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Global CLF stabilization of nonlinear systems with positive/signed control components in a hyperbox

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***Abstract*— Our main objective in this work is to study how to render an affine control system *globally asymptotically stable* (GAS), when the *control value set* (CVS) is given by an m-hyperbox $\mathcal{B}_{\boldsymbol{r}}^m(\infty) := [-r_1^-, r_1^+] \times \cdots \times [-r_m^-, r_m^+]$ with $0 \in \mathcal{B}_{\boldsymbol{r}}^m(\infty)$. Hence we allow the null-control input in its boundary, $0 \in \partial\mathcal{B}_{\boldsymbol{r}}^m(\infty)$, *i.e. positive/signed control input components*. Working along the line of Artstein and Sontag's *control Lyapunov function* (CLF) approach, we study the conditions that feedback controls of the decentralized form $u(x) = (\rho_1(x)\,\overline{\omega}_1(x), \ldots, \rho_m(x)\,\overline{\omega}_m(x))^\top$, should satisfy in order to be *admissible* (regular and valued in $\mathcal{B}_{\boldsymbol{r}}^m(\infty)$) and render a system GAS, given a known CLF. Here, $\overline{\omega}(x)$ is an *optimal control w.r.t. a* CLF and $\rho_j(x)$ are *rescaling functions*.**

Keywords: constrained control, nonlinear control system, global stabilization, control Lyapunov function.

I. INTRODUCTION

Consider the multiple input continuous-time affine system

$$\dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x), \tag{1}$$

where $x \in \mathbb{R}^n$, $f, g_j : \mathbb{R}^n \to \mathbb{R}^n$, for $j = 1, \ldots, m$, are *regular* vector fields. Here, the word *regular* means continuous, of class $\mathcal{C}^s(\mathbb{R}^n)$ $(s \geq 1)$, smooth, etc. We shall assume that $f(0) = 0$. A *control value set* (CVS) is any *convex* set $U \subseteq \mathbb{R}^m$, $u = (u_1, \ldots, u_m)^\top \in U$, and $^\top$ denotes transposition. By an *admissible feedback control* we will understand any regular function $u : \mathbb{R}^n \to U$.

We say that a control input component u_j is *signed* if and only if (*iff*) u_j can take both signs; whereas it is *positive iff* $u_j \geq 0$. A control input u is called *positive iff* all $u_j \geq 0$.

The main aim of this paper is to study how to render an affine control system (1) *globally asymptotically stable* (GAS) via an admissible feedback control $u(x)$, when the CVS is given by an m-hyperbox $\mathcal{B}_{\boldsymbol{r}}^m(\infty) := [-r_1^-, r_1^+] \times \cdots \times [-r_m^-, r_m^+]$, with $r_j^- \geq 0$ & $r_j^+ > 0$, so that $0 \in \mathcal{B}_{\boldsymbol{r}}^m(\infty)$. Note that renaming $g_j(x) \leftarrow -g_j(x)$ & $u_j \leftarrow -u_j$ in (1), any component $u_j \leq 0$ is converted into positive, so $r_j^+ \neq 0$. Therefore, in view that (1) is affine in the control input, the case of negative components is already included.

Hence, we will allow that either $0 \in \mathrm{int}\mathcal{B}_{\boldsymbol{r}}^m(\infty)$ (*i.e. all* $r_j^- > 0$) or the possibility that the null-control input be in its boundary, $0 \in \partial\mathcal{B}_{\boldsymbol{r}}^m(\infty)$ (*i.e. some* $r_j^- = 0$), so we can have control inputs with an assortment of *signed* or *positive* components ranging between all *signed* to all *positive*.

In control theory, a *control Lyapunov function* (CLF) $V(x)$ is used to prove that a control system is *feedback stabilizable*. This concept was introduced in (Artstein, 1983), opening the possibility of using it to solve stabilization problems: The CLF *stabilization* approach. We say that $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF [for system (1) with controls taking values in U] *iff* it is a $\mathcal{C}^\kappa(\mathbb{R}^n)$ $(\kappa \geq 1)$ function which is *positive definite* ($V(0) = 0$ and $V(x) > 0$ *iff* $x \neq 0$) and *proper* ($V^{-1}(c)$ is compact, for any $c \geq 0$), such that

$$\forall x \neq 0\, \exists u \in U \;\; \dot{V}(x) < 0. \tag{2}$$

It is known that a system of ordinary differential equations is GAS *iff* there is a global strict Lyapunov function. An analogous result for affine systems is given by the so-called Artstein's theorem in (Artstein, 1983): *Assume that (1) is regular and $U \subseteq \mathbb{R}^m$ is a* CVS. *There exists a smooth* CLF $V(x)$ *iff there exists a continuous (except possibly at* 0*) control $u(x)$, taking values in U, that renders (1)* GAS. Now, let us restate (2) into the equivalent representation[1]

$$\forall x \neq 0 \; \inf_{u \in U} \dot{V}(x) = \inf_{u \in U} \{a(x) - b(x) \cdot u\} < 0, \tag{3}$$

where $\xi^1 \cdot \xi^2$ denotes the inner product of ξ^1 and ξ^2, and

$$\begin{aligned} &a(x) := L_f V(x) \quad \& \quad b(x) := (b_1(x), \ldots, b_m(x)), \\ &\text{with } b_j(x) := -L_{g_j} V(x), \quad \text{for } j = 1, 2, \ldots, m \end{aligned} \tag{4}$$

denote the Lie derivatives of $V(x)$ with respect to (w.r.t.) the vector fields that define the system (1). The feedback controls can also be made continuous at $x = 0$ under the additional assumption of the *small control property* (SCP) introduced in (Artstein, 1983). However, although Artstein's result made a great impact on stabilization theory, it cannot be used as a control design tool, since its proof is nonconstructive. Another obstacle consists on finding CLF's (fortunately, there are methods to construct CLF's for special classes of systems, *cf.* (Malisoff & Mazenc, 2009)). Nevertheless, there has been a great activity in designing feedback controls via CLF's due to an explicit formula when $U = \mathbb{R}^m$,

[1]W.l.g. we have made a slight modification on (3)-(4) changing the sign.

obtained in (Sontag, 1989): the *universal formula*. Motivated by Artstein and Sontag's results, increasing efforts have been made to design control formulæ w.r.t. more general CVS (see (Leyva *et al.*, 2013; Solís–Daun, 2013a) and the references therein). The following important *open* problem was stated in (Sontag, 1998): "*Find universal formulas for* CLF *stabilization, for general (convex) control-value sets* U", *i.e.* solve the synthesis problem for *almost smooth* (of class $\mathcal{C}^\infty(\mathbb{R}^n\backslash\{0\})$ and continuous on $\mathbb{R}^n$) or *almost real analytic* feedback controls valued in general (convex) CVS.

