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On The Generalized Synchronization Of Complex Networks

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***Resumen*— We investigate the emergence of generalized synchronization (GS) on networks of strictly different dynamical systems. GS is characterized by a functional relationship between the states of two systems, this notion has been extended for the case of networks. There are two basic approaches to detect GS between two systems: In the first, SG is inferred from the identical synchronization of two systems under the same driving force, the so-called auxiliary system approach. Alternatively, a functional relationship between the systems can be explicitly imposed by design, this is usually called the controlled synchronization approach. In this contribution, we take the latter approach to impose GS on a linearly and diffusively coupled network, where the nodes are different dynamical systems that are fully triangularizable. Further, we investigate how a robust controller can make the network synchronized in a generalized sense with respect to a given functional relationship. We illustrate our results with numerical simulations.**

Keywords: Complex networks, Generalized synchronization, Robust control.

I. Introduction

Over the last couple of decades, synchronization of complex dynamical systems has received a great deal of attention from the scientific community. Recently special attention has been given to the synchronization of large-scale networks of coupled oscillators (Wu, 2002; Boccaletti et al., 2006). In particular, studies on synchronization of chaotic systems coupled in complex small-world and scale-free topologies (Wang and Chen, 2002a; Wang and Chen, 2002b; Barajas and Femat, 2008). The main concert of these investigations has been to improved the understanding of the synchronization phenomenon in real-world complex networks, such as the Internet, the WWW, biological and even social networks (Arenas et al., 2008; Newman, 2010).

In most of the works referenced above, the authors have focus their attention on the emergence of identical synchronization in networks consisting of identical $n-$dimensional dynamical systems. However, in real-world situations a far more likely scenario is that the nodes be totally different systems, with parameter uncertainties and disturbances. In these situations, identical synchronization is unlikely. Yet, these networks exhibit other forms of temporal correlation, like interdependence, autoorganization, consensus and collaboration. Clearly these interactions are more complex that the simultaneous dynamical evolution, these phenomenons require a more general definition of synchronization. Generalized synchronization (GS) is defined as a functional relationship between states of two systems (Boccaletti et al., 2002). Different types of GS can be defined, depending how the state space of one node are mapped to the others. In this way, one can think of complete identical synchronization as a particular case of GS where the functional relationship is the identity. Another form of GS is achieved when the functional relationship is defined in terms of coordinate transformations, for example a diffeomorphism define on a feedback linearization (Femat et al., 2005).

In the literature we have basically to approaches to achieve GS: An indirect method, in which synchronization in generalized terms is inferred from the identical synchronization of two systems under the same driving force, the so-called auxiliary system approach (Abarbanel et al., 1996). Alternatively, GS is directly achieved by controllers that impose a prescribed functional relationship between the systems, for that reason this is usually called the controlled synchronization approach (Barajas-Ramirez et al., 2012). Recently the concept of GS which was originally utilized for the study of synchronization on master-slave configurations of chaotic systems has been generalizes to the problem of network synchronization. In some of the initial works (Hung et al., 2008; Xu et al., 2008; Liu et al., 2010) the auxiliary systems approach was considered, while other focus on the controlled synchronization approach (Guan et al., 2009). The main difference between these approaches is whether or not the description of the functional relationship between the nodes is of significance. In the case of auxiliary system approach, its existence is implied, while in the controlled synchronization is a requirement.

In this contribution we take the controlled synchronization approach to achieved GS in a network of linear and diffusively coupled triangularizable nonidentical n-dynamical systems. We proposed robust controllers designed such that a given functional relationship between the states of different groups of nodes are imposed.

The rest of the manuscript is organized as follows. In Section II, GS problem for a network is express as a stability analysis and a robust controller design problems. In Section III, our proposed controlled synchronization design for GS is presented for a particular class of dynamical networks. In Section IV, the effectiveness of the proposed design is illustrated with numerical simulations of network with different groups of chaotic nodes. Finally in Section V, some comments and conclusions are presented.

