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Nonsmooth $\mathcal{L}_2$-Gain Analysis of Super-Twisting Algorithm

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***Resumen*— The present work extends the $\mathcal{L}_2$-gain analysis towards sliding mode dynamic systems and it is tested on the super-twisting algorithm to illustrate that the resulting closed-loop system is capable not only of rejecting matching uniformly bounded disturbances, but also of attenuating unbounded ones.**

Keywords: Nonsmooth systems, $\mathcal{L}_2$-gain, Super-twisting algorithm.

I. Introduction

Sliding mode control algorithms are well recognized for their useful robustness features against matching disturbances with *a priori* known bounds on their magnitudes. Their capability of attenuating disturbances with *a priori* unknown bounds on their magnitudes, which remain unattended in the literature, constitute the topic of the present investigation. First, the $\mathcal{L}_2$-gain analysis is extended toward sliding mode dynamic systems and then it is tested on a pre-selected sliding mode control algorithm, being the popular super-twisting controller. It is thus demonstrated that the super-twisting algorithm controller is capable of not only rejecting matching bounded disturbances but also of attenuating the ones of class $\mathcal{L}_2$.

II. Nonsmooth $\mathcal{L}_2$-Gain Analysis

The $\mathcal{L}_2$-gain analysis, presented here, is based on the game-theoric approach from (Basar and Bernhard, 1995) and extends the results from (Isidori and Astolfi, 1992), (Van Der Shaft, 1992), where investigations were confined to smooth autonomous systems, towards locally Lipschitz continuous autonomous systems.

A. Basic assumptions and definitions

The $\mathcal{L}_2$-gain analysis is developed for an autonomous system of the form

$$\dot{x} = \varphi(x) + \psi(x)w(t) \tag{1}$$

and is made with respect to the output

$$z = h(x). \tag{2}$$

Hereinafter, $x(t) \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{R}$ is the time variable, $w(t) \in \mathbb{R}^r$ are the unknown disturbances, $z(t) \in \mathbb{R}^p$, $\varphi(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$ are vector functions, and $\psi(x) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times r}$ is a matrix function. The following *assumptions* are imposed on the system.

1) The functions $\varphi(x)$, $\psi(x)$, and $h(x)$ are piecewise continuous locally Lipschitz in x behind the discontinuity manifold.
2) $\varphi(0) = 0$ and $h(0) = 0$ for almost all t.

Recall that the function $\varphi(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is piece-wise (locally Lipschitz) continuous iff $\mathbb{R}^n$ is partitioned into a finite number of domains $G_j \subset \mathbb{R}^n$, $j = 1, \dots, N$, with disjoint interiors and boundaries ∂G_j of measure zero such that $\varphi(x)$ is (locally Lipschitz) continuous within each of these domains and for all $j = 1, \dots, N$ it has a finite limit $\varphi^j(x)$ as the argument $x^* \in G_j$ approaches a boundary point $x \in \partial G_j$.

Assumption 2 is made to ensure that the origin is an equilibrium point of the nominal (i.e., disturbance-free) system whereas Assumption 1 admits the underlying system to undergo discontinuities on the boundaries ∂G_j of measure zero, which is why the precise meaning of the differential equation (3) with a piece-wise continuous right-hand side is throughout defined in the sense of Filippov. For convenience of the reader, the following definition is recalled from (Filippov, 1988).

Definition 1 *Given the differential equation*

$$\dot{x} = \varphi(x), \tag{3}$$

let us introduce for each point $x \in \mathbb{R}^n$ the smallest convex closed set $\Phi(x)$ which contains all the limit points of $\varphi(x^*)$ as $x^* \to x$, and $x^* \in \mathbb{R}^n \setminus (\cup_{j=1}^N \partial G_j)$. An absolutely continuous function x, is said to be a solution of (3) if it satisfies the differential inclusion

$$\dot{x} \in \Phi(x). \tag{4}$$

Apart from this, we extend the $\mathcal{L}_2$-gain concept to the above discontinuous system.

Definition 2 *Given a real number $\gamma > 0$, further referred to as a disturbance attenuation level, it is said that the system (1) (locally) possesses $\mathcal{L}_2$-gain less than γ with respect to output (2) (or, simply, system (1), (2) (respectively, locally) possess $\mathcal{L}_2$-gain less than γ if the response $z(t)$, resulting from $w(t)$ for initial state $x(t_0) = 0$, satisfies*

$$\int_{t_0}^{t_1} \|z(t)\|^2 dt < \gamma^2 \int_{t_0}^{t_1} \|w(t)\|^2 dt \tag{5}$$

for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$ (locally around the origin).

