Towards the Synchronization of Under–actuated Lagrangian Systems: the Flexible Joint Robots with Uncertain Communication Delays case

Sofía Avila–Becerril and Gerardo Espinosa–Pérez

***Abstract*— Synchronization of networks composed by *fully–actuated* Lagrangian systems has received a lot of attention from the control theory community. Unfortunately, the case of networks composed by *under–actuated* systems of the same kind has not been deeply studied and the related literature is very reduced. The aim of this paper is to contribute towards the establishment of control scheme for this class of networks by solving the particular case of a network composed by agents defined by Flexible–joint robots. The proposed schemes consider the presence of unknown delays in the communication channels. The usefulness of the controllers is validated through numerical simulations.**

I. Introduction

Synchronization of networks composed by dynamical systems has become in an interesting challenge for the control theory community since more than one decade ago [1], [2]. Roughly speaking, the purpose is to design a control scheme that achieves some agreement between the outputs of the different systems (usually called *agents*) involved in the network. Illustrative examples of this behavior are, the tracking trajectory problem, when all the agents must track a common desired (time–varying) behavior, or the consensus problem, when the output of all the agents must tend to a given (non-zero) constant value.

Starting by considering collections of identical agents described by linear dynamics without delays in the communication channels [3], [4], the study of the aforementioned problem has evolved in a remarkable way giving as a result that currently different control strategies can be found for more complicated scenarios, namely, nonidentical and discrete–time agents, fixed and switched topologies and existence of delays in the transmitted information, among other [5], [6], [7], [8].

Although the number of techniques that have been used to approach the control problem of networks is very large, one that has produced quite attractive results is the related with the notion of passivity [9], [10], [11], [12] where the advantages offered by this concept to deal with interconnected systems are exploited. In particular the problem of synchronizing networks composed by Euler–Lagrange or Hamiltonian dynamical systems has been deeply studied from this perspective. In this context, the class of *fully actuated* systems has received a lot of attention. Indeed, after stating several results for general affine nonlinear passive systems in [13], the particular case of mechanical systems is approached to solve the multi-robot coordination problem for balanced graphs. This class of systems is also studied in [14] to solve the tracking problem allowing parametric uncertainty. Solution for several configurations, including leaderless and leader–follower structures, are included in [15] while in [16] the consensus and coordination control problems are approached under a Port–Hamiltonian framework. Unfortunately and to the best of the author's knowledge, the case of networks composed by *under–actuated* (Lagrangian) systems has not been studied in an exhaustive way and the number of publications related with this topic reduces to [17] where, under the Controlled–Lagrangian approach, the synchronization control problem is solved *without* delays in the communication channels.

The aim of this paper is to contribute towards the solution of the control problem for networks composed by *under–actuated* Lagrangian systems by extending the result presented in [18] to treat the case when the agents are Flexible–joint robots that operate in presence of delays in the communication channels. Even that in [18] both tracking and consensus objectives are achieved a simultaneous way (for fully–actuated systems) considering parametric uncertainty, in this paper (for ease of presentation) only the consensus problem is analyzed under the same scenario than in [18], namely, nonidentical agents are considered, the associated graph is only connected, the delays in the channels are *unknown* and the ”synchronization to zero” is avoided.

As usual, the proposed solution for the aforementioned problem is composed by two elements, the control the individual dynamics and the interconnection pattern among the agents. While the latter is of the classical type where information is exchanged among each agent and its neighbors, control of each robot is carried using a scheme proposed in [19] and [20] for single robots. The advantage of this algorithm lies in the facility that it offers for proving that consensus is achieved following a similar procedure than the proposed in [18].

The rest of the paper is organized as follows: In Section II, the considered model for the network is presented. The proposed controller is presented in Sections III and IV, including in the former the case without delays and in the latter the case with delays in the communication channels. A numerical evaluation of the proposed controllers is presentes in Section V while Section VI is devoted to state some concluding remarks.

S. Avila–Becerril and G. Espinosa–Pérez are with Facultad de Ingeniería – UNAM, A.P. 70-256, 04510 México D.F., MEXICO. gerardoe@unam.mx. Part of their work was supported by DGAPA–UNAM and CONACYT (51050).

II. Problem formulation

Throughout the paper to vectors and matrices will be associated up to three subindexes ijk. The first will denote each of the robots that compose the network, hence $i \in \bar{N} = \{1, \ldots, N\}$ where N is the number of robots. The second subindex j stands for actuated and under–actuated coordinates of each robot. If $j = 1$ then it refers to under–actuated coordinates while $j = 2$ refers to actuated coordinates. Finally, the third subindex k is related with $k-th$ actuated or underactuated coordinate of a given robot, for example q_{111} denotes the first under–actuated coordinate of robot 1. In this context $k \in \bar{n} = \{1, \ldots, n\}$ since it will be assumed in the sequel that each robot has n degrees of freedom.