II. Problem Statement

Consider a network of N non-identical nodes, with each one been a dynamical system described by

$$\dot{x}_i = f_i(x_i) \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in})^\top \in \mathbf{R}^n$ are state variables of node i (all nodes are assume to have the same dimension n); and $f_i : \mathbf{R}^n \to \mathbf{R}^n$ is a known nonlinear function describing the dynamical evolution of node i.

The state equation of the entire dynamical network is

$$\dot{x}_i = f_i(x_i) + g_i(X) + u_i \tag{2}$$

for $i = 1, 2, ..., N$ where $X = (x_1, x_2, ..., x_N) \in \mathbf{R}^{n \times N}$ are form by the state variables of all the nodes; $g_i : \mathbf{R}^{n\times N} \to \mathbf{R}^n$ are the coupling functions describing the connections to node i from the rest of the network; and $u_i \in \mathbf{R}^m$ $(m \leq n)$ is a local controller to be designed.

A dynamical network is said to be identically synchronized, if the state solutions of every node move in unison, in the sense that

$$\lim_{t\to\infty} \|x_i - x_j\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{3}$$

The synchronization criterion for complete identical synchronization, can be interpreted as a requiring that the states variables of any node in the network be exactly mapped to the state variables of any other. A generalization of this interpretation of synchronization can be introduced by considering mappings between the state variables of the nodes to be different from the identity, in this way more complicated interactions between the network components can be considered (Boccaletti et al., 2002). Then, the network in (2) will be synchronized in a generalized sense with respect to the functional relation H_i if the condition

$$\lim_{t\to\infty} \|x_i - H_i(x_j)\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{4}$$

is satisfied. Note that, the functional H_i maybe the same for all the nodes or it can be different for each pair of nodes. Additionally, as presented in (Barajas-Ramirez et al., 2012), potentially each system can have its own transformation H_{Mi} and H_{Si}, with a GS condition

$$\lim_{t\to\infty} \|H_{Mi}(x_i) - H_{Si}(x_j)\| = 0, \text{ for } i, j = 1, 2, ..., N \tag{5}$$

where $H_i = H_{Mi} \circ H_{Si}$.

In the sense of (Abarbanel et al., 1996), GS is achieved by the existence of H_i not by its exact description. That is, if an auxiliary system is consider to experience the same driving forces as our system and they identically synchronize to each other; then, the existence of a functional relationship can be inferred for the original system. Similarly to (Hung et al., 2008; Xu et al., 2008; Liu et al., 2010), we can extend this approach to determine GS in a dynamical network. Considering the coupling to each node in the network as an external driving, an exact replica of the network dynamics

$$\dot{\hat{x}}_i = f_i(\hat{x}_i) + g_i(\hat{X}) + u_i \tag{6}$$

for $i = 1, 2, ..., N$ where $\hat{X} = (\hat{x}_1, ..., \hat{x}_N) \in \mathbf{R}^{n\times N}$ can be taken to be an auxiliary network system.

Following the auxiliary system approach the network (2) achieves GS if (Abarbanel et al., 1996)

$$\lim_{t\to\infty} |x_i - \hat{x}_i| = 0 \tag{7}$$

for $i = 1, ..., N$ with the initial conditions $x_i(0) \neq \hat{x}_i(0)$.

From (2) and (6) the dynamics of the error $\epsilon_i = \hat{x}_i - x_i$ are given by

$$\dot{\epsilon}_i = f_i(x_i) - f_i(\hat{x}_i) + g_i(X) - g_i(\hat{X}) \tag{8}$$

for $i = 1, ..., N$. The emergence of GS is equivalent to the stability of the zero equilibrium point of (8). There are different results in the literature where GS is assured for dynamical networks under some very standard assumptions like, global Lipschitz condition for all nodes with linear and diffusive couplings; e.g. in (Liu et al., 2010) adaptive coupling strengths are used to achieve GS.

