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On the stabilization of linear delayed systems by static predicted feedback

J.F. Marquez-Rubio*, B. del Muro-Cuéllar**, M. Velasco-Villa*** and J.J. Álvarez-Ramírez*
*Universidad Autónoma Metropolitana-Iztapalapa, Iztapalapa, DF 09340 México.
**Instituto Politécnico Nacional ESIME-Unidad Culhuacan, Coyoacán, DF 04430, México.
***CINVESTAV-IPN, Gustavo A. Madero, DF 07000, México
jfcomr23@yahoo.com.mx, bdelmuro@yahoo.com, velasco@cinvestav.mx, jjar@xanum.uam.mx

Abstract— **This work considers the stabilization of unstable first order linear processes with I/O delay term. A new result is present including a simple static estimated (predicted) "state" feedback scheme. In this approach, an observer is designed including the plant model with two static gains and the controller consists of only two static gains. The compete schema allows to stabilize first order delayed systems with delays restricted to $\tau < 4\tau_{un}$, where τ_{un} is the unstable time constant in the process.**

I. Introduction

Time delays are present practically in any dynamical systems due mainly to phenomenon like information, material or energy transport; however, they take central importance when the delay are large enough when compared to the dominant time constant in the system. The stabilization of unstable processes with I/O delay is a challenging problem from both design and analysis standpoints. The key issue is to design a feedback control input acting sufficiently fast to counteract the unstable process dynamics. The general problem is still poorly understood and, in contrast to the stable counterpart, only a limited number of results are available in the literature. Some reports have considered the first order process,

$$\frac{Y(s)}{U(s)} = \frac{b}{s-a}e^{-\tau s} \tag{1}$$

with $a > 0$ and $b > 0$, as a benchmark for establishing necessary and/or sufficient conditions for process stabilization under a designed feedback control strategy. By considering that $\tau_{un} = a^{-1}$ can be seen as the unstable time-constant of the process, (Seshagiri R.A. et al., 2007) proposed a modified Smith predictor to show sufficient stabilization conditions for $\tau < 1.5\tau_{un}$. (Nesimioglu B.S and Soylemez M.T., 2010) computed all stabilization proportional controllers for Eq. (1). (Hwang C and Hwang J.H., 2004) used the D-partition technique to estimate the stabilization limits of PID compensation, showing that the process given by Eq. (1) can be stabilized if $\tau < \tau_{un}$. (Michiels W. et al., 2002) proposed a partition of the delay operator $e^{-\tau s}$ into $e^{-\tau_1 s}$ and $e^{-\tau_2 s}$ acting respectively in the input and output channels to show that stabilization of Eq. (1) can be achieved if $\tau < 2\tau_{un}$. (Silva G.J and Bhattacharyya S.P., 2005) provided a complete characterization of the PID compensators for Eq. (1).

In (Del Muro Cuéllar B. et al., 2012) is analyzed the use of observer (predictor) based controllers, with special attention to the case of continuous first order linear unstable processes subject to large input-output time delays, with special interest to the case $\tau < 2\tau_{un}$. In this work a controller is provided allowing the stabilization of systems with $\tau < 3\tau_{un}$. In (Marquez Rubio J.F. et. al, 2012a)[Marquez et al 2012 (agregar cita)], the same approach is used including the use of PID controllers and allowing to stabilize systems with delay restricted to $\tau < 4\tau_{un}$.

This work uses the delay splitting strategy (Michiels W, et al., 2002) to extend the recent results in (Del Muro Cuéllar et al., 2012) by showing that stabilization can be achieved if $\tau < 4\tau_{un}$. The feedback control design is based on the estimation of intermediate delayed states at the input and output channels to fed-back these signals in order to counteract effects of the unstable pole. A numerical example are used for illustrating the parametric stability region of the observer-based compensator.