Definition 3 *Respectively, system* (1), (2) *is said to have* $\mathcal{L}_2$-gain less than γ, locally around the origin, *if there exists a neighborhood U of the origin such that the inequality* (5) *is satisfied for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$ for which the state trajectory of the closed-loop system starting from the initial point $x(t_0) = 0$ remains in U for all $t \in [t_0, t_1]$.*

For later use, the following instrumental results, inspired from (Clarke, 1988), are involved.

Lemma 1 *Let $x \in \mathbb{R}^n$ be an absolutely continuous function of time variable t and let $V(x)$ be a scalar locally Lipschitz function around $x \in \mathbb{R}^n$. Then the composite function $V(x)$ is absolutely continuous and its time derivative is given by*

$$\frac{d}{dt} V(x(t)) = DV(x(t), \dot{x}(t)) \tag{6}$$

almost everywhere. Furthermore,

$$DV(x(t), \dot{x}(t)) \leq \frac{\partial V}{\partial x} \dot{x}(t) \tag{7}$$

for almost all t and for all supergradients $\left(\frac{\partial V}{\partial x}\right)^T \in \partial V(x)$, if any.

Lemma 2 *Let system (3) possess a Lyapunov function $V(x)$. Then system (3) is stable. If in addition, the function $V(x)$ is a strict Lyapunov function (and radially unbounded) then system (3) is (globally) asymptotically stable.*

B. Hamilton-Jacobi inequality and their proximal solutions

System (1)–(2) is subsequently analyzed under the hypothesis that

H) there exists a piecewise locally Lipschitz continuous, positive definite, radially unbounded proximal solution $V(x)$ of the Hamilton-Jacobi inequality

$$\frac{\partial V}{\partial x}\varphi(x) + \frac{1}{4\gamma^2}\frac{\partial V}{\partial x}\psi(x)\psi^T(x)\left(\frac{\partial V}{\partial x}\right)^T + h(x)^T h(x) \leq -v(x) \tag{8}$$

under some positive γ and some positive definite function $v(x)$.

A locally Lipschitz continuous function $V(x)$ is said to be a *proximal solution* of the partial differential inequality (8) iff its proximal superdifferential $\partial^P V(x)$ is everywhere non-empty and (8) holds with $V(x)$ for all $x \in \mathbb{R}^n$, $\varphi(x) \in \Phi(x)$, and for all proximal supergradients $\frac{\partial V}{\partial x} \in \partial^P V(x)$. The interested reader may refer (Clarke, 1988) for the proximal superdifferential concept.

C. Global analysis

The following result presents sufficient conditions of the nonsmooth system (1), (2) to be internally asymptotically stable and to possess $\mathcal{L}_2$-gain less than γ.

Theorem 1 *Let Assumptions 1 and 2 be in force, and let Hypothesis H be satisfied (locally). Then the nominal system (3) is globally (locally) asymptotically stable whereas its disturbed version (1) possesses (locally) $\mathcal{L}_2$-gain less than γ with respect to output (2).*

Proof: It is clear that Lemma 1 is applicable to a proximal solution $V(x)$ of the Hamilton-Jacobi inequality (8) viewed on the solutions $x(t)$ of the disturbance-free system (3). Then relations (6)–(8), coupled together, result in

$$\frac{d}{dt}V(x) = DV(x, \dot{x}) \leq \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}\varphi(x) \leq -v(x). \tag{9}$$

where $v(x)$ is some positive definite function.

Taking into account that (9) holds almost everywhere, Hypothesis **H** thus ensures that $V(x)$ is a strict radially unbounded Lyapunov function of the nominal system (3). By Lemma 2, system (3) is globally (locally) asymptotically stable.

It remains to show that the disturbed system (1) (locally) possesses $\mathcal{L}_2$-gain less than γ with respect to output (2). For this purpose, let us introduce the multivalued function

$$H(x, w) = \frac{\partial V(x)}{\partial x}\left[\phi(x) + \psi(x)w\right] + h^T(x)h(x) - \gamma^2 w^T w \tag{10}$$

where $\frac{\partial V}{\partial x} \in \partial^P V(x)$. Clearly, the multi-valued function (10) is quadratic in w. Then

$$\left.\frac{\partial H(x, w)}{\partial w}\right|_{w=\alpha(x)} = \frac{\partial V(x)}{\partial x}\psi(x) - 2\gamma^2\alpha^T(x) = 0 \tag{11}$$