The auxiliary system approach for GS can be applied to establish that the network is synchronized in a generalized sense. However, although the functional relations H_i exist, its not possible to determine its specific form. If we want to impose a functional relationship among the nodes in the network a controlled synchronization approach is needed. We define a GS error $e_{kj} = H_{Mk}(x_k) - H_{Sk}(x_j)$, for $k, j = 1, ..., N$, which has the dynamics

$$\dot{e}_{kj} = H_{Mk}(f_k(x_k) + g_k(X)) - H_{Sk}(f_j(x_j) + g_j(X)) + \nu_k \tag{9}$$

for $k, j = 1, ..., N$ with $\nu_k = H_{Mk}(u_k) - H_{Sk}(u_j)$.

The total number of SG errors in the network can be reduced to $N - 1$ by defining $j = k + 1$, then we have

$$\dot{e}_i = H_{Mi}(f_i(x_i) + g_i(X)) - H_{Si}(f_{i+1}(x_{i+1}) + g_{i+1}(X)) + \nu_i \tag{10}$$

for $i = 1, ..., N - 1$. To stabilize (10) the design of ν_i can be undertaken from different points of view. In the following Section, we design the controllers from a coordinate transformation approach.

III. Generalized synchronization design

The design of the local controllers ν_i strongly depends on the nature of the nodes and the network topology. In the first place we consider that all nodes can be taken, via a coordinate transformation, to a triangularized form. Although this may seem a very restrictive condition, a large number of chaotic systems can in fact be transformed to a triangular or at least partially triagularized form with internal dynamics by linearizing feedback (Barajas-Ramirez et al., 2012). Additionally, chaotic dynamics can be generated from piecewise linear systems that are easily triangularizable (Sprott, 2000; Campos et al., 2010).

We assume that an adequate coordinate transform $\mathcal{T}_i$ exist such that (1) can be rewritten as:

$$\dot{z}_i = A_i z_i + B\psi_i \tag{11}$$

where $z_i = \mathcal{T}_i x_i = (z_{i1}, z_{i2}, ..., z_{in})^\top \in \mathbf{R}^n$ are the transform state coordinates of node i; the constant matrices A_i and B have the controller-type companion form

$$A_i = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ : & : & : & ... & : \\ 0 & 0 & 0 & ... & 1 \\ a_{i,1} & a_{i,2} & a_{i,3} & ... & a_{i,n} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

with ψ_i the linearizing feedback controller if that is the case.

The network topology is assumed to be linear and diffusive, such that,

$$g_i(Z) = \gamma \sum_{j=1}^{N} c_{ij} \Gamma z_j \tag{13}$$

for $i = 1, 2, ..., N$ where $Z = (z_1, ..., z_N) \in \mathbf{R}^{n \times N}$ are the node states in the transformed coordinates; $\Gamma \in \mathbf{R}^{n \times n}$ is a $0-1$ inner connection matrix describing the manner in which the state variables of node i and j are connected; and $C = \{c_{ij}\} \in \mathbf{R}^{N \times N}$ is the $0-1$ coupling matrix, which captures the topological structure of the network, if $c_{ij} = 1$ ($i \neq j$), there is a connection of strength γ between the nodes i and j, otherwise the nodes are disconnected. For diffusive coupling the diagonal entries of the coupling matrix satisfy the following equality

$$c_{ii} = -\sum_{j=1, j \neq i}^{N} c_{ij} = -\sum_{j=1, j \neq i}^{N} c_{ji} \tag{14}$$

for $i = 1, 2, ..., N$. If there are no isolated nodes in the network, using Gershgorin circle theorem we can establish that the eigenvalues of C are real negative values with zero an eigenvalue of multiplicity one.