II. Main Results

Consider the process Eq. (1) rewritten as follows,

$$\frac{Y(s)}{U(s)} = e^{-\tau_1 s}G(s)e^{-\tau_2 s} \tag{2}$$

where $\tau_1 = \tau_2 = \tau/2$. For simplicity in notation, let $G(s) = \frac{b}{s-a}$ denote the delay-free process. Regarding the state feedback depicted in Figure 1, the following preliminary result can be established.

Lemma 1: Consider the stabilizing scheme shown in Figure 1. There exist constants f_1 and f_2 such that the closed-loop system,

$$\frac{Y(s)}{U(s)} = \frac{be^{-\tau s}}{(s-a)(1+g_1e^{-\tau_2 s}) + g_2be^{-\tau_2 s}} \tag{3}$$

is stable, if and only if, $\tau_2 < \frac{2}{a}$.

Proof: The proof of this lemma is provided in Appendix. ∎

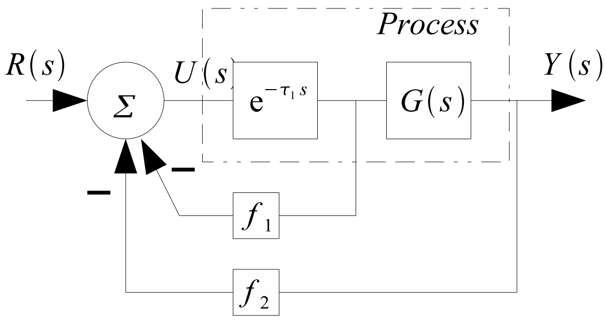

Figura 1. Static state feedback.

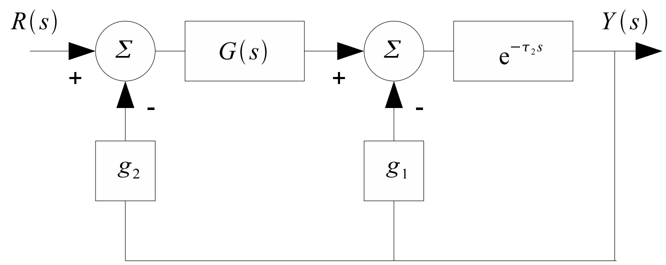

Figura 2. Output injection feedback.

Corollary 1: Consider the stabilizing scheme shown in Figure 1. If $\tau_2 < \frac{2}{a}$, then the parameters f_1 and f_2 such that the closed-loop transfer function 3 is stable can be computed by considering the inequalities $a\tau_2 - 1 < f_1 \leq a\tau_2 - 1 + \epsilon$, $\frac{a}{b}(f_1+1) < f_2 \leq \frac{a}{b}(f_1+1)+\bar{\epsilon}$, for some constants $\epsilon, \bar{\epsilon} > 0$.

Proof: The proof of this result is given in Appendix ■

The main idea of Corollary 1 is that, if $\tau < 2/a$, then the stabilizing region for parameter g_1 is completely determined by $a\tau_2 - 1 < g_1 < 1$. Once a particular g_1 value has been selected, a simple frequency domain analysis can be done for the transfer function,

$$\bar{G}(s) = \frac{b}{s-a}\frac{e^{-\tau_1 s}}{1+g_1 e^{-\tau_1 s}} \tag{4}$$

obtaining the stability gain margins for g_2; getting the already known lower bound $(a/b)(g_1+1)$ and the unknown upper bound θ. Note that this transfer function can easily be analyzed using the computational software MATLAB. Applying this procedure to the set of g_1 values along $(a\tau_2 - 1, 1)$ it is possible to obtain the complete stability region of parameters g_1 and g_2.

The result presented in Lemma 1 is the fundamental key in this work. A dual version of the proof was presented in (Marquez Rubio J.F. et. al, 2012b). In a dual way to the previous lemma, it can be stated the following result.

Lemma 2: Consider the stabilizing output injection scheme shown in Figure 2. Then, there exist constants g_1 and g_2 such that the closed loop system,

$$\frac{Y(s)}{U(s)} = \frac{be^{-\tau_2 s}}{(s-a)(1+g_1 e^{-\tau_2 s}) + g_2 b e^{-\tau_2 s}} \tag{5}$$

is stable if and only if $\tau_2 < \frac{2}{a}$.