for $\alpha(x) = \frac{1}{2\gamma^2}\psi^T(x)\left(\frac{\partial V(x)}{\partial x}\right)^T$ and $\frac{\partial V}{\partial x} \in \partial^P V(x)$. Expanding the quadratic function $H(x, w)$ in Taylor series, we derive that

$$H(x, w) = H(x, \alpha(x)) - \gamma^2\|w - \alpha(x)\|^2 \tag{12}$$

where $H(x, \alpha(x)) \leq -v(x)$ due to (8). Hence,

$$H(x, w) \leq -\gamma^2\|w - \alpha(x)\|^2 - v(x) \tag{13}$$

and employing (10) and (12) we arrive at

$$\frac{\partial V(x)}{\partial x}[\phi(x)+\psi(x)w] \le -\gamma^2\|w-\alpha(x)\|^2 \\ -v(x)-\|h(x)\|^2+\gamma^2\|w\|^2 \quad (14)$$

By applying Lemma 1 and taking (14) into account, the time derivative of the solution $V(x)$ of the Hamilton-Jacobi inequality (8) on the trajectories of (1) is estimated as follows

$$\frac{d}{dt}V(x) \le -\gamma^2\|w-\alpha(x)\|^2 - v(x) - \|z\|^2 + \gamma^2\|w\|^2 \quad (15)$$

As a matter of fact, the latter inequality ensures that

$$\int_{t_0}^{t_1}(\gamma^2\|w(t)\|^2-\|z(t)\|^2)dt \ge V(x(t_1))-V(x(t_0)) \\ +\gamma^2\int_{t_0}^{t_1}[\|w(t)-\alpha(x(t))\|^2+v(x(t))]dt > 0 \quad (16)$$

for any trajectory of (1), (2), initialized with $x(t_0)=0$. Thus, inequality (5) is established thereby completing the proof of Theorem 1.

■

III. $\mathcal{L}_2$-Gain Analysis for the Super-Twisting Algorithm

In this section, we will develop the $\mathcal{L}_2$-gain analysis of the algorithm governed by the following second-order system based on the super-twisting algorithm (Levant, 1993):

$$\begin{aligned}\dot{x}_1 &= x_2 - k_1|x_1|^{\frac{1}{2}}\,\mathrm{sign}(x_1) + \mu_1(x)w_1,\\ \dot{x}_2 &= -k_3\,\mathrm{sign}(x_1) + \mu_2(x)w_2,\end{aligned} \quad (17)$$

where $x=[x_1,x_2]^T \in \mathbb{R}^2$ is the state vector, $w=[w_1,w_2]^T \in \mathcal{L}_2$ is the disturbance vector which is assumed to be unknown, k_1 and k_3 are positive constants, and $\mu_i(x):\mathbb{R}^2 \mapsto \mathbb{R}, i=1,2$, are continuous functions. The analysis will be made with respect to the output

$$z=[x_1,x_2]^T. \quad (18)$$

If the system is referred to the observer, then the actuator error is adsorbed in the term $\mu_2(x)w_2$ only, whereas the term $\mu_1(x)w_1$ takes into account the measurement error only.

It should be noted that the above system is represented in the form (1), (2) if specified with

$$\varphi(x)=\begin{bmatrix}x_2-k_1|x_1|^{\frac{1}{2}}\,\mathrm{sign}(x_1)\\ -k_3\mathrm{sign}(x_1)\end{bmatrix}, \quad (19)$$

$$\psi(x)=\begin{bmatrix}\mu_1(x) & 0\\ 0 & \mu_2(x)\end{bmatrix}, \quad (20)$$

$$h(x)=[x_1,x_2]^T. \quad (21)$$

A. $\mathcal{L}_2$-Gain Analysis

Consider the positive definite function extracted from (Moreno and Osorio, 2008)

$$V=2k_3|x_1|+\frac{1}{2}x_2^2+\frac{1}{2}s^2 \quad (22)$$

where

$$s=x_2-k_1|x_1|^{\frac{1}{2}}\,\mathrm{sign}(x_1). \quad (23)$$

Let us verify that the Hamilton-Jacobi inequality (8) is satisfied with the positive definite function (22)–(23). For this purpose, let us denote $\mathcal{H}=H(x,\alpha(x))$, that is, the notation $\mathcal{H}$ stands for the left-hand side of the Hamilton-Jacobi inequality (8). Then by inspection, one derives that

$$\mathcal{H}=-\frac{1}{2}k_1s^2|x_1|^{-\frac{1}{2}}+k_3(s-x_2)\mathrm{sign}(x_1) \\ +\frac{1}{4\gamma^2}\left(2k_3\mathrm{sign}(x_1)-\frac{1}{2}k_1s|x_1|^{-\frac{1}{2}}\right)^2\mu_1^2(x) \\ +\frac{1}{4\gamma^2}(x_2+s)^2\mu_2^2(x)+x_1^2+x_2^2 \quad (24)$$

