Nonlinear $\mathcal{H}_\infty$-control of mechanical systems under unilateral constraints on the position

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***Abstract*— The work focuses on the study of hybrid mechanical systems under unilateral constraints on the position. The problem of robust control of mechanical systems is addressed under unilateral constraints by designing a nonlinear $\mathcal{H}_\infty$-controller developed in the nonsmooth setting, covering impact phenomena. Performance issues of the nonlinear $\mathcal{H}_\infty$-tracking controller are illustrated in a numerical simulation**

Keywords: hybrid systems, robust control, nonlinear control, tracking, mechanical systems.

I. Introduction

The study of hybrid dynamical systems has recently attracted a significant research interest, basically, due to the wide variety of applications and the complexity that arises from the analysis of this type of systems. See, e.g., the relevant works by Hamed and Grizzle (in press), Goebel, Sanfelice and Teel (2009), Savkin and Evans (2002) and Antsaklis (2000).

Description of hybrid systems involve both continuous-valued and discrete-valued variables. Their evolution is given by equations of motion that generally depend on both variables. In turn, these equations contain mixtures of logic and discrete-valued or digital dynamics and continuous-variable or analog dynamics. The continuous and discrete dynamics interact at "event" or "trigger" times when the continuous state hits certain prescribed sets in the continuous state space (Branicky, Borkar & Mitter, 1998).

The focus of this work is centered on the study of a subclass of hybrid systems, namely, the autonomous-impulse hybrid systems, also recognized as dynamical systems under unilateral constraints (Brogliato, 1996).

More precisely, mechanical systems of the general form $\ddot{q} = \Phi(q,\dot{q}) + \Psi(q)u,\ F(q) \geq 0$ are under study, where $q \in R^n$ is the vector of generalized coordinates of the system; $u \in R^n$ is the vector of inputs (or controllers) that generally involves a state feedback loop; and the function $F(\cdot,\cdot)$ represents a unilateral constraint that is imposed on the state (specifically, the position). A general property of these systems is that their solution is nonsmooth, which arises from the occurrence of impacts when trajectories attain the surface $F(q,t) = 0$. Some authors such as Nešić, Zaccarian and Teel (2008), Haddad et al. (2005), Orlov and Acho (2001) and Nguang and Shi (2000) to name a few have addressed the disturbance attenuation problem for hybrid dynamical systems. Typically, a pair of Riccati equations, coming from continuous and discrete dynamics, are separately involved and strict conditions are thus imposed on their solutions to simultaneously satisfy both equations. Apart from this, an unrealistic use of an impulsive control is admitted at the impact times.

Motivated by the fact that impulsive inputs, applied at the impact time instants, cause the well-posedness problem of defining dynamics of such a system (Brogliato, 1996), and they are in addition hardly possible to be physically implemented, this work intends to introduce a new control strategy that avoids using impulsive control inputs while ensuring asymptotical stability for the undisturbed system, and at the same time, possessing the $\mathcal{L}_2$-gain of the disturbed system to be less than an appropriate disturbance attenuation level γ. A periodic trajectory is designed such that only a finite number of impacts occur, i.e., it does not contain impact accumulation points. An essential feature, adding the value to the present investigation, is that not only standard external disturbances are in play but also their discrete-time counterpart, typically ignored in the existing literature, that occurs due to imperfect knowledge of the restitution rule at the impact time instants. The control strategy is synthetized for a n-DOF mechanical system, and simulated numerically on a simple model to prove its effectiveness and robustness.

II. Problem Statement

Consider a nonlinear mechanical system (1)-(4) of the form

$$\begin{gathered}\dot{x}_1 = x_2 \\ \dot{x}_2 = \Phi(x_1,x_2,t) + \Psi_1(x_1,x_2,t)w + \Psi_2(x_1,x_2,t)u, \\ t \neq t_k, \qquad F(x) \geq 0\end{gathered} \tag{1}$$

$$\begin{gathered}z = h(x_1,x_2,t) + k_{12}(x_1,x_2,t)u, \\ t \neq t_k, \qquad F(x) \geq 0\end{gathered} \tag{2}$$

$$\begin{gathered}x_2(t_k^+)^T \nabla_{\mathrm{x}} F\big(x_1(t_k)\big) = -e x_2(t_k^-)^T \nabla_{\mathrm{x}} F\big(x_1(t_k)\big) + w_d(t_k), \\ t = t_k, \qquad F\big(x_1(t_k)\big) = 0\end{gathered} \tag{3}$$

$$\begin{gathered}z_d = -e x_2(t_k^-)^T \nabla_{\mathrm{x}} F\big(x_2(t_k)\big) + w_d(t_k), \\ t = t_k, \qquad F\big(x_1(t_k)\big) = 0, \\ k = 1,2,\dots\end{gathered} \tag{4}$$

With functions Φ, Ψ_1, Ψ_2, h, k_{12}, F of appropriate dimensions which are piece-wise continuous in t and twice continuously differentiable in **x.** For ease of reference, (1) is presented as follows

$$\dot{\mathbf{x}} = f(\mathbf{x},t) + g_1(\mathbf{x},t)w + g_2(\mathbf{x},t)u \quad (5)$$

with $f(\mathbf{x},t) = [x_2, \Phi(x_1,x_2,t)]^T$, $g_1(\mathbf{x},t) = [0, \Psi_1(x_1,x_2,t)]^T$, and $g_2(\mathbf{x},t) = [0, \Psi_2(x_1,x_2,t)]^T$.