Under the notation introduced above, the $i-th$ robot involved in the network is denoted by

$$\mathcal{D}_i(q_{i1})\ddot{q}_i + \mathcal{C}_i(q_{i1}, \dot{q}_{i1})\dot{q}_i + \mathcal{G}_i(q_{i1}) + \mathcal{K}_i q_i = M_i u_i \quad (1)$$

where $i \in \bar{N}$ while $q_i = \text{col}(q_{i1}, q_{i2}) \in \mathbb{R}^{2n}$ and

$$\mathcal{D}_i(q_{i1}) = \begin{bmatrix} D_i(q_{i1}) & 0 \\ 0 & J_i \end{bmatrix}; \ \mathcal{K}_i = \begin{bmatrix} K_i & -K_i \\ -K_i & K_i \end{bmatrix}$$

$$\mathcal{C}_i(q_{i1}, \dot{q}_{i1}) = \begin{bmatrix} C_i(q_{i1}, \dot{q}_{i1}) & 0 \\ 0 & 0 \end{bmatrix}; \ M_i = \begin{bmatrix} 0 \\ I_n \end{bmatrix}$$

with I_n the identity matrix of dimension $n \times n$ while $\mathcal{D}_i$, $\mathcal{C}_i$ and $\mathcal{K}_i$ stands for the Inertia, Coriolis and Stiffness matrices (all of them of belonging to $\mathbb{R}^{2n \times 2n}$), respectively, and $\mathcal{G}_i = \text{col}(G_i(q_{i1}, 0) \in \mathbb{R}^{2n}$ the vector of Gravitational forces.

Remark. It is well–known that model (1) enjoys several useful properties related with its passivity nature and stability, e.g. the skew–symmetric structure of matrix $\dot{\mathcal{D}}_i - 2\mathcal{C}_i$ (see for example [20]). These properties will be cited when necessary along the paper.

Concerning the interconnection of the robots, it is assumed that the exchange of information is carried out over a directed simply connected graph and is subject to a constant unknown delay $T_{ij} \geq 0$ between the $i-th$ and the $j-th$ agent. Furthermore, this graph is modeled by a Laplacian matrix $\mathbf{L} \in \mathbb{R}^{N \times N}$ whose elements are defined as

$$l_{ij} = \begin{cases} \Sigma_{k=1}^{N} a_{ik} & i = j \\ -a_{ij} & i \neq j \end{cases} \quad (2)$$

with $a_{ij} = 1$ if $j \in \mathcal{N}_i$, $a_{ij} = 0$ otherwise, with $\mathcal{N}_i$ the set of neighbor agents transmitting information to the $i-th$ robot.

Remark. Besides its immediate physical interpretation, the considered Laplacian matrix (2) exhibits several structural properties [5] that will be used below, namely, $\mathbf{L}\mathbf{1}_N = 0$, where $\mathbf{1}_N \in \mathbb{R}^N$ is a vector filled with ones, $rank\{\mathbf{L}\} = N-1$, meaning that it has a single zero-eigenvalue, and that the rest of the spectrum of $\mathbf{L}$ has positive real parts.

Under the conditions stated in the previous paragraphs, it is possible to formulate the problem approached in this paper as

Problem formulation: Consider a network composed by N flexible–joint robots of the form (1) interconnected through a communication protocol described by (2) and subject to delays $T_{ij} \geq 0$. Find control laws u_i such that the underactuated coordinates q_{i1}, $i \in \bar{N}$, of all the robots reach consensus, i.e. $\lim_{t\to\infty} q_{i1} = q_c$, for some constant vector $q_c \in \mathbb{R}^n$, satisfying that $\lim_{t\to\infty} |\dot{q}_{i1}| = 0$ and guaranteeing internal stability.

III. Case without delays

In this section, with the aim of illustrating in a better way the results presented in the paper, the problem stated in Section II is solved assuming that there are no delays in the communication channels. In this sense, it will be useful to introduce two equivalent representations for the set of N robots. The first is given by

$$\begin{aligned} D(\bar{q}_1)\ddot{\bar{q}}_1 + C(\bar{q}_1, \dot{\bar{q}}_1)\dot{\bar{q}}_1 + G(\bar{q}_1) + K(\bar{q}_1 - \bar{q}_2) = 0 \\ J\ddot{\bar{q}}_2 - K(\bar{q}_1 - \bar{q}_2) = \bar{u} \end{aligned} \quad (3)$$

where $\bar{q}_1 \in \mathbb{R}^{Nn}$, $\bar{q}_2 \in \mathbb{R}^{Nn}$, are the direct composition of the under–actuated and actuated coordinates of the N robots, respectively, while $D = \text{diag}\{D_1, D_2, \ldots, D_N\} \in \mathbb{R}^{Nn \times Nn}$, $C = \text{diag}\{C_1, C_2, \ldots, C_N\} \in \mathbb{R}^{Nn \times Nn}$, $G = \text{col}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N) \in \mathbb{R}^{Nn}$, $K = \text{diag}\{K_1, K_2, \ldots, K_N\} \in \mathbb{R}^{Nn \times Nn}$, $J = \text{diag}\{J_1, J_2, \ldots, J_N\} \in \mathbb{R}^{Nn \times Nn}$ and $\bar{u} = \text{col}(u_1, u_2, \ldots, u_N) \in \mathbb{R}^{Nn}$.