The latter problem has been addressed by Sontag and co-workers for specific *compact* CVS: First, it was proposed an explicit universal formula for feedback controls taking values in the Euclidean open unit ball; and then in (Malisoff & Sontag, 2000), that result was extended to p-normed open unit balls, $\text{int}\mathcal{B}_1^m(p) := \{u \in \mathbb{R}^m : \|u\|_p < 1\}$, where $\|u\|_p := \sqrt[p]{|u_1|^p + \ldots + |u_m|^p}$, with $p = 2k/(2k-1)$ for $k = 1, 2, \ldots$ (so, $1 < p \leq 2$). Moreover, they proved that their designed universal formula is *almost smooth* for these specific values of p, whenever $a(x)$ and $b(x)$ are smooth.

In (Suárez *et al.*, 2002), it was defined a family of global stabilizers $u_\varepsilon(x)$ taking values in the (*asymmetric*) CVS $\mathcal{B}_{\boldsymbol{r}}^m(p) := \{u \in \mathbb{R}^m : \psi_{p,\boldsymbol{r}}(u) \leq 1\}$, where $\psi_{p,\boldsymbol{r}}(u) := \sqrt[p]{|u_1/r_1(u_1)|^p + \ldots + |u_m/r_m(u_m)|^p}$, for $1 < p < \infty$, and each $r_j(u_j)$ is a function defined by

$$r_j(\zeta) := \begin{cases} r_j^+, & \text{if } \zeta \geq 0, \\ r_j^-, & \text{if } \zeta < 0, \end{cases} \tag{5}$$

with $r_j^\pm > 0$, for $j = 1, \ldots, m$. The designed controls $u_\varepsilon(x)$ are continuous for any $p > 1$. Furthermore, continuous feedback controls were also derived for the $\boldsymbol{r}$-weighted m-hyperbox $\mathcal{B}_{\boldsymbol{r}}^m(\infty) := [-r_1^-, r_1^+] \times \cdots \times [-r_m^-, r_m^+]$, with $r_j^\pm > 0$. Then, in (Suárez *et al.*, 2001) this control design was generalized proposing an explicit formula for a one-parameterized family of continuous controls $u_\varepsilon(x)$ that render a system GAS w.r.t. more general CVS. Recently, in (Solís–Daun, 2013a; Solís–Daun, 2013b), it was proposed a general form of admissible feedback controls ($u(x) = \rho(x)\,\overline{\omega}(x)$, where $\rho(x)$ is a rescaling function and $\overline{\omega}(x)$ is an optimal control w.r.t. a CLF), that comprehends many of the control formulæ found in the literature. Moreover, it was shown how the regularity of $\overline{\omega}(x)$ depends on the geometry of U. Explicit control formulæ for feedbacks w.r.t. general compact CVS U with $0 \in \text{int}U$ were designed (*practically smooth* if $a(x)$ and $b(x)$ are smooth) that render (1) GAS, but at the expense of small overflows in the control values.

Considering polytopic CVS, we have: In (Curtis, 2003), it was introduced a method for algorithmically parameterizing stabilizing controls subject to polytopic CVS, given a known CLF. Then, in (Solís–Daun & Leyva, 2011), it was studied how to obtain admissible feedback controls that renders a system (1) GAS w.r.t. polytopic CVS U with $0 \in \text{int}U$.

In all the aforementioned papers, the control input components are *all signed*. Hence, in the case of *positive* control inputs, we have: In (Lin & Sontag, 1995), it was addressed the scalar control design problem w.r.t. CVS $(0,1)$ or $(0,\infty)$, but their control formulæ are not necessarily continuous at $x = 0$. In (Leyva *et al.*, 2009), it was proposed a formula for continuous feedbacks taking values in $[-r^-, r^+]$, also addressing the case of positive controls. Finally, in (Leyva *et al.*, 2013), it was proposed an explicit formula for *regular* feedback controls taking values in $\mathcal{B}_{\boldsymbol{r}}^m(\infty) = [-r_1^-, r_1^+] \times \cdots \times [-r_m^-, r_m^+]$, to render systems GAS. Moreover, it was studied the problem of positive feedback controls taking values in $\mathcal{B}_{\boldsymbol{r}}^m(\infty)$ (*i.e. all* $r_j^- = 0$). The feedback controls proposed in (Solís–Daun & Leyva, 2011; Leyva *et al.*, 2013) share the control scheme $u(x) = (u_1, \ldots, u_m)^\top$, with $u_j(x) = \rho_j(x)\,\overline{\omega}_j(x)$, where $\overline{\omega}(x)$ is an *optimal control w.r.t. a* CLF and $\rho_j(x)$ are *rescaling functions*, $j = 1, \ldots, m$.

In this paper, we generalize the results achieved in (Leyva *et al.*, 2009; Leyva *et al.*, 2013). In general, the feedback controls are *continuous*, in accordance with Artstein's theorem, and take values in $\mathcal{B}_{\boldsymbol{r}}^m(\infty)$ with $0 \in \mathcal{B}_{\boldsymbol{r}}^m(\infty)$, *i.e.* we allow control inputs with *signed/positive* components.

The paper is organized as follows. In §II, we obtain some convexity results for polytopes and hyperboxes that are needed in this work. In §III, we study properties of the optimal control $\overline{\omega}(x)$; and then, $\overline{\omega}(x)$ is analyzed for an $\boldsymbol{r}$-weighted m-hyperbox $\mathcal{B}_{\boldsymbol{r}}^m(\infty)$, finding that it is a *bang-bang* type control. Hence, inasmuch as $\overline{\omega}(x)$ is discontinuous with values on $\partial\mathcal{B}_{\boldsymbol{r}}^m(\infty)$, in §IV we propose feedback controls of the decentralized form $u(x) = (u_1, \ldots, u_m)^\top$, with $u_j(x) = \rho_j(x)\,\overline{\omega}_j(x)$, and $\boldsymbol{\rho}(x)$ a *rescaling function*. We search conditions that controls $u(x)$ should satisfy in order to be admissible (using functions $\rho_j(x)$ to regularize each control component $\overline{\omega}_j(x)$ at its *singular* switching hypersurface $\mathcal{N}_j$, for $j = 1, \ldots, m$), and render (1) GAS.