Hereinafter, $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^{2n}$ represents the state vector with the components $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, being the position and velocity, respectively; $u \in \mathbb{R}^n$ is the control input of dimension n (thus confining investigation to the fully actuated case), and $w \in \mathbb{R}^l$ collects exogenous signals affecting the system. The outputs to be controlled are represented by variables z and z_d. The scalar inequality $F(x_1) \geq 0$ stands for a unilateral constraint within which the system evolves. The restitution law given by equation *(3)* describes a contact between rigid bodies and establishes the interaction between the continuous dynamics (1) and the surface $F(x_1) = 0$, reached at $t = t_k$; $e \in [0,1]$ is the restitution coefficient, whereas $w_d \in \mathbb{R}^s$ is a perturbation accounting for inadequacies of the restitution law (Brogliato, Nicolescu and Orhant, 1997). For making physical sense of the energy dissipation, it is assumed that

$$-(1-e)\left|x_2(t_i^-)^T \nabla_{\mathbf{x}} F(x_1(t_i))\right| < w_d < e\left|x_2(t_i^-)^T \nabla_{\mathbf{x}} F(x_1(t_i))\right| \quad (6)$$

Equations (1) and (2) describe the continuous dynamics before the system hits the reset surface $F(x_1) = 0$, and equations (3) and (4) govern the way that the states are instantaneously changed when the resetting surface is hit. This model is restricted to surfaces of co-dimension one. Under certain assumptions (Brogliato, Nicolescu and Orhant, 1997), this restriction can be relaxed to surfaces of higher co-dimensions.

The $\mathcal{H}_\infty$-control problem consists in finding a controller, if any, such that the undisturbed, closed-loop system (1)-(4) is asymptotically stable, and such that the $\mathcal{L}_2$-gain of the disturbed system is less than γ, that is the inequality

$$\int_{t_0}^{T} \|z(t)\|^2 \mathrm{d}t + \sum_{i=1}^{N} \|z_d(t_i)\|^2 \leq \gamma^2 \left[\int_{t_0}^{T} \|w(t)\|^2 \mathrm{d}t + \sum_{i=1}^{N} \|w_d(t_i)\|^2\right] + \sum_{i=0}^{N} \beta_i\left(\mathbf{x}(t_i^-)\right) \quad (7)$$

holds with some positive definite functions $\beta_i(\mathbf{x}), i = 0, \ldots, N$ for all $T > 0$ and $N \in \mathbb{Z}$ such that $t_N \leq T$. This definition is consistent with the notion of dissipativity introduced by Willems (1972) and Hill & Moyan (1980), that has become standard in the literature, and represents a natural extension for hybrid systems (see, e.g., the works by Nešić, Zaccarian & Teel (2008), Yuliar, James & Helton (1998), Lin & Byrnes (1996) and Baras & James (1993)).

III. NONLINEAR $\mathcal{H}_\infty$-CONTROL SYNTHESIS

A. Global state-space solution

The main result of the present work is given below.

Theorem 1. Suppose that in a domain $(\mathbf{x} \in B_\delta, t \in \mathbb{R})$ there is a Lipchitz continuous, positive definite, decrescent function $V(\mathbf{x},t)$, a positive definite function $G(\mathbf{x})$ and a constant $\gamma > 0$ such that for the system (1)-(4) with assumptions above and the initial conditions within B_δ, the following conditions

C1. $V(\mathbf{x}(t_i^+), t_i) < V(\mathbf{x}(t_i^-), t_i), i = 1, \ldots, N$ provided that $w_d = 0$

C2. $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\left(f(\mathbf{x},t) + g_1(\mathbf{x},t)\alpha_1 + g_2(\mathbf{x},t)\alpha_2\right) + h^T h + {\alpha_2}^T \alpha_2 - \gamma^2 {\alpha_1}^T \alpha_1 \leq -G(\mathbf{x})$

hold under $\alpha_1 = \frac{1}{2\gamma^2} g_1^T(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T$, $\alpha_2 = -\frac{1}{2} g_2^T(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T$.

Then driven by the controller

$$u = \alpha_2(\mathbf{x},t), \quad (8)$$

the closed-loop undisturbed system (1)-(4) is asymptotically stable, while its disturbed version possesses a $\mathcal{L}_2$-gain less than γ. If in addition, $V(\mathbf{x},t)$ is radially unbounded, then the result becomes global.