where we used $\varphi(x)=[s,-k_3\mathrm{sign}(x_1)]^T$. Taking into account that $x_2=s+k_1|x_1|\mathrm{sign}(x_1)$ due to (23) and using the well known inequality $(a+b)^2 \le 2a^2+2b^2$, the following upper bound

$$\mathcal{H}\le-\frac{1}{2}k_1s^2|x_1|^{-\frac{1}{2}}-k_1k_3|x_1|^{\frac{1}{2}}+\frac{2}{\gamma^2}k_3^2\mu_1^2(x) \\ +\frac{1}{8\gamma^2}k_1^2s^2|x_1|^{-1}\mu_1^2(x)+\frac{1}{2}\gamma^2s^2\mu_2^2(x) \\ +\frac{1}{2\gamma^2}k_1^2|x_1|\mu_2^2(x)+x_1^2+2s^2+2k_1^2|x_1| \quad (25)$$

is obtained.

In what follows, the perturbation term $\mu_2(x)w_2$ is specified with $\mu_2=\eta_2^{\frac{1}{2}}$ for ease of reference, whereas the perturbation term $\mu_1(x)w_1$ is of the form

$$\mu_1(x)=\eta_1^{\frac{1}{2}}|x_1|^{\frac{1}{2}}, \qquad \eta_{1,2}\in\mathbb{R}^+ \quad (26)$$

for ensuring that it escapes to zero as x goes to zero because only such perturbations are recognized in the literature (Moreno and Osorio, 2008) to admit attenuation. Then

$$\mathcal{H}\le-\left(\frac{1}{2}k_1|x_1|^{-\frac{1}{2}}-\frac{1}{8\gamma^2}\eta_1k_1^2-\frac{2}{\gamma^2}\eta_2-2\right)s^2 \\ -\left[k_1k_3-\left(\frac{2}{\gamma^2}\eta_1k_3^2-\frac{1}{2\gamma^2}\eta_2k_1^2-|x_1|-2k_1^2\right)|x_1|^{\frac{1}{2}}\right]|x_1|^{\frac{1}{2}} \\ \le-\underbrace{\varepsilon\|\begin{bmatrix}s & |x_1|^{\frac{1}{4}}\end{bmatrix}^T\|_2^2}_{v(x)} \quad (27)$$

where

$$0 < \varepsilon \leq \min\left\{\left(\frac{1}{2}k_1|x_1|^{-\frac{1}{2}} - \frac{1}{8\gamma^2}\eta_1 k_1^2 - \frac{2}{\gamma^2}\eta_2 - 2\right),\right.$$
$$\left.\left[k_1 k_3 - \left(\frac{2}{\gamma^2}\eta_1 k_3^2 - \frac{1}{2\gamma^2}\eta_2 k_1^2 - |x_1| - 2k_1^2\right)|x_1|^{\frac{1}{2}}\right]\right\}. \tag{28}$$

The Hypothesis **H** is thus locally satisfied for all $x \in D_{\mathcal{G}}$ where

$$D_{\mathcal{G}} = \left\{x \in \mathbb{R}^2 : |x_1|^{\frac{1}{2}} < \frac{4\gamma^2 k_1}{\eta_1 k_1^2 + 16\eta_2 + 16\gamma^2}\right\}. \tag{29}$$

and

$$k_1 k_3 > \left(\frac{2}{\gamma^2}\eta_1 k_3^2 + \frac{1}{2\gamma^2}\eta_2 k_1^2 + |x_1| + 2k_1^2\right)|x_1|^{\frac{1}{2}}. \tag{30}$$

From the above inequality and using (29), we can get

$$k_3 > \left(\frac{2}{\gamma^2}\eta_1 k_3^2 + \frac{1}{2\gamma^2}\eta_2 k_1^2 + 2k_1^2 + \left(\frac{4\gamma^2 k_1}{\eta_1 k_1^2 + 16\eta_2 + 16\gamma^2}\right)^2\right)\left(\frac{4\gamma^2}{\eta_1 k_1^2 + 16\eta_2 + 16\gamma^2}\right). \tag{31}$$

If we assume that the perturbation w_1 does not affect the system, i.e., $\eta_1 = 0$ then

$$|x_1|^{\frac{1}{2}} < \frac{4\gamma^2 k_1}{16\eta_2 + 16\gamma^2} \tag{32}$$
$$k_3 > \frac{k_1^2}{8(\eta_2 + \gamma^2)}\left(\eta_2 + 4\gamma^2 + \frac{\gamma^6}{8\left(\eta_2 + \gamma^2\right)^2}\right), \tag{33}$$

will be necessary in order to meet the inequality (27). Thus, we arrive at the following result.

