_1 + 2sgn(x_1) + u \end{aligned} \quad (5)$$

which can be written in the form of (4) with $A_1 = A_2 = A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.2 & -1 & -0.6 \end{pmatrix}$; $B_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $B_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$; for $h_1(x) =$ 'if $x_1 > 0$'; $h_2(x) =$ 'if $x_1 = 0$'; and $h_3(x) =$ 'if $x_1 < 0$'.

Analyzing this PWL system we can identify that two conditions must be satisfied for chaotic behavior to emerge. In the one hand, the local dynamics at each section $h_i(x(t))$ must be unstable and oscillatory. On the other hand, the overall trajectory must remain bounded. This gives the system a local stretching and global folding mechanism. The eigenvalues of the system matrix A_i for Sprott's circuit are $\{-0.9237, 0.1619 \pm 1.1282i\}$. Which implies that locally the system has a stable direction, while the unstable direction escapes oscillating around the equilibrium point of the local subsystem. The transition generated by the partition of the state space given by $h_i(x_1)$ along with the location of the fixed points force the trajectories to move from the unstable direction of one subsystem to the stable direction of the next. Then, as the trajectory moves away from the current fixed point, it crosses to the neighboring section, then initially is attracted to the new fixed point along the stable mode, to latter be pushed away from it along the unstable oscillatory mode. The attractor is generated as this process is repeated as the dynamics evolve. In this way, local stretching and global folding mechanisms result on bounded chaotic behavior.

Motivated by the observations described above, we propose to design of multiscroll chaotic attractors by choosing the components of (4) in the following manner:

A. Local Dynamics.

A.1 Eigenvalues of matrix A_i

For unstable local dynamics we propose to have:

- A purely real eigenvalue (λ_{RP}), with negative real part ($\Re(\lambda_{RP}) < 0$), and
- A pair of complex conjugate eigenvalues (λ_{CC}, λ_{CC}^*) with positive real parts ($\Re(\lambda_{CC}) > 0$).

In particular, we propose to use $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix}$ for (1) and (2), respectively. Where the values of a_i are to be designed such that the eigenvalues of each subsystem satisfy the restrictions expressed above.

B. Global Dynamics.

B.1 Location of equilibrium points

From (4), we have that the equilibrium points of each subsystem are given by: $\mathcal{X}_i(t) = -A_i^{-1}B_i$ with $i = 1, 2, 3$. Then, for each section of state space the corresponding equilibrium point can be placed at an arbitrary location if the input matrix is chosen as:

$$B_i = -A_i\mathcal{X}_i(t), \text{ for } i = 1, 2, 3 \quad (6)$$

B.2 Location of partitions of state space

The conditions $h_i(x(t))$ divide the state space of (4) into sections (S_i), to have transitions between them, the minimum number of sections is two. At each section of (4) the trajectory has a combination of stable (λ_{PR}) and unstable (λ_{CC}) modes. To have a bounded chaotic behavior in which scrolls are describe around each $\mathcal{X}_i(t)$, the distance between equilibrium points must be large enough to have oscillation, and small enough to avoid unstable exponential growth. We want an equilibrium point for each section and the transition (c_{ij}) between sections at the middle point between their corresponding equilibrium points. In particular, we propose the following:

• The distance between two fixed points (d_{ij}) is chosen such that when starting near one of them, without switching, the trajectory describes a sufficiently large number of oscillations before reaching the transition between sections.

• The condition $h_i(x(t))$ is bounded from one side by c_{ij}, this value also bounds the section $h_j(x(t))$ from the opposite side. Moreover, c_{ij} is place about $\frac{d_{ij}}{2}$, between the equilibrium points $\mathcal{X}_i$ and $\mathcal{X}_j$.

In the following Section we use these guidelines to generate multiple stable attractors in the state space of a PWL system.

III. Attractors with different number of scrolls and orientations

III-A. Two-scrolls symmetrically along the x_1-axis

We start with a two-scroll attractor aline along the x_1-axis.