Under this connection structure in the transformed variables the network in (2) becomes:

$$\dot{z}_i = A_i z_i + B\psi_i + \gamma \sum_{j=1}^{N} c_{ij} \Gamma z_j + \nu_i, \tag{15}$$

for $i = 1, 2, ..., N$. where ν_i are the local controllers for the network in the transformed variables. To achieve GS in the original variables, in the transformed variables we look for complete identical synchronization of the network. In particular, since the nodes in the transformed variables have a very similar structure (11)-(12), we can argue that their differences are bounded and we can define the average node as reference for synchronization $\bar{s} = \frac{1}{N}\sum_{j=1}^{N} z_j$ (Barajas, 2012). The dynamics of the average node are given by

$$\begin{aligned} \dot{\bar{s}} = \; & \tfrac{1}{N}\textstyle\sum_{k=1}^{N}(A_k z_k + B\psi_k) \\ & + \tfrac{1}{N}\textstyle\sum_{k=1}^{N}(\gamma \sum_{j=1}^{N} c_{kj}\Gamma z_j) \\ & + \tfrac{1}{N}\textstyle\sum_{k=1}^{N}(\nu_i) \end{aligned} \tag{16}$$

Notice that under the assumption that the control actions vanish at the synchronized solution ($z_1 = z_2 = ... = z_N = \bar{s}$) and as a consequence of the diffusive nature of the coupling (14), once the network is synchronized the second term of (16) is also zero. Then, at the synchronized solution the dynamics of the reference node is the average of the nodes isolated from the network.

$$\dot{\bar{s}} = \frac{1}{N}\sum_{k=1}^{N}(A_k z_k + B\psi_k) \tag{17}$$

We define the synchronization error as $e_i = z_i - \bar{s}$, from (15) and (16) the error dynamics are given by:

$$\dot{e}_i = \mathcal{A}_i(Z, \bar{s}) + \gamma \sum_{j=1}^{N} c_{ij}\Gamma e_j + \nu_i \tag{18}$$

for $i = 1, 2, ..., N$ with

$$\begin{aligned} \mathcal{A}_i(Z, \bar{s}) = \; & A_i z_i + B\psi_i - \tfrac{1}{N}\textstyle\sum_{k=1}^{N}(A_k z_k + B\psi_k) \\ & -\tfrac{1}{N}\textstyle\sum_{k=1}^{N}(\gamma \sum_{j=1}^{N} c_{kj}\Gamma z_j) \\ & -\tfrac{1}{N}\textstyle\sum_{k=1}^{N}(\nu_i) \end{aligned}$$

Given that the term $\mathcal{A}_i$ in 18) is the difference between the the current node and the reference node ($z_i - \bar{s}$). If we restrict our attention to chaotic nodes which can be traingularized. Then, is reasonable to expect that this term is bounded, that is, we assume that

$$|\mathcal{A}_i(Z, \bar{s})| \leq \beta_i \tag{19}$$

for $i = 1, 2, ..., N$ with β_i nonnegative constants.

To stabilize the error dynamics (18) in the presence the non-vanishing perturbations due to the nodes differences (19) we propose a discontinuous feedback controller as described as follows.

Theorem 1: If the local controllers ν_i are constructed as

$$\nu_i = -\gamma\, \kappa\, \Gamma\, e_i - \delta sgn(e_i) \tag{20}$$

for $i = 1, 2, ..., N$ where $sgn(e_i) = [e_{i,1}, e_{i,2}, ..., e_{i,N}]^\top \in \mathbf{R}^n$ with $sgn(\cdot)$ is the discontinuous function signum, given by

$$sgn(x) = \begin{cases} 1, & if \quad x > 0 \\ 0, & if \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$$

and the smooth $\kappa > 0$ and discontinuous $\delta > 0$ controller gains, are designed to satisfy the bounds

$$\begin{aligned} \kappa & \geq \gamma\lambda_{min} \\ \delta & \geq \frac{\sum_{i=1}^{N}\beta_i\|\omega_{ji}\|}{\sum_{i=1}^{N}\|\omega_{ji}\|} \end{aligned} \tag{21}$$

where $\lambda_{min} = \min(|\lambda_i|)$ and ω_{ji} are the elements of a coordinate transformation that diagonalize the error coupling matrix $C = \Omega\Lambda\Omega$ with Λ is a diagonal matrix composed by the eigenvalues of C.