Theorem 2 *Let the parameter gains be such that* $k_1 > 0$ *and (31) holds. Then the nominal system (3), (19) is locally asymptotically stable within the attraction area* $x \in D_{\mathcal{G}} \subset \mathbb{R}^2$*, whereas its perturbed version (1), (19), (20) subject to* $\mu_1(x) = \eta_1^{\frac{1}{2}}|x_1|^{\frac{1}{2}}$ *and* $\mu_2(x) = \eta_2^{\frac{1}{2}}$ *possesses* $\mathcal{L}_2$*-gain less than an arbitrarily pre-specified* $\gamma > 0$ *with respect to output* $z = x$*, locally within the region* $D_{\mathcal{G}}$*.*

IV. $\mathcal{L}_2$-Gain Analysis for the Super-Twisting Algorithm with Proportional Terms

Now, we analyze the super-twisting algorithm with proportional terms, where the system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 - k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) - k_2 x_1 + \mu_1(x)w_1 \\ \dot{x}_2 &= -k_3\operatorname{sign}(x_1) - k_4 x_1 + \mu_2(x)w_2 \end{aligned} \tag{34}$$

with k_2 and k_4 being positive constants. The analysis will be made with respect to the output (18). The above system expressed in the form (1)–(2), is given by

$$\varphi(x) = \begin{bmatrix} x_2 - k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) - k_2 x_1 \\ -k_3\operatorname{sign}(x_1) - k_4 x_1 \end{bmatrix}, \tag{35}$$
$$\psi(x) = \begin{bmatrix} \mu_1(x) & 0 \\ 0 & \mu_2(x) \end{bmatrix} \tag{36}$$
$$h(x) = [x_1, x_2]^T. \tag{37}$$

A. $\mathcal{L}_2$-Gain Analysis

In order to verify whether system (34), affected by the disturbance w, possesses $\mathcal{L}_2$-gain less than γ with respect to output (37), we consider the following positive definite function, extracted from (Moreno and Osorio, 2008)

$$V = 2k_3|x_1| + k_4 x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}s^2 \tag{38}$$

where

$$s = x_2 - k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) - k_2 x_1. \tag{39}$$

First, we establish stability of the unperturbed system (34) using the candidate Lyapunov function (38). Differentiating this function along the system trajectories yields the time derivative in the form

$$\begin{aligned} \frac{dV}{dt} &= -k_2 k_4 x_1^2 - k_1 k_4|x_1|^{\frac{3}{2}} - k_2 k_3|x_1| \\ &\quad - k_1 k_3|x_1|^{\frac{1}{2}} - \frac{1}{2}k_1|x_1|^{-\frac{1}{2}}s^2 - k_2 s^2 \end{aligned} \tag{40}$$

which is negative definite for all $x \in \mathbb{R}^n$. Thus, we establish global asymptotic stability of the unperturbed system.

Now, let us verify that the Hamilton-Jacobi inequality (8) is satisfied with the positive definite function (38)

$$\begin{aligned} \mathcal{H} &= -k_2 k_4 x_1^2 - k_1 k_4|x_1|^{\frac{3}{2}} - k_2 k_3|x_1| - k_1 k_3|x_1|^{\frac{1}{2}} \\ &- \frac{1}{2}k_1|x_1|^{-\frac{1}{2}}s^2 - k_2 s^2 + \frac{1}{4\gamma^2}\left(x_2 + s\right)^2\mu_2^2(x) + x_1^2 + x_2^2 \\ &+ \frac{\mu_1^2(x)}{4\gamma^2}\left(2k_3\operatorname{sign}(x_1) + 2k_4 x_1 - s\left(\frac{1}{2}k_1|x_1|^{-\frac{1}{2}} + k_2\right)\right)^2. \end{aligned}$$

From (39) we have that $x_2 = s + k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) + k_2 x_1$, hence

$$\begin{aligned} \mathcal{H} &= -k_2 k_4 x_1^2 - k_1 k_4|x_1|^{\frac{3}{2}} - k_2 k_3|x_1| - k_1 k_3|x_1|^{\frac{1}{2}} \\ &- k_2 s^2 + \frac{1}{4\gamma^2}\left(2s + k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) + k_2 x_1\right)^2\mu_2^2(x) \\ &- \frac{1}{2}k_1|x_1|^{-\frac{1}{2}}s^2 + x_1^2 + \left(s + k_1|x_1|^{\frac{1}{2}}\operatorname{sign}(x_1) + k_2 x_1\right)^2 \\ &+ \frac{\mu_1^2(x)}{4\gamma^2}\left(2k_3\operatorname{sign}(x_1) + 2k_4 x_1 - \frac{1}{2}k_1|x_1|^{-\frac{1}{2}}s - k_2 s\right)^2. \end{aligned}$$