First, we chose the entries of Λ as $a_1 = a_2 = a_3 = 0{,}2$, then the system matrices of (4) are:

$$A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.8 & -0.8 & -0.8 \end{pmatrix} \tag{7}$$

Then, our local dynamics are given by the eigenvalues: $\lambda_{PR} = -0.8994$, and $\lambda_{CC} = 0{,}0497 \pm 0{,}9418i$.

Next, we chose the location of the nodes using rule B.1. Since the local dynamics are relatively slow (see positive real part of λ_{CC}) we chose a distance of two along x_1, with $\mathcal{X}_1 = [1, 0, 0]^T$ and $\mathcal{X}_2 = [-1, 0, 0]^T$. Then, from (6), the input gains are:

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 \\ 0 \\ -0.8 \end{pmatrix} \tag{8}$$

Finally, the location of the transition between sections is taken, as the simplest choice, the middle point between $\mathcal{X}_1$ and $\mathcal{X}_2$ along x_1 with zero taken in the positive side, then we have :

$$\begin{array}{rcl} h_1(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 0\} \text{ and} \\ h_2(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) < 0\} \end{array} \tag{9}$$

System (4) with (7)-(9) generates the two-scrolls attractor symmetrically locate along x_1 shown in Figura 1.

III-B. Three-scrolls symmetrically along the x_2-axis

A three-scroll attractor can be generated. For example, by considering the following equilibrium points $\mathcal{X}_1 = [5, 0, 0]^T$, $\mathcal{X}_2 = [7, 0, 0]^T$, and $\mathcal{X}_3 = [9, 0, 0]^T$. For simplicity, we use the same system matrices (7) as before. Then,

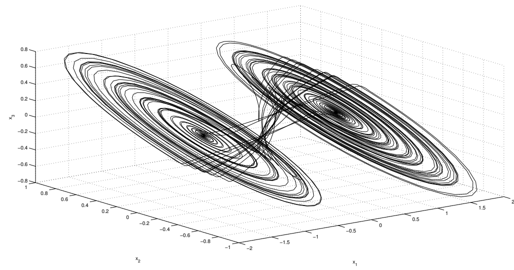

Figura 1. Two-scroll attractor symmetrical along x_1-axis

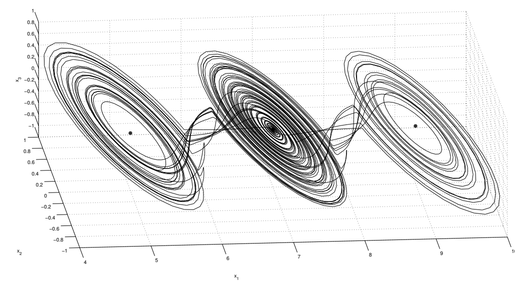

Figura 2. Three-scroll attractor along x_1-axis

we have the following input vectors:

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ 5.6 \end{pmatrix}, \text{ and } B_3 = \begin{pmatrix} 0 \\ 0 \\ 7.2 \end{pmatrix} \tag{10}$$

with the transition values placed in the middle point between the equilibrium points, such that:

$$\begin{array}{rcl} h_1(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) < 6\} \\ h_2(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 6 \wedge x_1(t) < 8\} \\ h_3(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 8\} \end{array} \tag{11}$$

Figura 2 shows the attractor generate by (7) with (5) and (10)-(11).

A change the orientation of the attractor can be achieved with a coordinate transformation. In particular, the three-scroll attractor generated with (10)-(11), can be oriented along the x_2-axis by the coordinate transformation

$$\begin{array}{rcl} A_i^{x_2} & = & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ & = & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -0.8 & -0.8 & -0.8 \end{pmatrix} \end{array} \tag{12}$$

with the input matrices transformed by:

$$B_i^{x_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B_i \tag{13}$$

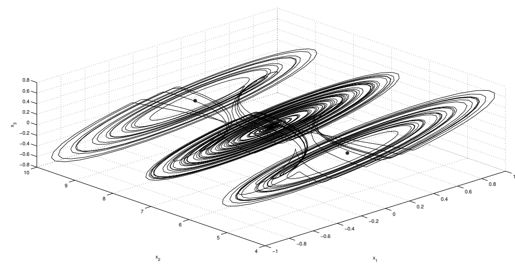

Figura 3. Three-scroll attractor along x_2-axis

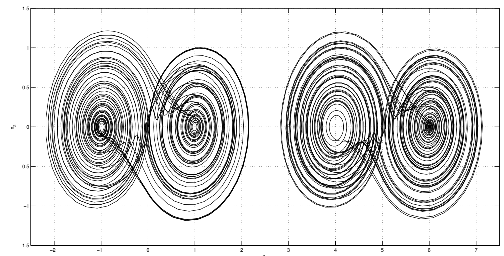

Figura 4. Two stable two-scroll attractors for the same system

In this case, the input vectors remain unchange as $B_1^{x_2} = (0,0,4)^T$, $B_2^{x_2} = (0,0,5.6)^T$, and $B_3^{x_2} = (0,0,7.2)^T$. Finally, the section of state space are redefine to:

$$\begin{array}{rcl} h_1(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_2(t) < 6\} \\ h_2(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_2(t) \geq 6 \wedge x_2(t) < 8\} \\ h_3(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_2(t) \geq 8\} \end{array} \quad (14)$$

The three-scroll attractor along x_2-axis given is shown in Figura 3.

IV. Multiple Attractors for the same system

The designs above can be combine into a single PWL system. The basic idea is to divide the state space in larger partitions, in turn, these partitions are re-partition to allocate desired scrolls with the desired orientation. For example, consider the case of two two-scroll attractors in the same system. Taking the system (4) with (7)-(9) as our base attractor. The state space is divided into two parts, namely $H_1(x(t)) = \{x(t) \in \mathbf{R}^3 : x_1(t) < 2.5\}$ and $H_2(x(t)) = \{x(t) \in \mathbf{R}^3 : x_1(t) \geq 2.5\}$. In each section we can allocate a two-scroll attractor, by setting the equilibrium points at $\mathcal{X}_1 = [1,0,0]^T$ and $\mathcal{X}_2 = [-1,0,0]^T$ for $H_1(x(t))$; and $\mathcal{X}_3 = [4,0,0]^T$ and $\mathcal{X}_4 = [6,0,0]^T$ for $H_2(x(t))$. Then, the input vectors become

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0.8 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ -0.8 \end{pmatrix}$$
$$B_3 = \begin{pmatrix} 0 \\ 0 \\ 3.2 \end{pmatrix}, B_4 = \begin{pmatrix} 0 \\ 0 \\ 4.8 \end{pmatrix} \quad (15)$$

With the transition between sections given by:

$$\begin{array}{rcl} h_1(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 0 \wedge x_1(t) < 2.5\} \\ h_2(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) < 0\} \\ h_3(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 2.5 \wedge x_1(t) < 5\} \\ h_4(x(t)) & = & \{x(t) \in \mathbf{R}^3 | x_1(t) \geq 5\} \end{array} \quad (16)$$

The system (7) with (15) and (16) has two stable two scroll attractors in that co-exist in the same state space. This attractors have their own basin of attraction and once a trajectory is within the attractor it remains there evermore. In particular, for our illustrative system for initial conditions that are located around $\mathcal{X}_1$ and $\mathcal{X}_2$ the trajectories remain in the left attractor of Figura 4, while for initial conditions around $\mathcal{X}_3$ and $\mathcal{X}_4$ the trajectories are in the right-side attractor.

V. Conclusions

A chaotification methodology previously reported is extended in this contribution to the case of multistable attractors. The proposed methodology has two main advantages: In the first place, its very simple to design the desire number of stable attractors for the same system. Secondly, the location and orientation of the attractors within the state space of the system imposed directly. In this sense, we provide simple guideline rules, to establish the local dynamics and the choices of locations and transitions in the global dynamics to achieve the desire multiple attractor. The applicability of multistable attractors is of significance, as it provides potential benefits in chaos applications like secure communications, and the design of reconfigurable logical gates, and for chaos computing. Current research efforts are devoted to exploit the simplicity of the multistable chaotification method proposed for such applications, our results will be reported elsewhere.

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