Proof: The proof of this lemma can be done in a dual way to the proof of Lemma 1. ■

Corollary 2: Consider the stabilizing output injection scheme shown in Figure 2. If $\tau_2 < \frac{2}{a}$, then the parameters g_1 and g_2 such that the closed-loop transfer function 5 is stable can be computed by considering the inequalities $a\tau_2 - 1 < g_1 \leq a\tau_2 - 1 + \epsilon$, $\frac{a}{b}(g_1+1) < g_2 \leq \frac{a}{b}(g_1+1)+\bar{\epsilon}$, for some constants $\epsilon, \bar{\epsilon} > 0$.

Proof: The proof of this result can be obtained from a dual view point of Corollary 1. ■

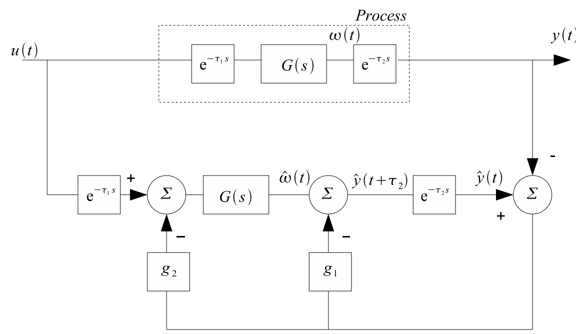

Figura 3. Proposed observer

As a consequence of the previous results we can state the following result.

Theorem 1: Consider the observer scheme shown in Figure 3. Then, there exist constants g_1 and g_2 such that $\lim_{t\to\infty}[\widehat{\omega}(t) - \omega(t)] = 0$ if and only if $\tau_2 < \frac{2}{a}$.

Proof: The proof can be done by taking into account the stability conditions given in Lemma 2. With this aim, consider the dynamics of the prediction scheme shown in Figure 3 that can be written in state space form as,

$$\begin{aligned} \begin{bmatrix} \dot{\omega}(t) \\ \dot{\widehat{\omega}}(t) \end{bmatrix} &= A_1 \begin{bmatrix} \omega(t) \\ \widehat{\omega}(t) \end{bmatrix} + \overline{A}_1 \begin{bmatrix} y(t) \\ \widehat{y}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} b \\ b \end{bmatrix} u(t-\tau_1) \\ \begin{bmatrix} y(t+\tau_2) \\ \widehat{y}(t+\tau_2) \end{bmatrix} &= A_2 \begin{bmatrix} \omega(t) \\ \widehat{\omega}(t) \end{bmatrix} + \overline{A}_2 \begin{bmatrix} y(t) \\ \widehat{y}(t) \end{bmatrix} \end{aligned}$$

with,

$$\begin{aligned} A_1 &= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} 0 & 0 \\ bg_2 & -bg_2 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} 0 & 0 \\ g_1 & -g_1 \end{bmatrix} \end{aligned}$$

and $\widehat{\omega}(t)$ the estimation of $\omega(t)$. Defining first the state prediction error $e_\omega(t) = \omega(t) - \widehat{\omega}(t)$ and the output estimation error $e_y(t) = y(t) - \widehat{y}(t)$ it is possible to describe

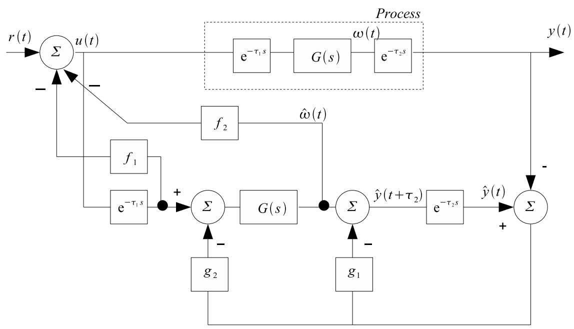

Figura 4. Proposed stabilization strategy.