II. Elements of Convex Theory

For the readers convenience and to keep the paper self-contained, we introduce some results from Convex Theory.

A. Polarity

A *Minkowski functional* (also known as (a.k.a.) *gauge*) $\mu : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is a positively homogeneous ($\mu(\lambda u) = \lambda\mu(u)$, for $\lambda \geq 0$) convex function. Hence, for some convex set $\varnothing \neq U \subset \mathbb{R}^m$, a gauge can be defined as

$$\mu(u) := \inf\{r \geq 0 : u \in rU\}, \tag{6}$$

and *vice versa* if $\mu(u)$ is *closed* (*i.e.* lower semi-continuous and its restriction $\mu\restriction_{\text{dom}\mu\neq\varnothing}$ is finite), then there exists a unique convex (level) set $\varnothing \neq U = \{u \in \mathbb{R}^m : \mu(u) \leq 1\}$.

Theorem 1: ((Rockafellar, 1972), pp. 79 & 125). Let $U \subset \mathbb{R}^m$ be a closed convex set with $0 \in U$. Then: (i) μ is closed and positive semi-definite; (ii) μ is positive definite *iff* U is bounded; and (iii) μ is finite *iff* $0 \in \text{int}U$.

If U is a compact convex set with $0 \in \text{int}U$, the *polar* of μ and the *polar* of U are defined, respectively, by

$$\mu^*(u^*) := \sup_{u\neq 0} \frac{u^* \cdot u}{\mu(u)} \;\&\; U^* := \{u^* \in (\mathbb{R}^m)^* : \mu^*(u^*) \leq 1\} \tag{7}$$

where $(\mathbb{R}^m)^*$ is the dual space of $\mathbb{R}^m$.

The *support function of* U is the *sublinear* (positively homogeneous and subadditive) function defined by

$$\varsigma_U(u^*) := \sup_{u\in U} u^* \cdot u, \tag{8}$$

and $\mathrm{dom}\,\varsigma_U$ is a cone in $(\mathbb{R}^m)^*$ with apex at 0.

Theorem 2: ((Rockafellar, 1972), p. 125). Assume that $\varnothing \neq U \subseteq \mathbb{R}^m$ is a closed convex set with $0 \in U$. Then: (i) U^* is a closed convex set with $0 \in U^*$, and $U^{**} = U$; (ii) if μ and μ^* are respectively the gauges of U and U^*, then $\mu^* = \varsigma_U$, and *vice versa*; and (iii) U is bounded *iff* U^* satisfies that $0 \in \mathrm{int}U^*$, and *vice versa*.

Observe that it is very important that $0 \in \mathrm{int}U$. Otherwise, if $0 \notin \mathrm{int}U$ then some properties are lost.

Corollary 1: If U is a compact convex set with $0 \in \partial U$, then: (i) U^* is an unbounded closed convex set with $0 \in \mathrm{int}U^*$; (ii) μ is positive definite and closed, but it is not finite everywhere; and (iii) ς_U is lower semi-continuous and finite everywhere, but it is only positive semi-definite.

B. Some convexity results for polytopes

The set of all convex combinations of points in A is the *convex hull* of A, denoted by $\mathrm{conv}\{A\}$. Analogously, we define the *affine hull* of A, $\mathrm{aff}(A)$. A set A is d-dimensional, a d-set for short, if $d = \dim \mathrm{aff}(A)$. The *relative interior* of A, $\mathrm{relint}A$, is the interior of A relative to $\mathrm{aff}(A)$.

Each hyperplane $H = \{p \in \mathbb{R}^m : v\cdot p = c\}$ separates the space $\mathbb{R}^m$ into two halfspaces $H^+ = \{p \in \mathbb{R}^m : v \cdot p \geq c\}$ and $H^- = \{p \in \mathbb{R}^m : v \cdot p \leq c\}$. We say that H is a *supporting hyperplane* to a closed convex set $A \subseteq \mathbb{R}^m$ if there is $a_0 \in A$ lying in H, and $A \subset H^+$ or $A \subset H^-$. The *supporting halfspace* of A is the halfspace containing A.

A compact convex set P that is the convex hull of a finite *point set* $\{v_1, \dots , v_r\} \subset \mathbb{R}^m$, $P = \mathrm{conv}\{v_1, \dots , v_r\}$, is a *polytope*. $H \cap A$ is an *exposed face* of A, if H is a supporting hyperplane to A. Faces of dimensions $0, 1, \dots , d, \dots , m - 1$ are called *vertex*, *edge*, d-*face* and *facet*.

A convex set $U \subseteq \mathbb{R}^m$ is said to be a *polyhedron iff* it is the intersection of finitely many closed half-spaces. The following result states an equivalent description of P.

Theorem 3: P is the convex hull of a finite point set (a V-polytope) *iff* P is a bounded polyhedron (an H-polytope).

Theorem 4: If U is a polytope, then U^* is a polyhedron.

Theorem 5: U is a polytope with $0 \in \mathrm{int}U$ *iff* U^* is also a polytope with $0 \in \mathrm{int}U^*$. Moreover, polarity provides a bijection between the faces of U and the faces of U^* that reverses the relation of inclusion.

Hereafter, we will identify the dual space $(\mathbb{R}^m)^*$ with $\mathbb{R}^m$ using the inner product, and denote covector u^* by b.

Assume that U is polytope with $0 \in U$. It is well known that U is a polytope *iff* its support function is continuous and piecewise linear. The domains of linearity correspond to the vertices of the polytope U (for the maximum of the scalar product that defines the support function is achieved at one of the vertices). Hence, assuming the *V-representation*, if U has k vertices, then $U = \mathrm{conv}\{v_1, v_2, \dots , v_k\}$ and

$$\varsigma_U(b) = \begin{cases} v_1 \cdot b, & \text{if } b \in C_1 \\ \vdots & \vdots \\ v_k \cdot b, & \text{if } b \in C_k \end{cases} \tag{9}$$

where C_i are polyhedral cones with apex at 0, $i = 1, \dots , k$, corresponding to the domains of linearity of ς_U. These cones tile $\mathbb{R}^m$, and this tiling is called the *fan* of the polytope U.