Using the inequality $(\sum_{i=1}^{4} a_i)^2 \leq 2\sum_{i=1}^{4} a_i^2$, we have

$$\mathcal{H} \leq -k_2k_4x_1^2 - k_1k_4|x_1|^{\frac{3}{2}} - k_2k_3|x_1| - k_1k_3|x_1|^{\frac{1}{2}} - k_2s^2 - \frac{1}{2}k_1|x_1|^{-\frac{1}{2}}s^2 + \frac{2}{\gamma^2}s^2\mu_2^2(x) + \frac{1}{2\gamma^2}k_1^2|x_1|\mu_2^2(x) + x_1^2 + \frac{1}{2\gamma^2}k_2^2x_1^2\mu_2^2(x) + 2s^2 + 2k_1^2|x_1| + 2k_2^2x_1^2 + \frac{2}{\gamma^2}k_3^2\mu_1^2(x) + \frac{2}{\gamma^2}k_4^2x_1^2\mu_1^2(x) + \frac{1}{8\gamma^2}k_1^2|x_1|^{-1}s^2\mu_1^2(x) + \frac{1}{2\gamma^2}k_2^2s^2\mu_1^2(x).$$

Setting $\mu_2 = \eta_2^{\frac{1}{2}}$ and taking $\mu_1(x)$ as in (26) one obtains

$$\begin{aligned}\mathcal{H} \leq &-\left(k_2k_4 - \frac{1}{2\gamma^2}k_2^2\eta_2 - 1 - 2k_2^2\right)x_1^2 \\ &-\left(k_1k_4 - \frac{2}{\gamma^2}k_4^2\eta_1|x_1|^{\frac{3}{2}}\right)|x_1|^{\frac{3}{2}} \\ &-\left(k_2k_3 - \frac{1}{2\gamma^2}k_1^2\eta_2 - 2k_1^2 - \frac{2}{\gamma^2}k_3^2\eta_1\right)|x_1| \\ &- k_1k_3|x_1|^{\frac{1}{2}} - \left(\frac{1}{2}k_1 - \frac{1}{2\gamma^2}k_2^2\eta_1|x_1|^{\frac{3}{2}}\right)|x_1|^{-\frac{1}{2}}s^2 \\ &-\left(k_2 - \frac{2}{\gamma^2}\eta_2 - 2 - \frac{1}{8\gamma^2}k_1^2\eta_1\right)s^2 \leq -\underbrace{\tilde{\varepsilon}\|\begin{bmatrix}x_1 & s\end{bmatrix}^T\|_2^2}_{v(x)}\end{aligned}$$

where

$$0 < \tilde{\varepsilon} \leq \min\left\{\left(k_2k_4 - \frac{1}{2\gamma^2}k_2^2\eta_2 - 1 - 2k_2^2\right), \left(k_2 - \frac{2}{\gamma^2}\eta_2 - 2 - \frac{1}{8\gamma^2}k_1^2\eta_1\right)\right\}. \tag{41}$$

In order to keep the above inequality, it is necessary to satisfy the following inequalities

$$|x_1|^{\frac{3}{2}} < \frac{k_1}{\eta_1}\gamma^2 \min\left(\frac{1}{2k_4}, \quad \frac{1}{k_2^2}\right) \tag{42}$$

$$k_2 > \max\left(\frac{2\eta_2}{\gamma^2} + \frac{k_1^2\eta_1}{8\gamma^2} + 2, \quad \frac{k_1^2\eta_2}{2\gamma^2k_3} + \frac{2k_3\eta_1}{\gamma^2} + \frac{2k_1^2}{k_3}\right) \tag{43}$$

$$k_4 > \frac{k_2\eta_2}{2\gamma^2} + \frac{1}{k_2} + 2k_2. \tag{44}$$

The Hypothesis **H** is thus locally satisfied for all $x \in D_{\mathcal{C}}$ where

$$D_{\mathcal{C}} = \left\{x \in \mathbb{R}^2 : |x_1|^{\frac{3}{2}} < \frac{k_1\gamma^2}{\eta_1}\min\left(\frac{1}{2k_4}, \quad \frac{1}{k_2^2}.\right)\right\} \tag{45}$$

Summarizing, the following result is obtained.