the behavior of the error signal as

$$\left[\begin{array}{c}\dot{e}_\omega(t)\\ e_y(t+\tau_2)\end{array}\right]=\left[\begin{array}{cc}a & -bg_2\\ 1 & -g_1\end{array}\right]\left[\begin{array}{c}e_\omega(t)\\ e_y(t)\end{array}\right]. \quad (6)$$

Consider now a state space realization of system (5) (described in Figure 2) that can be written as,

$$\left[\begin{array}{c}\dot{x}(t)\\ y(t+\tau_2)\end{array}\right]=\left[\begin{array}{cc}a & -bg_2\\ 1 & -g_1\end{array}\right]\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right]+\left[\begin{array}{c}b\\ 0\end{array}\right]u(t-\tau_1). \quad (7)$$

It is clear that the stability conditions of system (7), given in Lemma 2, are equivalent to the ones of system (6), from where, the result of the theorem follows. Considering the previous results, from Lemma 1, the state feedback system is stable if and only if $\tau_1 < \frac{2}{a}$ and from Lemma 2 the state can be estimated if and only if $\tau_2 < \frac{2}{a}$. ■

Based on this result, the following theorem can be established.

Theorem 2: Consider the estimated state feedback scheme shown in Figure 4. Then, there exist constants f_1, f_2, g_1 and g_2 such that such that the closed loop is stable if and only if $\tau < \frac{4}{a}$.

III. Simulation Results

The effectiveness of the proposed methodology will be now evaluated by means of a numerical example.

Example 1. Now, consider the unstable delayed process given by the transfer function,

$$\frac{Y(s)}{U(s)}=\frac{3}{s-1}e^{-\tau s} \quad (8)$$

with $\tau = 2.8$. Time delay satisfies $\tau < \frac{4}{a}$, from Theorem 2 there exist gains g_1, g_2, f_1 and f_2 that stabilize the closed-loop system shown in Figure 4. In this case it is considered $\tau = \tau_1 + \tau_2$ with $\tau_1 = \tau_2 = 1.4$. From Corollary 2, $0.4 < g_1 < 1$ and $(a/b)(g_1) + 1 < g_2 < \theta$. As mentioned previously, with the help of a frequency domain analysis, it is possible to get, for every g_1 value satisfying $0.4 < g_1 < 1$, all the corresponding values of g_2 that stabilize the scheme of Figure 2. The set of g_2 values were computed with MATLAB and the complete stability region for parameters g_1, g_2 is depicted in Figure 5 (or by duality