To every proper face F of a closed convex set $A \neq \varnothing$ corresponds a cone N_F of linear functions $v \in (\mathbb{R}^m)^*$ which are maximized in F on A. The cone N_F is called the *normal cone* of F and the normal cones of all faces of a polytope P form a complete *fan*, *the normal fan*, N_P, of P: Every face $F \neq \varnothing$ of a normal cone is also a normal cone of some face of P, the intersection of two normal cones is a face of both and the union of all cones covers $\mathbb{R}^m$.

For $\varsigma_U(b)$ defined in (9), the polar set U^* is given by

$$\begin{aligned} U^* &= \{b \in \mathbb{R}^m : \varsigma_U(b) \leq 1\} \\ &= \{b \in \mathbb{R}^m : v_1 \cdot b \leq 1\,\&\, \dots \,\&\, v_k \cdot b \leq 1\}, \end{aligned} \tag{10}$$

which is defined by a system of k linear inequalities.

For a closed convex set U, the *null-set* of ς_U is

$$N_\varsigma := \{b \in \mathbb{R}^m : \varsigma_U(b) = 0\}. \tag{11}$$

From Theorem 1, if U is compact with $0 \in \mathrm{int}U$, then $\varsigma_U(b)$ is finite everywhere and positive definite ($N_\varsigma = \{0\}$). However, if U is a polytope with $0 \in \partial U$, then Theorem 4 & Corollary 1, imply that U^* is an *unbounded* polyhedron with $0 \in \mathrm{int}U^*$, and $\varsigma_U(b)$ is only *positive semi-definite*.

Therefore, it is important to study the properties and the geometric structure of $\mathcal{N}_\varsigma$. Clearly, we have that $\{0\} \subseteq N_\varsigma$, with equality *iff* $0 \in \mathrm{int}U$ (from Theorems 1 (ii) and 2).

An important class of polytopes are the $\boldsymbol{r}$-weighted m-hyperboxes (a.k.a. *orthotopes*),

$$\begin{aligned} \mathcal{B}^m_{\boldsymbol{r}}(\infty) &:= [-r^-_1, r^+_1] \times \dots \times [-r^-_m, r^+_m] \\ &= \mathrm{conv}\left\{(-r^-_1, \dots , -r^-_m), \dots , (r^+_1, \dots , r^+_m)\right\}, \end{aligned} \tag{12}$$

with $r^-_j \geq 0$, $r^+_j > 0$, for $j = 1, \dots , m$.

First of all, in view that $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ is a compact convex set with $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$, then it admits a representation in terms of a Minkowski functional, $\mathcal{B}^m_{\boldsymbol{r}}(\infty) = \{u \in \mathbb{R}^m : \psi_{\infty,\boldsymbol{r}}(u) \leq 1\}$, where $\psi_{\infty,\boldsymbol{r}} : \mathbb{R}^m \to \mathbb{R}$ is

$$\begin{aligned} \psi_{\infty,\boldsymbol{r}}(u) &:= \sup_j \left\{r^{-1}_j(u_j)\ |u_j|\right\} \\ &= \left\|(r^{-1}_1(u_1)\ |u_1|, \dots , r^{-1}_m(u_m)\ |u_m|)\right\|_\infty, \end{aligned} \tag{13}$$

with $r_j(\zeta_j)$ defined in (5), and $r^-_j \geq 0$, $r^+_j > 0$, for $j = 1, \dots , m$. Corresponding to $\psi_{\infty,\boldsymbol{r}}$, we define the following $\boldsymbol{r}$-weighted l_1-type Minkowski functional

$$\begin{aligned} \psi_{1,1/\boldsymbol{r}}(b) &:= \textstyle\sum_{j=1}^m r_j(b_j)\ |b_j| \\ &= \|(r_1(b_1)\ |b_1|, \dots , r_m(b_m)\ |b_m|)\|_1, \end{aligned} \tag{14}$$

$r_j(\zeta_j)$ given by (5), and $r^-_j \geq 0$, $r^+_j > 0$, $j = 1, \dots , m$.

Proposition 1: (Leyva *et al.*, 2013). $\psi_{\infty,\boldsymbol{r}}(u)$ and $\psi_{1,1/\boldsymbol{r}}(b)$ are gauges which are polar to each other ($\psi^*_{\infty,\boldsymbol{r}} = \psi_{1,1/\boldsymbol{r}}$ and *vice versa*). Moreover, the polar set of (12) is

$$\mathcal{B}^m_{\boldsymbol{r}}(\infty)^* := \left\{b \in \mathbb{R}^m : \psi_{1,1/\boldsymbol{r}}(b) \leq 1\right\}, \tag{15}$$

which is an m-octahedron, whenever $0 \in \text{int}\mathcal{B}^m_{\boldsymbol{r}}(\infty)$.

The k-octants of the Euclidean space $\mathbb{R}^m$, for $k = 0, 1, 2, 3, \ldots, m$, are the origin, positive/negative semiaxes, quadrants, octants, and *orthants* (or *m-octants*), respectively. An *open orthant* can be defined as $C = \{b \in \mathbb{R}^m : \delta_j b_j > 0, \text{ where each } \delta_j = -1 \text{ or } 1, \text{ for } j = 1, \ldots, m\}$, so permutation of the signs δ_j yields 2^m different orthants, *e.g.* $\mathbb{R}^m_-$ and $\mathbb{R}^m_+$ are the negative and positive *open* orthants with $\delta_j = -1$ & $\delta_j = 1$, for all $j = 1, \ldots, m$, respectively.

Observe that in the case of an m-hyperbox, its support function is $\varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}(b) = \psi_{1,1/\boldsymbol{r}}(b)$, so it is a continuous and piecewise linear function, where the domains of linearity are given by the 2^m orthants. Moreover, the corresponding null-set $N_\varsigma = \{b \in \mathbb{R}^m : \psi_{1,1/\boldsymbol{r}}(b) = 0\}$, so that it is defined by the following system of 2^m homogeneous linear equations

$$\begin{cases} v_1 \cdot b = 0, & \text{for } b \in C_1 \\ \quad\vdots & \quad\vdots \\ v_k \cdot b = 0, & \text{for } b \in C_{2^m} \end{cases} \tag{16}$$

Note that each equation is solved in its orthant C_i, for $i = 1, \ldots, 2^m$ –the domains of linearity of $\varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}$.