Proof. The proof is brought up into two parts. First we demonstrate that the inequality

$$\int_{t_0}^{T} \|z(t)\|^2 \mathrm{d}t \leq \gamma^2 \left[\int_{t_0}^{T} \|w(t)\|^2 \mathrm{d}t\right] + \sum_{i=0}^{N} \beta'_i\left(\mathbf{x}(t_i)\right) \quad (9)$$

holds for all $T > 0$ and $N \in \mathbb{Z}$ such that $t_N \leq T$, and some positive definite functions $\beta'_i\left(\mathbf{x}(t_i)\right)$, $i = 0, \ldots, N$. Suppose there is a positive definite function $V(\mathbf{x},t)$ such that

$$V(\mathbf{x}(T),T) + \sum_{i=1}^{N} V(\mathbf{x}(t_i^-), t_i) - \sum_{i=0}^{N} V(\mathbf{x}(t_i^+), t_i) \leq -\int_{t_0}^{T} \|z(t)\|^2 \mathrm{d}t + \gamma^2 \int_{t_0}^{T} \|w(t)\|^2 \mathrm{d}t \quad (10)$$

holds. Then inequality (9) is achieved by setting

$$\beta'_i\left(\mathbf{x}(t_i)\right) = V(\mathbf{x}(t_i^+), t_i), \; i = 0 \ldots N. \quad (11)$$

In order to validate inequality (10) let us represent it in the equivalent differential form

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -z^T z + \gamma^2 w^T w, \qquad t \in (t_i, t_{i+1}) \quad (12)$$

between impact instants $t_i, i = 1, \ldots, N$. Then for the undisturbed system, $V(\mathbf{x},t)$ can be used as a Lyapunov function. Indeed, along the trajectories of such a system, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -z^T z \quad (13)$$

and

$$V(\mathbf{x}(t_i^+), t_i) < V(\mathbf{x}(t_i^-), t_i) \quad (14)$$

provided that C1 and C2 are satisfied. Just in case, inequalities (13)-(14), coupled to the assumption that $V(\mathbf{x},t)$ is decrescent, ensure that $\lim_{i\to\infty} V(\mathbf{x}(t_i^+), t_i) = 0$. The undisturbed system is thus asymptotically stable (Theorem 3.7, Orlov, 2009) and in addition, it is globally asymptotically stable if $V(\mathbf{x},t)$ is radially unbounded.

For the disturbed system, we verify (12) and (14) independently. For the continuous dynamics, the inequality

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\left(f(\mathbf{x},t) + g_1(\mathbf{x},t)w + g_2(\mathbf{x},t)u\right) + h^T h + u^T u - \gamma^2 w^T w \leq -G(\mathbf{x}) \quad (15)$$

is guaranteed by condition C2. To reproduce this conclusion, let us define the Hamiltonian function

$$H_c(\mathbf{x},w,u,t)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial \mathbf{x}}(f(\mathbf{x},t)+g_1(\mathbf{x},t)w+ g_2(\mathbf{x},t)u)+h^T h+u^T u-\gamma^2 w^T w \quad (16)$$

Then solving the equations

$$\left.\frac{\partial H_c}{\partial w}\right|_{(w,u)=(\alpha_1,\alpha_2)}=0, \qquad \left.\frac{\partial H_c}{\partial u}\right|_{(w,u)=(\alpha_1,\alpha_2)}=0\,,$$

we obtain $\alpha_1=\frac{1}{2\gamma^2}g_1^T(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T$, $\alpha_2=-\frac{1}{2}g_2^T(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T$.

Since $H_c(\mathbf{x},w,u,t)$ is quadratic in (w,u) its Taylor expansion around $(w=\alpha_1,u=\alpha_2)$ is expressed as

$$H_c(\mathbf{x},w,u,t)=H_c(\mathbf{x},\alpha_1,\alpha_2,t)+\|u-\alpha_2\|^2 -\gamma^2\|w-\alpha_1\|^2 \quad (17)$$

Thus, taking into account condition C2, we obtain

$$H_c(\mathbf{x},\alpha_1,\alpha_2,t)\le -G(\mathbf{x}) \quad (18)$$

and combining the result with (16)-(17) yields inequality (15) that in turn ensures (10).

The second part of the proof consists of demonstrating that the inequality

$$\sum_{i=1}^{N}\|z_d(t_i)\|^2\le\gamma^2\left[\sum_{i=1}^{N}\|w_d(t_i)\|^2\right]+\sum_{i=1}^{N}\left\|\beta''_i\left(\mathbf{x}(t_i^-)\right)\right\|^2 \quad (19)$$

holds for N such that $t_N\le T$, and some positive definite functions $\beta''_i(\cdot)$, $i=1,\dots,N$. Clearly, it suffices to prove its simplified version

$$\|z_d(t_i)\|^2\le\gamma^2\|w_d(t_i)\|^2+\left\|\beta''_i\left(\mathbf{x}(t_i^-)\right)\right\|^2 \quad (20)$$

for a single impact and all $i\in[1,N]$. Substituting (4) in (20) and applying the Cauchy-Schwartz and triangle inequalities to the left side, we obtain

$$2e^2\left\|x_2(t_i^-)^T\nabla_{\mathbf{x}}F\left(x_1(t_i)\right)\right\|^2+2\|w_d(t_i)\|^2 \le\gamma^2\|w_d(t_i)\|^2+\left\|\beta''_i\left(\mathbf{x}(t_i^-)\right)\right\|^2. \quad (21)$$