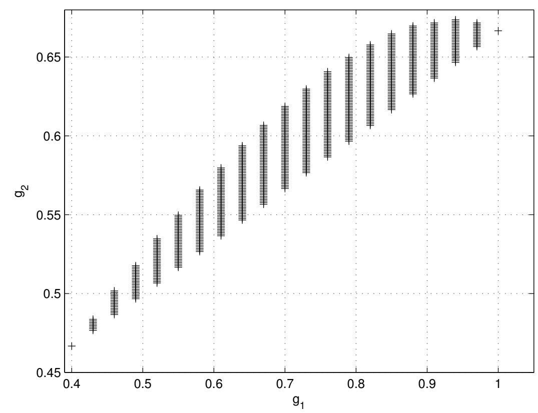

Figura 5. Stability region $g_1 - g_2$, Example 1

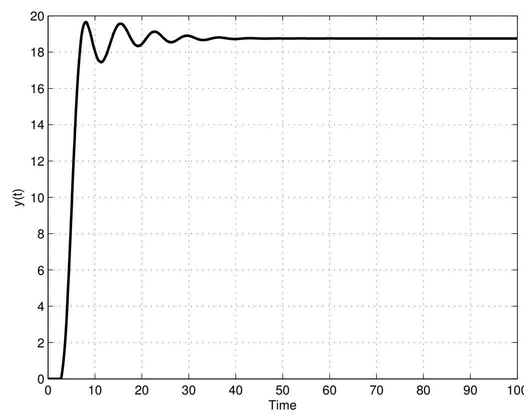

Figura 6. Output response, Example 1

$f_1 - f_2$) that stabilize the observer-controller scheme. For the simulation, $g_1 = f_1 = 0.75$ and $g_2 = f_2 = 0.61$ were used. Figure 6 shows the performance of the output signal $y(t)$ when a step input reference of magnitude 0.5 is regarded. For the simulation the exact value of the plant parameters is assumed to be known and initial conditions different from zero are also considered. Figure 7 presents the estimation error $e_\omega(t) = \omega(t) - \hat{\omega}(t)$.

IV. Conclusions

This work showed that the incorporation of intermediate delayed states within the feedback loop can enlarge the range of the delays for the stabilization of unstable processes. While simple proportional feedback can yield stabilization if $\tau < \tau_{un}$, the proposed observer-based compensation can lead to stabilization for $\tau < 4\tau_{un}$. This improvement motivates the development of systematic strategies for unstable systems with large I/O delay effects.

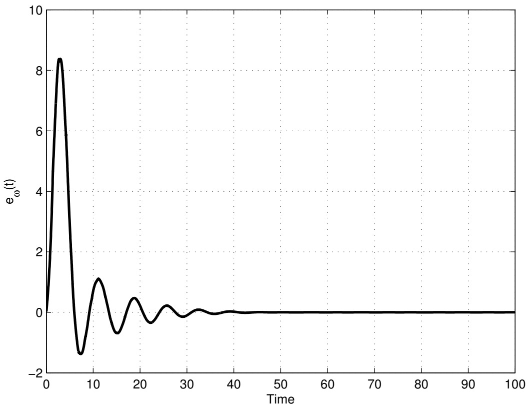

Figura 7. Error $e_\omega(t)$, Example 1

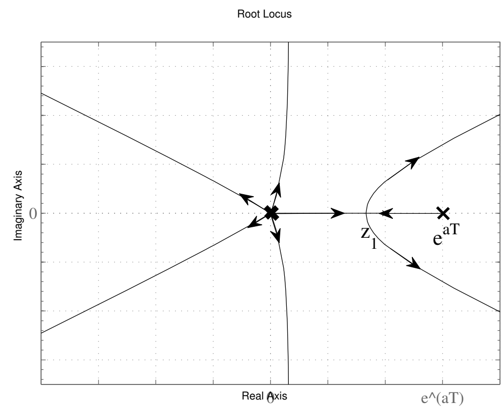

Figura 8. Root locus of equation (11) for $n = 5$.

APPENDIX

Proof of Lemma 1 Based on the splitting strategy of τ, system (3) can be rewritten as,

$$\frac{Y(s)}{U(s)} = \frac{be^{-\tau_2 s}}{(s-a)(1+f_1 e^{-\tau_2 s}) + f_2 b e^{-\tau_2 s}} e^{-\tau_1 s}$$

Note from that the delay term $e^{-\tau_1 s}$ does not affect the stability of the closed-loop system (since is on the direct loop). Therefore, it is considered the expression,

$$\frac{Y(s)}{U(s)} = \frac{be^{-\tau_2 s}}{(s-a)(1+f_1 e^{-\tau_2 s}) + f_2 b e^{-\tau_2 s}}$$

Consider now a discrete-time version of the original plant $G(s)e^{-\tau_2 s}$ together with the scheme given in Figure 1. To carry out this task, it is assumed that there exist a sampling period T that satisfies the condition $T = \frac{\tau_2}{n}$ for an integer n and that a zero order hold is located at the input of the system. Under these conditions, the discrete-time closed-loop transfer function is,

$$\frac{Y(z)}{U(z)} = \frac{(b/a)(e^{aT}-1)}{(z-e^{aT})(z^n + f_1) + f_2(b/a)(e^{aT}-1)}, \quad (9)$$

with the characteristic polynomial given by,

$$p_1(z) = (z-e^{aT})(z^n + f_1) + f_2(b/a)(e^{aT}-1). \quad (10)$$

The proof of the theorem is based on demonstrate that all roots in (10) lie inside the unit circle when it is considered, $\lim_{n\to\infty} \frac{\tau_2}{n}$, if and only if, $\tau_2 < \frac{2}{a}$.