Furthermore, the case of *positive* feedback controls is already included, if we consider the *positive* m-hyperbox

$$\begin{aligned} &\mathcal{B}^m_{\boldsymbol{r}^+}(\infty) = [0, r_1^+] \times \ldots \times [0, r_m^+] = \\ &\text{conv}\{(0, \ldots, 0), (0, \ldots, r_1^+), \ldots, (r_1^+, \ldots, r_m^+)\}, \end{aligned} \tag{17}$$

with $r_j^+ > 0$ and $r_j^- = 0$, for all $j = 1, \ldots, m$. In this case, we have that N_ς is the negative closed orthant $\overline{\mathbb{R}^m_-}$.

Now, let us illustrate the introduced results, and also how N_ς changes in terms of the location of 0 in a rectangle.

Example. Let us consider the asymmetrical rectangle $\mathcal{B}^2_{\boldsymbol{r}}(\infty) = [-r_1^-, r_1^+] \times [-r_2^-, r_2^+]$, with $r_j^- \geq 0$ and $r_j^+ > 0$. Then, its set of vertices taken counterclockwise is $V = \{(r_1^+, r_2^+), (-r_1^-, r_2^+), (-r_1^-, -r_2^-), (r_1^+, -r_2^-)\}$. The normal fan of $\mathcal{B}^2_{\boldsymbol{r}}(\infty)$ is given by the four quadrants of $\mathbb{R}^2$, where C_i denotes the i^{th} quadrant taken counterclockwise.

First of all, if both $r_1^-, r_2^- > 0$, then $(0,0) \in \text{int}\mathcal{B}^2_{\boldsymbol{r}}(\infty)$, and $N_\varsigma = \{(0,0)\}$. From (9)-(14), its support function is

$$\begin{aligned} \varsigma_{\mathcal{B}^2_{\boldsymbol{r}}(\infty)}(b_1, b_2) &= r_1(b_1)\ |b_1| + r_2(b_2)\ |b_2| \\ &= \begin{cases} r_1^+ b_1 + r_2^+ b_2, & \text{if } (b_1, b_2) \in C_1 \\ -r_1^- b_1 + r_2^+ b_2, & \text{if } (b_1, b_2) \in C_2 \\ -r_1^- b_1 - r_2^- b_2, & \text{if } (b_1, b_2) \in C_3 \\ r_1^+ b_1 - r_2^- b_2, & \text{if } (b_1, b_2) \in C_4 \end{cases} \end{aligned}$$

and we note that it is linear on each quadrant. Moreover, from (10) we obtain that $\mathcal{B}^2_{\boldsymbol{r}}(\infty)^*$ is a quadrilateral with vertices at $(1/r_1^+, 0), (0, 1/r_2^+), (-1/r_1^-, 0)$ & $(0, -1/r_2^-)$.

Assume that $r_1^- = 0$ but $r_2^- > 0$, so that u_1 is positive and u_2 is signed. Then, $(0,0) \in \text{relint}\left(\{0\} \times [-r_2^-, r_2^+]\right)$ –a vertical edge of $\mathcal{B}^2_{\boldsymbol{r}}(\infty)$, and $\varsigma_{\mathcal{B}^2_{\boldsymbol{r}}(\infty)}(b)$ is positive semi-definite: $N_\varsigma = C_2 \cap C_3 = \{(b_1, 0) : b_1 \leq 0\}$ is the (non-positive) b_1^-–semiaxis. Analogously, if $r_1^- > 0$ but $r_2^- = 0$, we obtain $(0,0) \in \text{relint}\left([-r_1^-, r_1^+] \times \{0\}\right) \subset \partial\mathcal{B}^2_{\boldsymbol{r}}(\infty)$, and $N_\varsigma = C_3 \cap C_4 = \{(0, b_2) : b_2 \leq 0\}$ is the b_2^-–semiaxis. Finally, if both $r_1^- = r_2^- = 0$, then u is *positive*, and the rectangle becomes the *positive* 2-hyperbox, $\mathcal{B}^2_{\boldsymbol{r}^+}(\infty)$, so that $(0,0)$ is a vertex of $\mathcal{B}^2_{\boldsymbol{r}^+}(\infty)$, and $N_\varsigma = C_3$.

In these cases, the polar $\mathcal{B}^2_{\boldsymbol{r}}(\infty)^*$ is an unbounded polygon with $(0,0) \in \text{int}\mathcal{B}^2_{\boldsymbol{r}}(\infty)^*$. *E.g.*, in the latter case

$$\begin{aligned} &\mathcal{B}^2_{\boldsymbol{r}^+}(\infty)^* = \varsigma_U^{-1}[0,1] = \{b \in \mathbb{R}^2 : \varsigma_U(b_1, b_2) \leq 1\} = \\ &\{(r_1^+ b_1 + r_2^+ b_2 \leq 1) \,\&\, (r_2^+ b_2 \leq 1) \,\&\, \& \,(r_1^+ b_1 \leq 1)\}, \end{aligned}$$

which is a polygon with vertices at $(1/r_1^+, 0)$ & $(0, 1/r_2^+)$, containing properly the quadrant C_3, and limited by the line $b_2 = -r_1^+/r_2^+\, b_1 + 1/r_2^+$, the horizontal line $b_2 = 1/r_2^+$ and the vertical line $b_1 = 1/r_1^+$. In order to figure out how this set looks like, take the limit of the quadrilateral polar set $\mathcal{B}^2_{\boldsymbol{r}}(\infty)^*$ as its vertices at $(-1/r_1^-, 0)$ & $(0, -1/r_2^-)$ tend to $-\infty$ on their corresponding axes, or as $r_1^-, r_2^- \to 0^+$.□

Remark 2.1. Our findings in this example, namely of how N_ς changes as the locus of $(0,0)$ moves as an interior point through the faces F of $\mathcal{B}^2_{\boldsymbol{r}}(\infty)$, are summarized in the next table. Observe that $\dim N_\varsigma = m - \dim F$, with $m = 2$. □

TABLE I

N_ς VS. LOCUS OF $(0,0)$ IN THE FACES OF $\mathcal{B}^2_{\boldsymbol{r}}(\infty)$

Face F	$\dim F$	N_ς	$\dim N_\varsigma$
$\mathcal{B}^2_{\boldsymbol{r}}(\infty)$	2	$\{(0,0)\}$	0
$\{0\} \times [-r_2^-, r_2^+]$	1	b_1^-–semiaxis	1
$[-r_1^-, r_1^+] \times \{0\}$	1	b_2^-–semiaxis	1
$\{(0,0)\}$	0	C_3	2

The following result shows the geometric structure of N_ς in terms of the locus of the origin as a relative interior point of the faces of an m-hyperbox $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$.