Then, the linear and diffusively connected network (15) will identically synchronized with the reference node $\bar{s}$. Equivalently, the original network (2) synchronizes in the generalized sense (5) in terms of the coordinate transformations $\mathcal{T}_i$.

`Proof`: Defining the vector variables

$$\bar{e} = [e_1, e_2, ..., e_N] \in \mathbf{R}^{n\times N}$$

$$\bar{\mathcal{A}}(\cdot) = [\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_N] \in \mathbf{R}^{n\times N}$$

$$\bar{\psi} = [\psi_1, \psi_2, ..., \psi_N] = -\gamma\Gamma\bar{e}D - \delta sgn(\bar{e}) \in \mathbf{R}^{n\times N}$$

where $D = diag(\kappa, ..., \kappa) \in \mathbf{R}^{N\times N}$ and $sgn(\bar{e}) = [sgn(e_1), sgn(e_2), ..., sgn(e_N)] \in \mathbf{R}^{n\times N}$; the error dynamics (18) can be rewritten as

$$\dot{\bar{e}} = \bar{\mathcal{A}}(Z, \bar{s}) + \gamma\Gamma\bar{e}\left(C^\top - D\right) - \delta sgn(\bar{e})$$

Since the coupling matrix is symmetric ($C = C^\top$), there exist a unitary matrix Ω, that satisfies:

$$C = \Omega\Lambda\Omega^\top$$

where $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbf{R}^{N\times N}$ with λ_i the eigenvalues of C; and $\Omega = (\omega_1, \omega_2, ..., \omega_N) \in \mathbf{R}^{N\times N}$, with $\Omega^\top\Omega = I_N$, where I_N represents the N-dimensional identity matrix.

Using a change of variables $\bar{\eta} = \bar{e}\,\Omega$, the error dynamics become

$$\dot{\bar{\eta}} = \bar{\mathcal{A}}(Z, \bar{s})\Omega + \gamma\Gamma\bar{\eta}\left(\Lambda - D\right) - \delta sgn(\bar{e})\Omega$$

where $\bar{\eta} = (\eta_1, \eta_2, ..., \eta_N)$, with $\eta_i = \bar{e}\ \omega_i \in \mathbf{R}^n$; or equivalently,

$$\dot{\eta_i} = \bar{\mathcal{A}}(Z, \bar{s})\omega_i + \gamma\left(\lambda_i - k\right)\Gamma\eta_i - \delta sgn(\bar{e})\omega_i, \qquad (22)$$

for $i = 1, 2, ..., N$.

The stability of the error dynamics around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{j=1}^{N} \eta_j^\top \eta_j,$$

The time derivative of V along the trajectories of the error dynamics in (22) is given by

$$\begin{array}{rcl} \dot{V} & = & \sum_{j=1}^{N} \eta_j^\top \bar{\mathcal{A}}(Z, \bar{s})\omega_j + \sum_{j=1}^{N} \eta_j^\top \gamma\left(\lambda_i - \kappa\right)\Gamma\eta_j \\ & & -\sum_{j=1}^{N} \eta_j^\top \delta sgn(\bar{e})\omega_j, \\ & \leq & \sum_{j=1}^{N} \|\eta_j\|^\top \|\bar{\mathcal{A}}(Z, \bar{s})\omega_j\| \\ & & +\sum_{j=1}^{N} \gamma(\lambda_i - \kappa)\|\eta_j\|^\top\Gamma\|\eta_j\| \\ & & -\sum_{j=1}^{N} \delta\|\eta_j\|^\top\|sgn(\bar{e})\omega_j\| \end{array}$$