To begin with, consider first the simple case when $f_1 = 0$ in (10), this produces the characteristic equation,

$$(z-e^{aT})z^n - f_2(b/a)(e^{aT}-1) = 0. \quad (11)$$

The root locus diagram (W. R. Evans, 1954) associated to (11) shows that the open-loop system has n poles at the origin and one at $z = e^{aT}$. Then, there exist $n+1$ branches to infinity, $n-1$ of them starting at the origin and going directly to infinity. The two remaining branches starting at a breaking point z_1 located over the real axis between the origin and $z = e^{aT}$ (this situation is illustrated in Figure 8 for the case $n = 5$). z_1 can be found by considering the equation,

$$\frac{df_2}{dz} = \frac{d}{dz}\left[-\frac{z^n(z-e^{aT})}{\frac{b}{a}(1-e^{aT})}\right] = 0,$$

that produces,

$$(n+1)z^n - nz^{n-1}e^{aT} = 0,$$

which has $n-1$ roots at the origin and one at,

$$z_1 = \frac{n}{n+1} e^{a\frac{\tau_2}{n}}.$$

If the breaking point z_1 over the real axis is located inside the unit circle, the closed loop system could have a region of stability, otherwise will be unstable for any f_2. The stability properties of the continuous system (3) are obtained by considering the limit as $n \to \infty$, or equivalently, when $T \to 0$, this is,

$$\lim_{n\to\infty} z_1 = \lim_{n\to\infty} \frac{n}{n+1} e^{a\frac{\tau_2}{n}} = 1. \quad (12)$$

It is important to note that any point $s = \theta$, over the real axis on the complex plane s is mapped to $z = e^{\theta T}$ on the z plane and as a consequence this point converges to $z = 1$ when T tends to zero. Notice also that any real point $s = \theta$ on the left half side of the complex plane ($\theta < 0$) is mapped to a point $e^{\theta T}$ that tends to one over the stable region of the z plane. On the contrary, if θ is on the right side of the complex plane over the real axis ($\theta > 0$), the point $e^{\theta T}$ tends to one over the unstable region. Then, from (11), it is not difficult to see that if $a\tau_2 < 1$ (i.e., $\tau_2 < 1/a$) there exists a gain f_2 that stabilizes the closed loop system (i.e., the limit tends to one from the left). In the case that $a\tau_2 \geq 1$ (always considering $f_1 = 0$) it is not possible to get f_2 that stabilize the system.

Consider now the case $f_1 \neq 0$. Applying again a root locus analysis for system (9) and its characteristic equation

(10), as f_1 grows from zero, the breaking point over the real axis moves in the root locus diagram (indeed, goes to the left). This point can be found by taking into account the equation,

$$\frac{df_2}{dz} = \frac{d}{dz}\left[-\frac{(z-e^{aT})(z^n+f_1)}{(b/a)(e^{aT}-1)}\right] = 0, \qquad (13)$$

yielding,

$$(n+1)z^n - nz^{n-1}e^{aT} + f_1 = 0. \qquad (14)$$

Expression (14) corresponds to the characteristic equation of a fictitious system of the form,

$$\frac{Y(z)}{V(z)} = G(z) = \frac{1/(n+1)}{z^{n-1}(z-e^{aT}n/(n+1))} \qquad (15)$$

in closed loop with the feedback,

$$V(z) = U(z) - f_1 Y(z). \qquad (16)$$

The open loop system (15) has $n-1$ root at the origin and one at

$$z = \frac{n}{n+1}e^{a\frac{\tau_2}{n}}.$$

If the breaking point over the real axis is located inside the unit circle, the closed loop system (15)-(16) could have a region of stability (once proved that the others $n-2$ poles are inside the unitary circle), otherwise the system will be unstable for any f_1. This point can be found by considering,