Theorem 6: Assume that $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ is an m-hyperbox with $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$ given by (12). Then, the null set N_ς of $\varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}$ is a polyhedral cone with apex at 0. Moreover, (a) if $0 \in \partial\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ is a vertex, then $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ is the positive hyperbox, and thus $N_\varsigma = \overline{\mathbb{R}^m_-}$; otherwise, (b) N_ς is an $(m-d)$-octant of $\mathbb{R}^m$, defined as the normal cone of the d-face F of $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ $(1 \leq d \leq m)$, including the hyperbox itself, such that $0 \in \text{relint}F$, and given by the intersection of the orthants corresponding to all the vertices of F.

Remark 2.2. Note that if $F = \{0\}$ –a vertex of the hyperbox, then we obtain the positive hyperbox $\mathcal{B}^m_{\boldsymbol{r}^+}(\infty)$ and thus $N_\varsigma = \overline{\mathbb{R}^m_-}$; whereas if F is the d-face $(1 \leq d \leq m)$ of the hyperbox such that $0 \in \text{relint}F$, then N_ς is a proper subset of an intersection of Cartesian hyperplanes. □

III. The global stabilization w.r.t. a hyperbox

In this section, we explore the geometry behind the CLF stabilization of system (1) with CVS given by hyperboxes containing the origin not necessarily as an interior point. In particular, the important case of positive controls.

Now, let us return to our control problem. Assume that U is a closed CVS with $0 \in U$. Observe that (3) is solved *if* there is a feedback $u(x)$ taking values in ∂U such that $a(x) < b(x) \cdot u(x)$, $\forall x \neq 0$. On the other hand, for any control $u(x)$ taking values in U, we have that[2]: $b(x)\cdot u(x) \leq \mu^*(b(x))\,\mu(u(x))$. Then, for $u = \overline{\omega}(x)$, with $\mu(\overline{\omega}(x)) \equiv 1$ (*i.e.* $\overline{\omega}(x)$ is valued in ∂U) and recalling that $\mu^*(b) = \varsigma_U(b)$,

$$b(x)\cdot \overline{\omega}(x) = \varsigma_U(b(x)). \tag{18}$$

Hence, given a CLF, then any control $\overline{\omega}(x)$ satisfying (18) accomplishes the equivalence between (3) and inequality

$$\forall x \neq 0,\ a(x) < \varsigma_U(b(x)). \tag{19}$$

We will call a feedback $\overline{\omega}(x)$ to be an *optimal* (a.k.a. *best rate*) *control law w.r.t. a* CLF $V(x)$ [for system (1) with controls taking values in U] *iff* it satisfies

$$\forall x \neq 0,\ a(x) - b(x)\cdot\overline{\omega}(x) = \inf_{u\in U}\{a(x) - b(x)\cdot u\} < 0. \tag{20}$$

Hence, problem (3) is satisfied *if* there exists an optimal control $\overline{\omega}(x)$. However, from (18), it follows that $\overline{\omega}(x)$ is **not admissible** since it is *singular* at the null set

$$\mathcal{N}_b := \{x \in \mathbb{R}^n : b(x) = 0\}. \tag{21}$$

In (Solís–Daun, 2013a), it was shown that the *existence*, *uniqueness* and *continuity* of the optimal control $\overline{\omega}(x)$ are guaranteed, whenever U belongs to the class of all compact *strictly convex* (no line segment is contained in ∂U) CVS $U \subset \mathbb{R}^m$ with $0 \in \mathrm{int}U$, denoted $\mathcal{U}(\mathbb{R}^m)$. Specifically, it was shown that if $U \in \mathcal{U}(\mathbb{R}^m)$, then $\varsigma_U(b)$ is $\mathcal{C}^1(\mathbb{R}^m\backslash\{0\})$, and $\overline{\omega}(x)$ is a *gradient-based* feedback control of the form

$$\overline{\omega}(x) := \omega(b(x)), \quad \text{where } \omega(b) = (\nabla\varsigma_U(b))^\top, \tag{22}$$

and $b(x)$ is given by (4). Observe that $\omega(b)$ is continuous for $b \neq 0$ and homogeneous of degree 0. Hence, note that if we drop x, (18) becomes into the so-called Euler's theorem for (positively) homogeneous functions: $b\cdot\nabla\varsigma_U(b) = \varsigma_U(b)$.

Now, if $\varsigma_U(b)$ is differentiable at b, then formula (22) is still valid. Thus, if $U = \mathrm{conv}\{v_1,\ldots,v_k\}$ –a polytope, then $\varsigma_U(b)$ is piecewise linear, so that from (22), $\omega(b)$ is constant on the interior of each polyhedral cone $\mathrm{int}C_i$, and singular at the switching surfaces ∂C_i, for $i = 1,\ldots,k$, *i.e.*

$$\omega(b) = (\nabla\varsigma_U(b))^\top = \begin{cases} v_1, & \text{if } b \in \mathrm{int}\,C_1 \\ \vdots & \vdots \\ v_k, & \text{if } b \in \mathrm{int}\,C_k. \end{cases} \tag{23}$$

Now, consider the m-hyperbox $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ defined in (12), with $r_j^- \geq 0$ and $r_j^+ > 0$, for all $j = 1,\ldots,m$, so that $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$. Recall that $\varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}(b) = \psi_{1,1/\boldsymbol{r}}(b)$, which is a *positive definite* function *iff* $0 \in \mathrm{int}\mathcal{B}^m_{\boldsymbol{r}}(\infty)$, but it is only *positive semi-definite if* $0 \in \partial\mathcal{B}^m_{\boldsymbol{r}}(\infty)$. Moreover, from Theorem 6, we have that N_ς is an $(m-d)$-octant in $\mathbb{R}^m$, being the normal cone of the d-face F $(1 \leq d \leq m)$ of $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ such that $0 \in \mathrm{relint}F$. On the other hand, if $F = \{0\}$ –a vertex, then we have the positive hyperbox $\mathcal{B}^m_{\boldsymbol{r}+}(\infty)$, and $N_\varsigma = \overline{\mathbb{R}^m_-}$. In any case, from (22), we obtain

$$\omega(b) = \left(\nabla\,\Sigma_{j=1}^m\, r_j\ |b_j|\right)^\top = (r_1\,\mathrm{sign}\,b_1,\ldots,r_m\,\mathrm{sign}\,b_m)^\top \tag{24}$$

which is constant on each of the 2^m open orthants of $\mathbb{R}^m$. For instance, in the case of $\mathcal{B}^m_{\boldsymbol{r}+}(\infty)$, we have $\omega(b)\restriction_{\mathbb{R}^m_-}\equiv 0$. Moreover, the switching surfaces N_j of $\omega(b)$ are the Cartesian hyperplanes of $\mathbb{R}^m$, $N_j = \{b \in \mathbb{R}^m : b_j = 0\,\&\,b_i \neq 0,\ i \neq j\}$, $j = 1,\ldots,m$, and besides $\cap_j N_j = \{0\}$.