Considering the bounds of each term, we have that the second term is quadratic and will be negative if its coefficient $-\gamma(\lambda_i - \kappa)$ is negative for every i. Since by construction the eigenvalues of C are all nonpositive, letting $\lambda_{min} = \text{mín}(|\lambda_i|)$ the bound on the smooth control gain κ becomes

$$\kappa \geq \gamma\lambda_{min}$$

From (19) one has the bound of the first term:

$$\|\bar{\mathcal{A}}(Z, \bar{s})\omega_j\| \leq \|\sum_{i=1}^{N} \mathcal{A}_i(Z, \bar{s})\omega_{ji}\| \leq \sum_{i=1}^{N} \beta_i\|\omega_{ji}\|$$

The bound for the third term is given by:

$$\|sgn(\bar{e})\omega_j\| \leq \sum_{i=1}^{N} sgn(e_i)\|\omega_{ji}\| \leq \sum_{i=1}^{N} \|\omega_{ji}\|$$

From the above results the time derivative of the Lyapunov function is bounded by

$$\begin{array}{rcl} \dot{V} & \leq & \sum_{j=1}^{N} \|\eta_j\|^\top \left(\sum_{i=1}^{N} \beta_i\|\omega_{ji}\| - \delta\sum_{i=1}^{N} \|\omega_{ji}\|\right) \\ & & -\gamma(\lambda_i - \kappa)\sum_{j=1}^{N} \|\eta_j\|^\top\|\eta_j\| \end{array}$$

For the time derivative of V to be negative, the discontinuous gain must satisfy

$$\delta \geq \frac{\sum_{i=1}^{N} \beta_i\|\omega_{ji}\|}{\sum_{i=1}^{N} \|\omega_{ji}\|}$$

Then, the error dynamics in (18) are globally uniformly asymptotically stable about the zero fixed point ($\bar{\eta} = 0$), which implies that

$$\text{lím}_{t\to\infty} \bar{e} = \text{lím}_{t\to\infty} \{z_1 - \bar{s}, z_2 - \bar{s}, ..., z_N - \bar{s}, \} = 0$$

In consequence, the dynamical network (2) under the controller (20), achieves GS in the sense of (5), with respect to the coordinate transformations $\mathcal{T}_i$.

Q.E.D.

IV. Illustrative example

We consider a network with two different types of nodes which can be triangularized. Namely, Sprott circuits ($\bigcirc$)(Sprott, 2000):

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & -0.6x_3 - x_2 - 1.2x_1 + 2sgn(x_1) + u \end{array} \qquad (23)$$

which is already in triangular form. The other type of nodes are Rössler systems ($\square$) (Barajas-Ramirez et al., 2012):

$$\begin{array}{rcl} \dot{x}_1 & = & -x_2 - x_3 \\ \dot{x}_2 & = & x_1 + 0.1x_2 \\ \dot{x}_3 & = & x_3(x_1 - 14) + 0.1 + u \end{array} \qquad (24)$$

The different nodes are connected randomly according to the ER network model (Newman, 2010). A possible realization with twenty nodes is shown in Figure 1. To achieve GS in the network we use a coordinate transformation for the Rössler system. Assuming that the output of (24) is $y = x_2$, the following coordinate transformation takes the Rössler

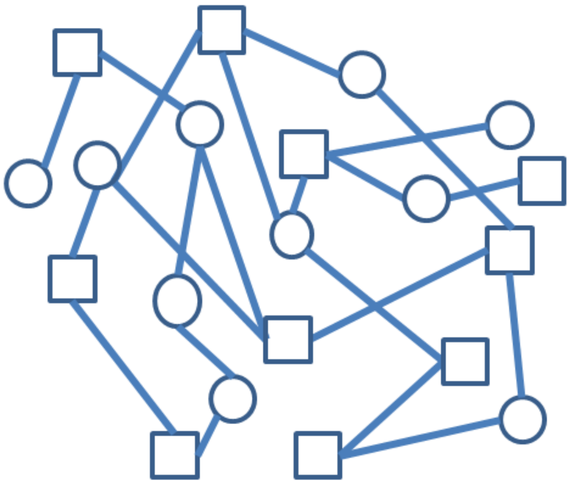

Figura 1. Network of non-identical nodes (Sprotts:◯ and Rössler:□).