By setting $\beta''_i\left(\mathbf{x}(t_i^-)\right)=2x_2(t_i^-)^T\nabla_{\mathbf{x}}F\left(x_1(t_i)\right)$, then inequality (19) is achieved for $\gamma^2\ge 2$. Combining this result with (11), we establish the dissipativity inequality (5) with

$$\beta_i(\mathbf{x}(t_i))=\begin{cases}V(\mathbf{x}(t_0),t_0), & i=0\\ V(\mathbf{x}(t_i^+),t_i)+2x_2(t_i^-)^T\nabla_{\mathbf{x}}F\left(x_1(t_i)\right), & i=1,\dots,N\end{cases} \quad (22)$$

This completes the proof. ■

B. Local state-space solution

The subsequent local analysis involves the linear $\mathcal{H}_\infty$-control problem for the system

$$\dot{\mathbf{x}}=A(t)\mathbf{x}+B_1(t)w+B_2(t)u, \qquad t\ne t_k, F(x_1)\ge 0 \quad (23)$$

$$z=C(t)\mathbf{x}+D_{12}(t)u, \qquad t\ne t_k, F(x_1)\ge 0 \quad (24)$$

$$x_2(t_k^+)^T\nabla_{\mathbf{x}}F\left(x_1(t_k)\right)=-ex_2(t_k^-)^T\nabla_{\mathbf{x}}F\left(x_1(t_k)\right)+w_d, \quad t=t_k, F\left(x_1(t_k)\right)=0 \quad (25)$$

$$z_d=-ex_2(t_k^-)^T\nabla_{\mathbf{x}}F\left(x_1(t_k)\right)+w_d(t_k), \quad t=t_k, F\left(x_1(t_k)\right)=0 \quad (26)$$

$\forall k\in\mathbb{Z}^+$, where $A(t)=\left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=0}$, $B_1(t)=g_1(0,t)$, $B_2(t)=g_2(0,t)$, $C(t)=\left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}=0}$, $D_{12}(t)=k_{12}(0,t)$.

Theorem 2. Given the system linearization (23)-(26) and some $0<\varepsilon<\varepsilon_0$, then conditions C1-C2 hold locally around the equilibrium $\mathbf{x}=0$ of the nonlinear system (1)-(4) with

$$V(\mathbf{x},t)=\mathbf{x}^T P_\varepsilon(t)\mathbf{x} \quad (27)$$

$$G(\mathbf{x})=\frac{\varepsilon}{2}\|\mathbf{x}\|^2 \quad (28)$$

and the state feedback

$$u=-g_2(\mathbf{x},t)^T P_\varepsilon(t)\mathbf{x}, \quad (29)$$

is a local solution of the $\mathcal{H}_\infty$-control problem for the nonlinear system (1)-(4) provided that $P_\varepsilon(t)$ is a bounded, symmetrical, positive definite solution of the differential Riccati equation

$$-\dot{P}_\varepsilon(t)=P_\varepsilon(t)A(t)+A^T(t)P_\varepsilon(t)+C^T(t)C(t) +P_\varepsilon(t)\left[\frac{1}{\gamma^2}B_1B_1^T-B_2B_2^T\right](t)P_\varepsilon(t)+\varepsilon I. \quad (30)$$

Proof. It should be noted that the time-varying strict bounded real lemma (Orlov, Acho & Solis, 1999) yields a constructive tool of verifying the existence of an appropriate solution of the differential Riccati equation (30). Recall that in accordance with this lemma, once the equation (31)

$$-\dot{P}(t)=P(t)A(t)+A^T(t)P(t)+C^T(t)C(t) +P(t)\left[\frac{1}{\gamma^2}B_1B_1^T-B_2B_2^T\right](t)P(t) \quad (31)$$

possesses a symmetrical, positive semidefinite solution $P(t)$ then there exists a positive constant ε_0 such that the perturbed Riccati equation (30) has a unique bounded, positive definite symmetric solution $P_\varepsilon(t)$ for each $\varepsilon\in(0,\varepsilon_0)$.

It should also be noted that by setting $V(\mathbf{x},t)=\mathbf{x}^T P(t)\mathbf{x}$ the Hamilton-Jacobi-Isaacs inequality (18) subject to $G(\mathbf{x})=0$ degenerates to the differential Riccati equation (31).