Let us return to the dependence on the state variable $x \in \mathbb{R}^n$, and let $b(x)$ be defined in (4). Observe that the set $\mathcal{N}_b$ given by (21) can be defined as the preimage of 0 under the mapping $b(x)$, *i.e.* $\mathcal{N}_b = b^{-1}[0] = \{x \in \mathbb{R}^n : b(x) = 0\}$. Analogously, we define the representation of the orthant corresponding to each vertex v_i of $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$, as $\mathcal{C}_i = b^{-1}[C_i] = \{x \in \mathbb{R}^n : b(x) \in C_i\}$, for $i = 1,\ldots,2^m$.

Now, we denote the representation in $\mathbb{R}^n$ of the switching surfaces N_j of $\omega(b)$ –the Cartesian hyperplanes of $\mathbb{R}^m$, by

$$\mathcal{N}_j = b^{-1}[N_j] = \{x \in \mathbb{R}^n : b_j(x) = 0\,\&\,b_i(x) \neq 0,\ i \neq j\} \tag{25}$$

$j = 1,\ldots,m$, and the null set of $\varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}$ given by (11), by

$$\mathcal{N}_\varsigma = b^{-1}[N_\varsigma] = \left\{x \in \mathbb{R}^n : \varsigma_{\mathcal{B}^m_{\boldsymbol{r}}(\infty)}(b(x)) = 0\right\}. \tag{26}$$

Remark 3.1. Note that $\mathcal{N}_b = \cap_j\mathcal{N}_j$, and $\mathcal{N}_b \subseteq \mathcal{N}_\varsigma$, with equality if $0 \in \mathrm{int}U$. Moreover, based on Remark 2.2, we have that: (a) either $\mathcal{N}_\varsigma = b^{-1}\left[\overline{\mathbb{R}^m_-}\right]$, whenever 0 is a vertex of the (positive) hyperbox; or (b) $\mathcal{N}_\varsigma \subset \mathcal{N}_j$ for some j, whenever $0 \in \mathrm{relint}F$, for a d-face F of the hyperbox. □

Hereafter, we denote by $r_j(x) := r_j(b_j(x))$, with $r_j(\zeta_j)$ given by (5), for $j = 1,\ldots,m$, $\beta(x) := \psi_{1,1/\mathbf{r}}(b(x))$ and $\overline{\omega}(x)$ is defined by (22)-(24). Moreover, it is clear that $\overline{\omega}(x)$ is a singular function on $\cup_j\mathcal{N}_j$. In particular, for the positive hyperbox $\mathcal{B}^m_{\boldsymbol{r}+}(\infty)$, we have that $\overline{\omega}(x)\restriction_{\mathrm{int}\mathcal{N}_\varsigma}\equiv 0$.

Assume that $V(x)$ a CLF w.r.t. system (1) with controls taking values in an m-hyperbox $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ containing 0. Then, based on (18) and assuming the optimal control $\overline{\omega}(x)$ defined by (22)-(24), we have that for all $x \neq 0$,

$$dV/dt = a(x) - b(x)\cdot\overline{\omega}(x) < 0 \text{ iff } a(x) < \beta(x). \tag{27}$$

Remark 3.2. Observe from (27) that[3] $\forall x \neq 0$, *if* $\beta(x) = 0$ *then* $a(x) < 0$. Moreover, $\beta(x) = 0$ *iff* $x \in \mathcal{N}_\varsigma$. □

IV. A FEEDBACK CONTROL DESIGN FOR A HYPERBOX

However, inasmuch as the optimal control $\overline{\omega}(x)$ is singular, we study the conditions that feedback controls should satisfy in order to be regular, take values in an m-hyperbox $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ with $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$, and render system (1) GAS, provided an appropriate CLF is known.

Now, assuming that $U \subset \mathbb{R}^m$ is a compact and strictly convex set with $0 \in \mathrm{int}U$, in (Solís–Daun, 2013a) it was considered general feedback controls of the form $u(x) := \rho(x)\,\overline{\omega}(x)$, where $\rho(x)$ is an rescaling function and $\overline{\omega}(x)$ is the best rate control. However, that designing can only deal

[2]Gauges polar to each other satisfy the following important property: $u^*\cdot u \leq \mu^*(u^*)\,\mu(u)$, $\forall u \in \mathrm{dom}\mu$ & $\forall u^* \in \mathrm{dom}\mu^*$. This expression is the "best" inequality in the sense that it cannot be tightened by replacing μ or μ^* by lesser functions on larger domains. *E.g.*, if μ is a p-norm, then μ^* is a q-norm $(1/p + 1/q = 1)$ and it reduces to Hölder's inequality.

[3]In the definition of CLF w.r.t. $U = \mathbb{R}^m$, this condition replaces (3).

with the singularities of the control $\overline{\omega}(x)$ at $\mathcal{N}_b$. Henceforth, following (Leyva *et al.*, 2013; Solís–Daun & Leyva, 2011), we propose feedback controls of the *decentralized* form

$$u(x) = (u_1(x), \ldots, u_m(x))^\top, \; u_j(x) := \rho_j(x)\,\overline{\omega}_j(x), \tag{28}$$

for $j = 1, \ldots, m$, where $\overline{\omega}(x)$ is defined by $\omega(b)$ given by (23) and $b(x)$ given by (4), and $\boldsymbol{\rho}(x) = (\rho_1(x), \ldots, \rho_m(x))$ is a *rescaling vector function* to be determined. Therefore, the control scheme (28) will be used to address the global CLF stabilization of (1) via feedback controls taking values in $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$, either if $0 \in \text{int}\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ or if $0 \in \partial\mathcal{B}^m_{\boldsymbol{r}}(\infty)$.