Theorem 3 *Let the positive parameter gains be such that inequalities (43), (44) hold for an arbitrary $\gamma > 0$, fixed a priori. Then the nominal system (3), (35) is globally asymptotically stable and its perturbed version (1), (35), (36) with $\mu_1(x) = \eta_1^{\frac{1}{2}}|x_1|^{\frac{1}{2}}$ and $\mu_2(x) = \eta_2^{\frac{1}{2}}$ possesses $\mathcal{L}_2$-gain less than $\gamma > 0$ with respect to output $z = x$, locally within the region $D_{\mathcal{C}}$.*

Following the same line of reasoning, the global version of Theorem 3 is established under the absence of the disturbance w_1.

Theorem 4 *Let the parameter gains be such that k_1, $k_3 > 0$, and k_2, k_4 satisfies (43), (44), respectively; for an arbitrary $\gamma > 0$. Then the nominal system (3), (35) is globally asymptotically stable whereas its perturbed version (1), (35), (36) with $w_1(x) = 0$ and $\mu_2(x) = \eta_2^{\frac{1}{2}}$ globally possesses a $\mathcal{L}_2$-gain less than an arbitrary $\gamma > 0$ with respect to output $z = x$.*

V. Simulation Results

We run numerical simulations in *Simulink* in order to corroborate that external harmonic disturbances $w(t) \in \mathbb{R}^2$, affecting the super-twisting algorithm (17) and the super-twisting algorithm with proportional part (34), are attenuated.

Figures 1(a) and 1(b) show the responses of the super-twisting algorithm (17) initialized at $x_1(0) = 0.1$ and $x_2(0) = 1$, considering

$$\psi(x) = \begin{bmatrix}|x_1|^{\frac{1}{2}} & 0 \\ 0 & 1\end{bmatrix} \tag{46}$$

and affected by the following disturbances for Figure 1(a):

$$w_1 = \sin(2\pi t), \qquad w_2 = \begin{cases}0 & 0 \leq t < 3 \\ 60 & 3 \leq t < 3.01 \\ 0 & t \geq 3.01\end{cases} \tag{47}$$

and the following disturbances for Figure 1(b):

$$w_1 = \sin(2\pi t), \qquad w_2 = \frac{2}{t^{\frac{1}{3}}}\cos(5\pi t) \tag{48}$$

where w_2 is unbounded of class $\mathcal{L}_2$.

These figures show the responses under $\gamma = 0.5$ (solid line) and $\gamma = 5$ (dotted line).

Figures 2(a) and 2(b) show the responses of the super-twisting algorithm with proportional part (34) with $k_1 = k_3 = 3$ initialized at $x_1(0) = 0.1$ and $x_2(0) = 1$ considering (46) and affected by the following disturbances for Figure 2(a):

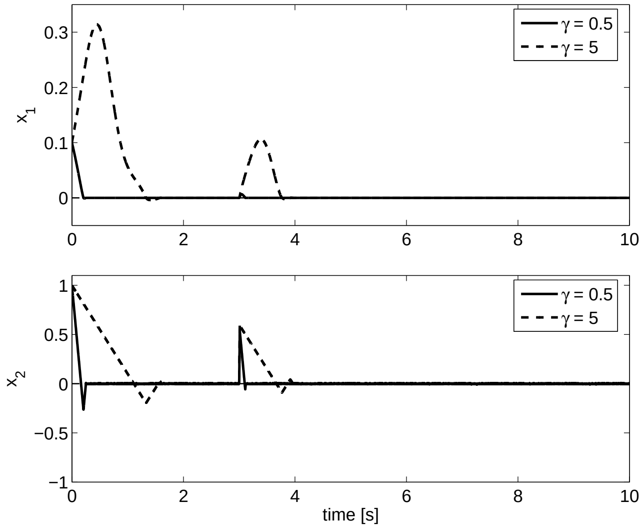
$$w_1 = 5\sin(2\pi t), \qquad w_2 = \begin{cases}0 & 0 \leq t < 3 \\ 60 & 3 \leq t < 3.01 \\ 0 & t \geq 3.01\end{cases} \tag{49}$$

and the following disturbances for Figure 2(b):