Thus, employing (30), we can set $V(\mathbf{x},t)=\mathbf{x}^T P_\varepsilon(t)\mathbf{x}$ to locally meet the Hamilton-Jacobi-Isaacs inequality (18) with the positive definite function $G(\mathbf{x})=-\frac{\varepsilon}{2}\|\mathbf{x}\|^2$. Finally, applying Theorem 1 to (23)-(26) subject to (27), (28), (30), the controller (29) is a local solution to the $\mathcal{H}_\infty$-control problem. This completes the proof. ■

Remark 1. For autonomous systems, where all functions in (1)-(4) and (23)-(26) are time-independent, the differential Riccati equations (31) and (30) degenerate to algebraic Riccati equations (ARE) by setting $\dot{P}_\varepsilon(t)=0$ and $\dot{P}(t)=0$.

C. Application to mechanical systems subject to unilateral constraints

In this section, the Lagrange model for mechanical manipulators will be used, in order to follow a trajectory composed of free-motion phases separated by transition phases, as follows:

Free-motion phase:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau + w \quad (32)$$

$$F(q) \geq 0 \quad (33)$$

Transition phases:

$$q(t_k^+) = q(t_k^-) \quad (34)$$

$$\dot{q}(t_k^+)^T \nabla_q F(q(t_k)) = -e\dot{q}(t_k^-)^T \nabla_q F(q(t_k)) + w_d(t_k) \quad (35)$$

$$F(q(t_k)) = 0 \quad (36)$$

where $q \in \mathbb{R}^n$ is a position, $\tau \in \mathbb{R}^n$ is a control input, $w \in \mathbb{R}^n$ is an external disturbance, w_d is a perturbation due to the modeling of the restitution rule (35), $M(q)$, $C(q,\dot{q})$, $G(q)$ are matrix functions of the appropriate dimensions. From the physical point of view, q is the vector of generalized coordinates, τ is the vector of external torques, $M(q)$ is the inertia matrix, symmetric and positive definite for all $q \in \mathbb{R}^n$, $C(q,\dot{q})\dot{q}$ is the vector of Coriolis, centrifugal torques and viscous friction and $G(q)$ is the vector of gravitational torques. As a matter of fact, the functions $M(q)$, $C(q,\dot{q})$, $G(q)$ are smooth functions of their arguments.

Remark 2: Notice that equations (32)-(36) do not provide a control action during the transition phase, mainly because such a control action would be impulsive in nature, whose implementation is challenging in practice.

Remark 3: In this fully-actuated case, it is clear that the aim of the robotic task is to follow a desired time-varying trajectory that will bounce in the surface $F(q) = 0$ at some instants $t = t_k, k = 1,2,\ldots$. An extension to the plant stabilization constrained to the surface $F(q) = 0$ is under study.

Now, suppose that there exists a discontinuous periodic solution $q(t) = q_d(t)$ of the undisturbed system (32)-(36), driven by an input torque $\tau = \tau_d$. In other words, suppose that there exis initial conditions of (32)-(36) with $w = w_d = 0$, $\tau = \tau_d$, such that it exhibits a periodic solution. Then, our objective is to design a controller of the form

$$\tau = \tau_d + u \quad (37)$$

$$\tau_d = M(q_d)\ddot{q}_d + C(q_d,\dot{q}_d)\dot{q}_d + G(q_d) \quad (38)$$

that imposes on the disturbance-free manipulator motion desired stability properties around $q_d(t)$ while also locally attenuating the effect of the disturbances. Thus, the controller to be constructed consists in the trajectory feedforward compensator design (38) and a disturbance attenuator synthesis $u(t)$, internally stabilizing the closed-loop system around the desired trajectory.

We confine our research to the position tracking control problem where the output to be controlled is given by

$$z = \begin{bmatrix} 0 \\ \rho_p(q_d - q) \\ \rho_v(\dot{q}_d - \dot{q}) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad (39)$$

$$z_d = -e\dot{q}(t_k^-)^T \nabla_q F(q(t_k)) + w_d(t_k) \quad (40)$$

with positive weight coefficients ρ_p, ρ_v.

The $\mathcal{H}_\infty$ position tracking control problem for robot manipulators subject to unilateral constraints on the position can formally be stated as follows. Given a mechanical system (32)-(36) a desired trajectory $q_d(t)$ to track, and a real number $\gamma > 0$, it is required to find (if any) a state feedback controller such that the undisturbed closed-loop system is uniformly asymptotically stable around $q_d(t)$ and its $\mathcal{L}_2$-gain is locally less than γ, for all T and all piecewise continuous functions $w(t)$, $w_d(t_k)$ for which the state trajectory of the closed-loop system starting in a neighborhood of the initial point $(q(0), \dot{q}(0)) = (q_d(0), \dot{q}_d(0))$ remains in a neighborhood of the desired trajectory $q_d(t)$ for all $t \in [0,T]$.

In order to accomplish this task, the following assumptions are made:

$$q_d(t_k) \in F(q) = 0, \text{ k=1,2,...} \quad (41)$$

$$\dot{q}_{di}(t) \neq 0, i = 1,\ldots,n \text{ for almost all } t. \quad (42)$$

To begin with, let us introduce the state deviation vector $\mathbf{x} = (x_1, x_2)^T$ where $x_1(t) = q_d(t) - q(t)$ is the position deviation from the desired trajectory $q_d(t)$, and $x_2(t) = \dot{q}_d(t) - \dot{q}(t)$ is the velocity deviation from the desired velocity $\dot{q}(t)$.