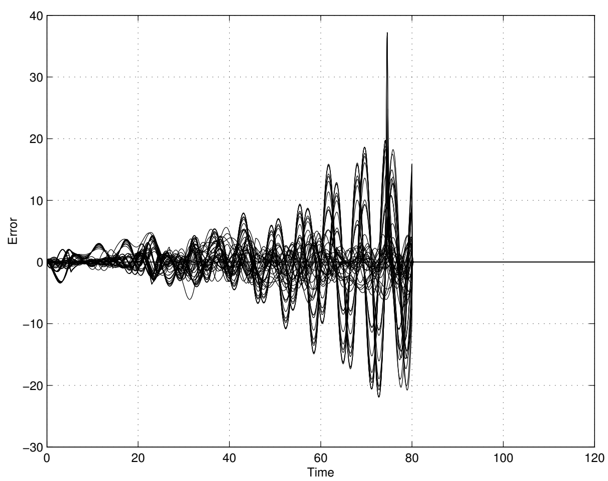

Figura 2. Synchronization error on the transformed coordinates $z(t)$

system to a triangular form with the transform variables $z = \phi(x)$ (Femat et al., 2005):

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + a x_2 \\ a x_1 + (a^2 - 1) x_2 - x_3 \end{pmatrix} \tag{25}$$

This coordinate transformation is a diffemorphism and its inverse is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a z_1 + z_2 \\ z_1 \\ (2a^2 - 1) z_1 + a z_2 - z_3 \end{pmatrix} \tag{26}$$

The local controllers such that GS is achieved in the network are design according to Theorem 1. In Figure 2 the trajectories of the Rössler and Sprott circuits in the transformed coordinates are shown. In Figure 3, we show their error in the original coordinates of the network.

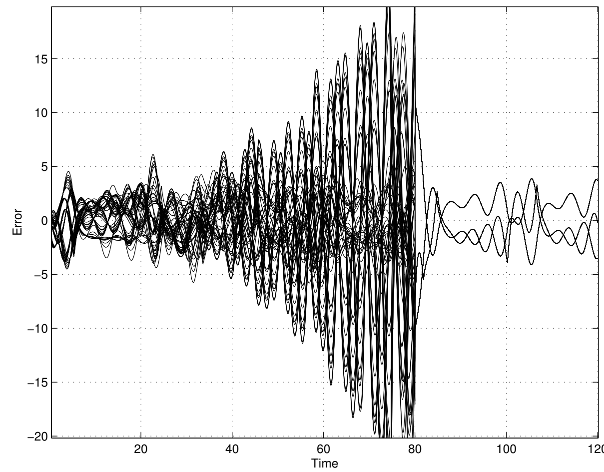

Figura 3. Synchronization error on the original coordinates $x(t)$

V. Conclusions

On networks with different dynamical nodes complete identical synchronization is not directly achievable. As such, alternative interpretations of the synchronization phenomena are possible. In this contribution, we investigate the emergence of GS in a network of non identical nodes. Unlike some of the current results in the literature, where the auxiliary system approach it taken, we use a controlled synchronization approach to impose a functional relationship between the nodes in the network. In particular, we consider nodes that can be triangularized by a coordinate transformation and possibly linearizing feedback. Under such conditions, GS can be achieved through robust synchronizing controllers, which naturally apply for triangularized chaotic systems, since their differences are bounded. We design robust controllers such that identical synchronization is achieved in the transformed coordinates, in this way, GS with respect to the corresponding coordinate transformations are achieved between the nodes in the network. Obvious limitations of the proposed method are the fact it requires the existence of a triangularizing coordinate transformation, and that the number of controllers is than of the nodes. Additionally, the imposed functional relationship between the nodes is fixed by the coordinate transformation. However, it seems possible to overcome this restrictions by considering alternative way to design the synchronizing controllers. These are considerations for future work and will be reported elsewhere.

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