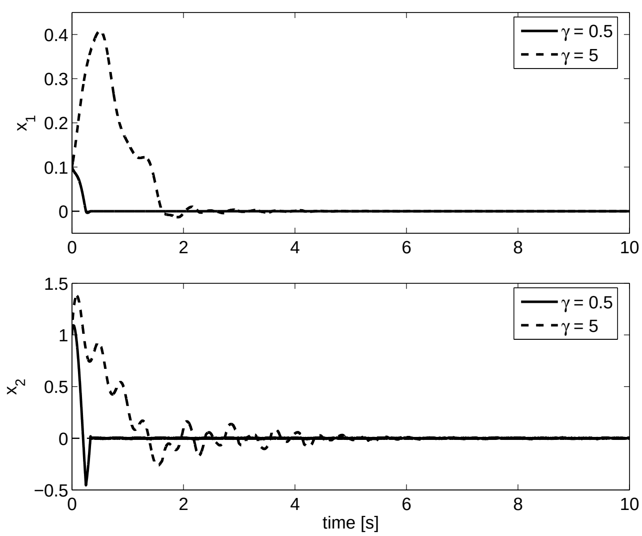
$$w_1 = 4\sin(2\pi t), \qquad w_2 = \frac{5}{t^{\frac{1}{3}}}\cos(5\pi t). \tag{50}$$

These figures show the responses under $\gamma = 0.2$ (solid line) and $\gamma = 5$ (dotted line).

It is concluded from these figures that as predicted by the theory, the disturbances are not rejected but only attenuated,



(a) Disturbances given by (47).



(b) Disturbances given by (49).

Figura 1. Responses of the perturbed super-twisting algorithm ($k_2 = k_4 = 0$) where $k_1 = 4.5$ and $k_3 = 6.1$ for $\gamma = 0.5$ and $k_1 = 1.4$ and $k_3 = 0.9$ for $\gamma = 5$.

that is, the system response is no longer rejected but it remains bounded.

VI. Conclusions

$\mathcal{L}_2$-gain analysis, developed for the super-twisting algorithm, has clearly shown its applicability to sliding mode dynamic systems and the capability of the popular algorithm to attenuate unbounded disturbances.

Acknowledgment

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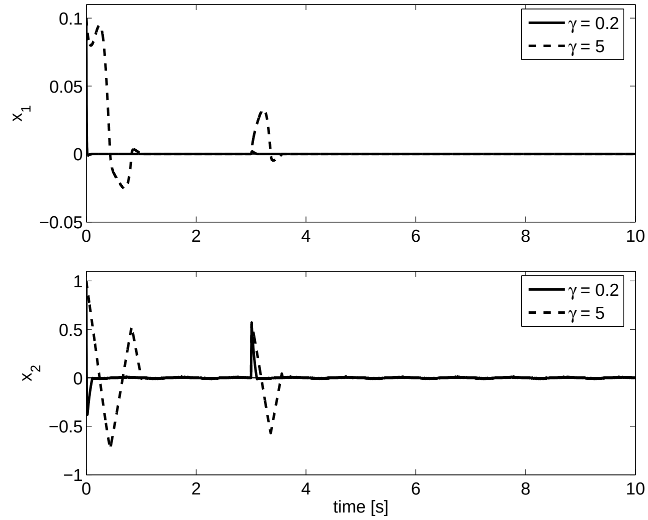
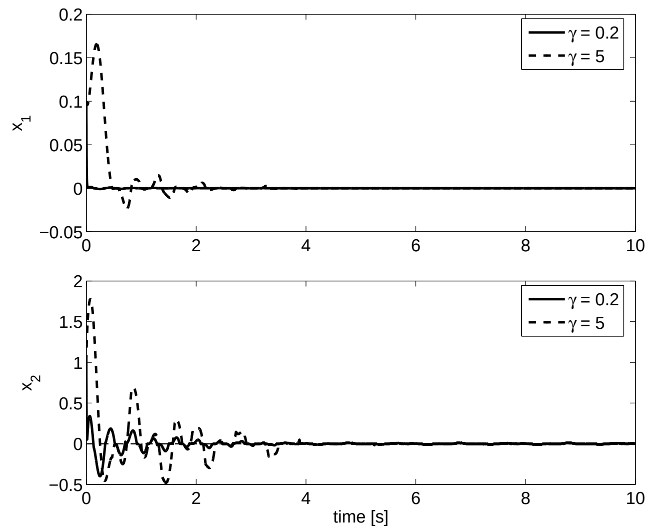


(a) Disturbances given by (48).



(b) Disturbances given by (50).

Figura 2. Responses of super-twisting algorithm with proportional terms where $k_1 = k_3 = 3$, $k_2 = 194$ and $k_4 = 2806$ for $\gamma = 0.2$ and $k_1 = k_3 = 3$, $k_2 = 6.5$ and $k_4 = 13$ for $\gamma = 5$.

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