After that, let us rewrite the state equations (32)-(36), (39)-(40) in terms of these deviations:

Free-motion phase errors:

$$\dot{x}_1 = x_2 \quad (43)$$

$$\begin{aligned}\dot{x}_2 = \ddot{q}_d + M^{-1}(q_d - x_1)[&C(q_d - x_1, \dot{q}_d - x_2)(\dot{q}_d \\ &- x_2) + G(q_d - x_1) + M(q_d)\ddot{q}_d \\ &- C(q_d,\dot{q}_d)\dot{q}_d - G(q_d) - u - w_1]\end{aligned} \quad (44)$$

$$F(x_1) \geq 0 \quad (45)$$

$$z = \begin{bmatrix} 0 \\ \rho_p x_1 \\ \rho_v x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \quad (46)$$

Transition phase errors:

$$x_1(t_k^+) = x_1(t_k^-) \quad (47)$$

$$x_2(t_k^+)^T \nabla_{x_1} F(x_1(t_k)) = -e x_2(t_k^-)^T \nabla_{x_1} F(x_1(t_k)) + w_d(t_k) \quad (48)$$

$$F(x_1(t_k)) = 0 \quad (49)$$

$$z_d = -e x_2(t_k^-)^T \nabla_{x_1} F(x_1(t_k)) + w_d(t_k) \quad (50)$$

The above $\mathcal{H}_\infty$-tracking control problem can be specified as follows:

$$\begin{aligned} f(\mathbf{x},t) &= \begin{bmatrix} x_2 \\ \ddot{q}_d + M^{-1}(q_d - x_1)[C(q_d - x_1, \dot{q}_d - x_2)(\dot{q}_2 - x_2)] \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ M^{-1}(q_d - x_1)[G(q_d - x_1) - M(q_d)\ddot{q}_d] \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ M^{-1}(q_d - x_1)[-C(q_d,\dot{q}_d)\dot{q}_d - G(q_d)] \end{bmatrix}, \end{aligned} \quad (51)$$

$$g_1(\mathbf{x},t) = \begin{bmatrix} 0 \\ -M^{-1}(q_d - x_1) \end{bmatrix}, \quad (52)$$

$$g_2(\mathbf{x},t) = \begin{bmatrix} 0 \\ -M^{-1}(q_d - x_1) \end{bmatrix} \quad (53)$$

$$h(\mathbf{x}) = \begin{bmatrix} 0 \\ \rho_p x_1 \\ \rho_v x_2 \end{bmatrix} \tag{54}$$

$$k_{12}(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

Theorem 3. Let the following conditions be satisfied

1) (41) and (42) hold for the desired trajectory to follow
2) There exists a symmetrical, positive definite solution $P_\varepsilon(t)$ to (30), where A, B_1, B_2, C are obtained by the linearization of (51)-(55), under some $\varepsilon > 0$.

Then, the state feedback

$$u = -g_2^T(\mathbf{x}, t) P_\varepsilon(t) \mathbf{x} \tag{56}$$

is a local solution of the $\mathcal{H}_\infty$-position tracking problem for the mechanical manipulator under unilateral constraints on the position (32)-(36).

Proof. By applying Theorem 2 to the error system (43)-(55) specified with a given trajectory subject to (41)-(42) the validity of the theorem is established. ■

IV. Numerical Simulation

The objective of this section is to demonstrate the effectiveness of the proposed control synthesis. To facilitate exposition, a simple model of mass-spring-damper-barrier that captures all the essential features of the general treatment is chosen to numerically support the theory. The work is in progress and an extension to the generation of a periodic walking gait of a biped robot will be reported elsewhere.

A. *Mass-spring-damper-barrier model*

Theorem 3 will be applied to a simple mass-spring-damper-barrier system as depicted in figure 1, where m represents the mass, k the spring constant, b a damping constant, τ is the applied control force, and q represents the position. The objective is to follow a trajectory that bounces against the wall located at $q = 0$.

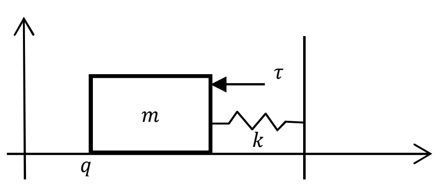

Figure 1. Mass-spring-damper-barrier system

For the free-motion dynamics ($q > 0$), the model is:

$$m\ddot{q} + b\dot{q} + kq = \tau + w$$
$$q_1 = q, \qquad q_2 = \dot{q}$$
$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -\frac{k}{m} q_1 - \frac{b}{m} q_2 + \frac{1}{m}\tau + \frac{1}{m} w$$

whereas for the transition phase ($q_1 = 0$):

$$q_1^+ = q_1^- \qquad\qquad q_{1d}^+ = q_{1d}^-$$
$$q_2^+ = -e q_2^- + w_d \qquad\qquad \dot{q}_{1d}^+ = -e\dot{q}_{1d}^-$$