Now, we ask the conditions that $\boldsymbol{\rho}(x)$ should satisfy in order to guarantee the existence of an *admissible* feedback control $u(x)$ of the form (28) that renders system (1) GAS.

Hypothesis H. Assume that $\boldsymbol{\rho} : \mathbb{R}^n \to \mathbb{R}^m$ is a regular function such that

(***i***) $\forall x \in \mathbb{R}^n$, $0 \leq \rho_j(x) \leq 1$, for $j = 1, \ldots, m$,

(***ii***) $\rho_j(x) = 0$ *iff* $x \in \mathcal{N}_j$, for $j = 1, \ldots, m$, and

(***iii***) $\forall x \in \mathbb{R}^n \backslash \mathcal{N}_\varsigma$, $\|\boldsymbol{\rho}(x)\|_\infty > \frac{a(x)}{\beta(x)}$.

Our main theorem in this section is the following.

Theorem 7: Assume $V(x)$ is a CLF [for system (1) with controls taking values in $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ with $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$ given by (12)] satisfying the SCP, $\overline{\omega}(x)$ is the optimal control defined in (22)-(24) and $\boldsymbol{\rho} : \mathbb{R}^n \to \mathbb{R}^m$ is a regular mapping satisfying *Hypothesis H.* Then, $u(x)$ given by (28) is an admissible feedback control that renders system (1) GAS.

Proof: First of all, we have that $u(x)$ given by (28) is admissible: In fact, from Condition (***i***) we have that $u_j(x) = \rho_j(x)\,\overline{\omega}_j(x) = \rho_j(x) r_j(x) \operatorname{sign} b_j(x)$ is equivalent to $-r_j^- \leq -r_j^- \rho_j(x)$, if $b_j(x) < 0$, or $r_j^+ \rho_j(x) \leq r_j^+$, if $b_j(x) > 0$; so that $u_j(x)$ is valued in $[-r_j^-, r_j^+]$, for $j = 1, \ldots, m$. Thus, $u(x)$ takes values in $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$.

Further, $\forall x \in \mathbb{R}^n \backslash \cup_j \mathcal{N}_j$, we have that both $\boldsymbol{\rho}(x)$ and $\overline{\omega}(x)$ are continuous (recall that $\overline{\omega}(x)$ is constant on each open orthant), so that $u(x)$ is continuous. In the case that $x \in \mathcal{N}_j$, since $\rho_j(x)$ is continuous and $\overline{\omega}_j(x) = r_j(x) \operatorname{sign} b_j(x)$ is bounded, then from the SCP and Condition (***ii***) it follows that $\forall x^* \in \mathcal{N}_j$, for $j = 1, \ldots, m$,

$$0 \leq \lim_{x \to x^*} |u_j(x)| = \lim_{x \to x^*} \rho_j(x)\; |\overline{\omega}_j(x)| = 0,$$

then each entry $u_j(x)$ is continuous at $\mathcal{N}_j$ and $u_j(x) \restriction_{\mathcal{N}_j} \equiv 0$. Based on Remark 3.1, we note that the regularity of $u(x)$ at the null set $\mathcal{N}_\varsigma$ has been already addressed above: Either $\mathcal{N}_\varsigma \subset \mathcal{N}_j$ for some j, or $\mathcal{N}_\varsigma = b^{-1}\left[\overline{\mathbb{R}^m_-}\right]$.

Finally, we show that the closed-loop system is GAS:

(a) If $x \in \mathcal{N}_\varsigma \backslash \{0\}$, then from Remark 3.2 it follows that both $a(x) < 0$ and $u(x) \restriction_{\mathcal{N}_\varsigma} \equiv 0$. Hence, we have that $dV/dt = a(x) - b(x) \cdot u(x) = a(x) < 0$, $\forall x \in \mathcal{N}_\varsigma \backslash \{0\}$.

(b) If $x \in \mathbb{R}^n \backslash \mathcal{N}_\varsigma$, then there is at least a j ($1 \leq j \leq m$) such that $b_j(x) \neq 0$. Then, from Proposition 1, the definitions of $\beta(x)$ and $u(x)$, and Condition (***iii***), we obtain

$$\begin{aligned}
&dV/dt = a(x) - b(x) \cdot u(x) < 0 \text{ iff} \\
&a(x) < \psi_{1,1/\boldsymbol{r}}(b(x))\, \psi_{\infty,\boldsymbol{r}}\left(\rho_1(x)\overline{\omega}_1(x), \ldots, \rho_m(x)\overline{\omega}_m(x)\right) \\
&\leq \beta(x)\; \|\boldsymbol{\rho}(x)\|_\infty\; \psi_{\infty,\boldsymbol{r}}(\overline{\omega}(x)) = \beta(x)\; \|\boldsymbol{\rho}(x)\|_\infty\,, \text{ where}
\end{aligned}$$

$\|\boldsymbol{\rho}(x)\|_\infty = \max_j \rho_j(x)$. Therefore, $dV/dt < 0$, $x \neq 0$. ■

V. Conclusions and future work

In this paper, we address the problem of the global CLF stabilization of affine control systems (1) w.r.t. $\boldsymbol{r}$-weighted m-hyperboxes $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$ with $0 \in \mathcal{B}^m_{\boldsymbol{r}}(\infty)$.

First, we show that for an m-hyperbox $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$, control $\overline{\omega}(x)$ is piecewise constant, with switching surfaces $\mathcal{N}_j$ defined by the level sets (25): It is a *bang-bang type* control.

However, inasmuch as the feedback control $\overline{\omega}(x)$ for an m-hyperbox is not admissible (it is discontinuous at $\cup_j \mathcal{N}_j$), we consider feedback controls of decentralized form $u(x) = (u_1(x), \ldots, u_m(x))^\top$, with components given by $u_j(x) = \rho_j(x)\,\overline{\omega}_j(x)$, where $\rho_j(x)$ is a rescaling function used to regularize $\overline{\omega}_j(x)$. Then, we study the conditions that such controls should satisfy in order to be *admissible* (continuous and valued in $\mathcal{B}^m_{\boldsymbol{r}}(\infty)$) and render system (1) GAS, provided a CLF is known. We pay special attention to the case when $0 \in \partial\mathcal{B}^m_{\boldsymbol{r}}(\infty)$, and in particular to positive controls.

Finally, the generalization of the results w.r.t. polytopic CVS U with $0 \in U$ and the design of an explicit control formula valued in U (with signed/positive input components) that renders system (1) GAS, are topics for future research.

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