$$\frac{df_1}{dz} = \frac{d}{dz}\left[-\frac{z^{n-1}\{z-e^{aT}n/(n+1)\}}{1/(n+1)}\right] = 0, \qquad (17)$$

that produces,

$$z^{n-2}(z - \frac{n-1}{n+1}e^{aT}) = 0,$$

which has $n-2$ roots at the origin and one at,

$$z = \frac{n-1}{n+1}e^{a\frac{\tau_2}{n}}.$$

As previously, the stability properties of the equivalent continuous system (3) are obtained by considering the limit as $n \to \infty$, or equivalently, when $T \to 0$. That is,

$$\lim_{n\to\infty} z = \lim_{n\to\infty} \frac{n-1}{n+1}e^{a\frac{\tau_2}{n}} = 1.$$

Again, since this limit point is located on the stability boundary, in this case it is possible to see that if $a\tau_2 \le 2$ (i.e., the limit tends to one from the left) there exists a gain f_1 that places the breaking point (two poles) inside the unit circle in the original discrete Root Locus diagram. Then, if the remaining $n-1$ roots are into the unit circle, the closed loop system is stable. In the case that $a\tau_2 > 2$ it is not possible to stabilize the system by static output injection (i.e., the limit goes to one from the right). Let us now prove that the remaining $n-1$ roots are into the unitary circle if and only if $a\tau_2 < 2$. Assume that $a\tau_2 \le 2$ and to take into account the continuous case, the characteristic equation (10) it is modified as,

$$\begin{aligned}\lim_{n\to\infty} p_1(z) &= \lim_{n\to\infty}[(z-e^{a\frac{\tau_2}{n}})(z^n+f_1) + f_2(b/a)(e^{a\frac{\tau_2}{n}}-1)] \\ &= (z-1)\lim_{n\to\infty}(z^n+f_1) = 0\end{aligned}$$

from where it is stated that while one pole is on the neighborhood of $z = 1$, the remaining poles are in a neighborhood of the points $(-f_1)^{1/n}$, inside the unit circle producing a stable closed loop system if, as it was previously stated, it is satisfied, $f_1 < 1$. From equation (17),

$$f_1 = -\frac{z^n\{z-e^{aT}n/(n+1)\}}{1/(n+1)},$$

then if $z = 1$,

$$f_1 = -\frac{\{1-e^{aT}n/(n+1)\}}{1/(n+1)} = -(n+1-ne^{aT}).$$

Taking into account the continuous case as previously done, it is obtained,

$$\lim_{n\to\infty} f_1 = \lim_{n\to\infty} -(n+1-ne^{a\tau_2/n}) = a\tau_2 - 1. \qquad (18)$$

As $f_1 < 1$ is a necessary condition for the stability, $a\tau_2 - 1 < 1$, then $a\tau_2 < 2$.

Proof of Corollary 1

From equation (18) in the proof of Lemma 1 we have:

$$\lim_{n\to\infty} f_1 = a\tau_2 - 1.$$

Therefore if $\tau_2 < \frac{2}{a}$, there exist f_2 that stabilizes the closed loop system (3), with $a\tau_2 - 1 < f_1 \le a\tau_2 - 1 + \epsilon$ for $\epsilon > 0$.

Now, from equation (13),

$$f_2 = -\frac{(z-e^{aT})(z^n+f_1)}{(b/a)(e^{aT}-1)},$$

then, if $z = 1$,

$$f_2 = \frac{f_1+1}{(b/a)} = (a/b)(f_1+1).$$

Therefore, the gain f_2 can be obtained by considering the condition $\frac{a}{b}(f_1+1) < f_2 \le \frac{a}{b}(f_1+1) + \bar{\epsilon}$, for some $\bar{\epsilon} > 0$.

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