The notation f^+ (f^-) is equivalent to $f(t_k^+)$ ($f(t_k^-)$). The variables w and w_d were introduced to account for model inadequacies, and non-modeled external forces, such as friction. Now, let's define the error variables $x_1 = q_1 - q_{1d}$ and $x_2 = q_2 - \dot{q}_{1d}$. Rewriting the system with these error variables, leads to the free-motion phase error system:

$$\tau = m\ddot{q}_{1d} + kq_{1d} + b\dot{q}_{1d} + u$$
$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B_1} w + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B_2} u$$
$$z = \underbrace{\begin{bmatrix} 0 & 0 \\ \rho_p & 0 \\ 0 & \rho_v \end{bmatrix}}_{C} \mathbf{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{D_{12}} u$$

And to the transition phase error equations:

$$\mathbf{x}^+ = \begin{bmatrix} 1 & 0 \\ 0 & -e \end{bmatrix} \mathbf{x}^- + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_d$$
$$z_d = -ex^- + w_d$$

From the expressions above, we can identify the terms , B_1, B_2, C, necessary to solve (30) (see remark 1).

B. *Simulation results*

The simulation shown in figure 2 was performed using Matlab and the parameters from table 1. The solution of (31) was obtained by iterating on γ, and the infimal achievable level attained was $\gamma^* \approx 0.73$. From theorem 1, it is known that $\gamma \geq \sqrt{2}$; however, $\gamma = 2$ was selected to avoid an undesirable high-gain controller design that would appear for a value of γ close to the optimum. With $\gamma = 2$, the value of $\varepsilon = 0.01$ was obtained so the corresponding perturbed Riccati equation (30) has a positive definite solution.

TABLE I

Simulation Parameters

Parameter	Value	Parameter	Value
k	10	ρ_v	1
b	1	ε	0.01
m	1	w_d	0.3
e	0.5	w	$0.1q_2 + 0.1sign(q_2)$ (coulomb + viscous friction model)
ρ_p	1		

The trajectory to follow was generated by a Van der Pol oscillator bouncing against a surface with a restitution coefficient of 0.5. The model used was:

Free-motion phase ($x < 0$):

$$\dot{x} = y$$
$$\dot{y} = \mu(1 - x^2)y - x$$

Transition phase ($x = 0$):

$$x(t_k^+) = x(t_k^-)$$
$$y(t_k^+) = -ey(t_k^-)$$

The parameters used for this oscillator were $\mu = 1$, $e = 0.5$, $x(0) = 0$ and $y(0) = 1.0126$. This reference system generates a hybrid periodic orbit (Grizzle et al., 1999). Thus, the planned trajectory to follow by the system will be

$$q_d(t) = x(t), \qquad \dot{q}_d(t) = y(t)$$

From figure 2 we can see that the system tracks the desired trajectory in a sound manner despite the disturbances affecting the free-motion (friction) and transition phases (deviation from restitution coefficient), while asymptotically stabilizing the error for the undisturbed system.

From figure 3 we can conclude that as the parameter γ approaches the limit value $\gamma = \sqrt{2}$, the system begin to decrease its disturbance attenuation property. For values of γ less than this limit value, this property is lost, as predicted.

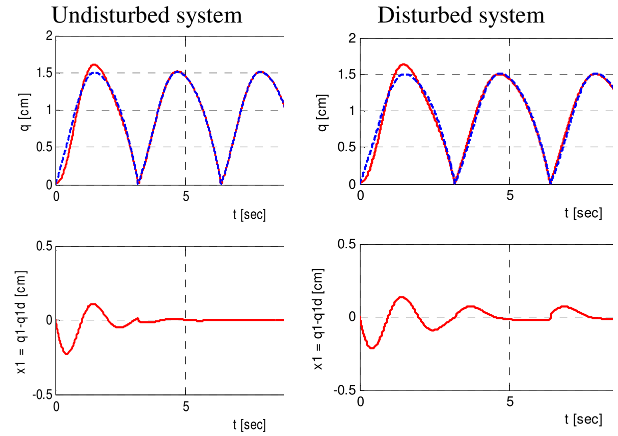

Figure 2. Trajectory tracking for $\gamma = 2$. Left: undisturbed system. Right: disturbed system.

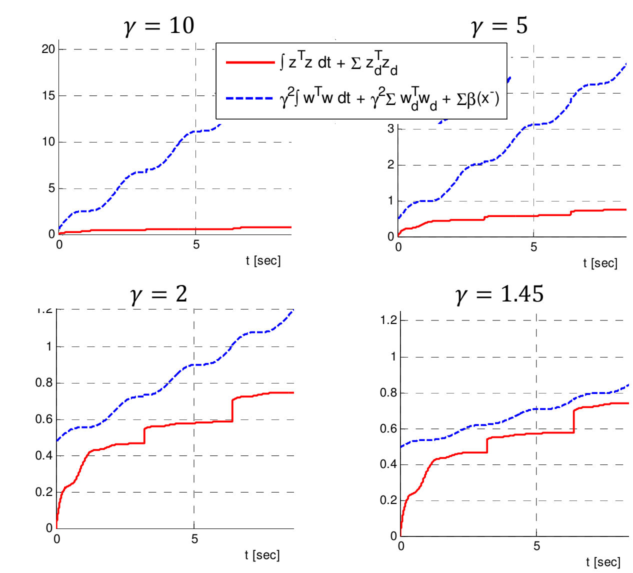

Figure 3. Behavior of the system's $\mathcal{L}_2$-gain while varying the parameter γ.

V. Conclusions

In this paper, the state feedback $\mathcal{H}_\infty$-control is solved for mechanical systems subject to unilateral constraints on the position. A global (local) solution for the tracking problem is found by solving only a unique Hamilton-Jacobi-Isaacs inequality (or differential Riccati equation for finding a local solution), which represents an advantage over solutions available in the existing literature. Effectiveness of the proposed disturbance attenuation design has been supported by the numerical simulations, made for a mass-spring-damper-barrier model operating in the presence of a coulomb friction force under an uncertain restitution coefficient.

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References

Antsaklis, P. J. (2000). A brief introduction to the theory and applications of hybrid systems.*In Proc IEEE, Special Issue on Hybrid Systems: Theory and Applications*.

Baras, J.S. & James, M.R. (1993). Robust output feedback control for discrete-time nonlinear systems: the finite-time case. *Proceedings of the 32nd IEEE Conference on Decision and Control,* 55(1), 15-17.

Branicky, M. S., Borkar, V. S., & Mitter, S. K. (1998). A unified framework for hybrid control: Model and optimal control theory. *Automatic Control, IEEE Transactions on*, 43(1), 31-45.

Brogliato, B. (1996). *Nonsmooth mechanics: models, dynamics, and control*. Springer Verlag.

Brogliato, B., Niculescu, S. I., & Orhant, P. (1997). On the control of finite-dimensional mechanical systems with unilateral constraints. *Automatic Control, IEEE Transactions on*, *42*(2), 200-215.

Goebel, R., Sanfelice, R. G., & Teel, A. (2009).Hybrid dynamical systems.*Control Systems, IEEE*, 29(2), 28-93

Grizzle, J. W., Plestan, F., & Abba, G. (1999). Poincare's method for systems with impulse effects: application to mechanical biped locomotion. In *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on* (Vol. 4, pp. 3869-3876). IEEE.

Haddad, W. M., Kablar, N. A., Chellaboina, V., & Nersesov, S. G. (2005). Optimal disturbance rejection control for nonlinear impulsive dynamical systems. *Nonlinear Analysis: Theory, Methods & Applications*, *62*(8), 1466-1489.

Hamed, K. A., Grizzle, J.W., (In Press). Robust Event-based Stabilization of Periodic Orbits for Hybrid Systems: Application to an Underactuated 3D Bipedal Robot. *In Press for American Control Conference*, Washington DC, June 2013.

Hill, D., & Moylan, P. (1980). Connections between finite-gain and asymptotic stability. *Automatic Control, IEEE Transactions on*, *25*(5), 931-936.

Lin, W., & Byrnes, C. I. (1996). $\mathcal{H}_\infty$-control of discrete-time nonlinear systems. *Automatic Control, IEEE Transactions on*, *41*(4), 494-510.

Nešić, D., Zaccarian, L., & Teel, A. R. (2008). Stability properties of reset systems. *Automatica*, *44*(8), 2019-2026.

Nguang, S. K., & Shi, P. (2000). Nonlinear $\mathcal{H}_\infty$ filtering of sampled-data systems. *Automatica*, *36*(2), 303-310.

Orlov, Y. V. (2009). Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions. Springer.

Orlov, Y., & Acho, L. (2001). Nonlinear $\mathcal{H}_\infty$-control of time-varying systems: a unified distribution-based formalism for continuous and sampled-data measurement feedback design. *Automatic Control, IEEE Transactions on*,*46*(4), 638-643.

Orlov, Y., Acho, L. & Solis V. (1999). Nonlinear $\mathcal{H}_\infty$-control of time-varying systems. *Decision and Control, Proceedings of the 38th IEEE Conference on*. IEEE, p. 3764-3769.

Savkin, A. V., & Evans, R. J. (2002).*Hybrid dynamical systems: controller and sensor switching problems*. Birkhauser.

Willems, J. C. (1972). Dissipative dynamical systems part I: General theory.*Archive for rational mechanics and analysis*, *45*(5), 321-351.

Yuliar, S., James, M. R., & Helton, J. W. (1998). Dissipative control systems synthesis with full state feedback. *Mathematics of Control, Signals and Systems*, *11*(4), 335-356.