The second representation[1] takes the form

$$\mathcal{D}(\bar{q}_1)\ddot{q} + \mathcal{C}(\bar{q}_1, \dot{\bar{q}}_1)\dot{q} + \mathcal{G}(\bar{q}_1) + \mathcal{K}q = M\bar{u} \quad (4)$$

where $q = \text{col}(\bar{q}_1, \bar{q}_2)$, $\mathcal{D} = \text{diag}\{D, J\}$, $\mathcal{C} = \text{diag}\{C, 0\}$, $\mathcal{G} = \text{col}(G, 0)$, $M = \text{col}(I_{Nn}, 0)$, and

$$\mathcal{K} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

With these representations, it is possible to introduce the *consensus error* as

$$e_1 = (\mathbf{L} \otimes I_n)\bar{q}_1 \quad (5)$$

with $\otimes$ the standard Kronecker product, and two auxiliary variables

$$\epsilon_1 = \dot{\bar{q}}_1 + \lambda_1 e_1; \ \ \epsilon_2 = \dot{\bar{q}}_2 + \Lambda_2 \tilde{q}_2 \quad (6)$$

where $\tilde{q}_2 = \bar{q}_2 - q_{2d}$, q_{2d} defined below, and $\lambda_1 \in \mathbb{R}$, $\Lambda_2 = \text{diag}\{\Lambda_{12}, \Lambda_{22}, \ldots, \Lambda_{N2}\}$ constant gains, which define the *control error* given by

$$\epsilon = \dot{\tilde{q}} + \Lambda E_1 \quad (7)$$

with $\epsilon = \text{col}(\epsilon_1, \epsilon_2)$, $\Lambda = \text{diag}\{\lambda_1 I_{Nn}, \Lambda_2\}$ and

$$\dot{\tilde{q}} = \begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 - \dot{q}_{2d} \end{bmatrix}; \ \ E_1 = \begin{bmatrix} e_1 \\ \tilde{q}_2 \end{bmatrix}$$

As in [18], the rational behind the definition of the errors (5), (7), is to show that convergence to zero of the latter implies achievement of the consensus objective. This is done in the following

[1] With the aim to avoid difficulties with the notation, in the sequel the dimension of vectors and matrices will be explicitely stated only when necessary.

Proposition 1.

Consider a network composed by N flexible–joint robots of the form (4). Assume that

A.1 *The robots are interconnected through a connected graph without delays.*

A.2 *The complete state of all the robots is available for measurement.*

Under these conditions, the control law

$$\begin{array}{c} u = -K_{d2}\epsilon_2 + J(\ddot{q}_{2d} - \Lambda_2\dot{\tilde{q}}_2) + \\ \mathcal{K}\left(\int_0^t (\lambda_1 e_1 - \Lambda_2 \tilde{q}_2)d\tau + q_{2d}\right) - \mathcal{K}(\bar{q}_{10} - \bar{q}_{20}) \\ q_{2d} = p(p+\Lambda_2)^{-1}\left\{\mathcal{K}^{-1}u_r - \int_0^t (\lambda_1 e_1 - \Lambda_2 q_2)d\tau\right\} \\ +p(p+\Lambda_2)^{-1}(\bar{q}_{10} - \bar{q}_{20}) \\ u_r = -D\lambda_1\dot{e}_1 - C\lambda_1 e_1 + G - K_{d1}\epsilon_1 \end{array} \tag{8}$$

with e_1, ϵ_1 and ϵ_2 defined above, and $p = \frac{d}{dt}$, $K_d = K_d^T = diag(K_{d1}, K_{d2}) \geq 0$, $q(0) = q_0$ achieves the consensus objective.

Remark. Before presenting the proof of the proposition it is important to point out that the result has been stated in its basic form depending on the knowledge of the initial conditions of the robots coordinates. Although this seems to be reasonable under assumption **A.2**, following the proof of the proposition some robustness properties with respect uncertainty on these values will be stated.

Proof of Proposition 1.

Noting that from (7) it is possible to write $q = \int_0^t (\epsilon(\tau) - \lambda_1 E_1(\tau))d\tau + \xi + q(0)$ with $\xi = \text{col}(0, q_{2d})$, the control error dynamics, under the control law (8), is described by

$$\mathcal{D}(\bar{q}_1)\dot{\epsilon} + (\mathcal{C}(\bar{q}_1, \dot{\bar{q}}_1) + K_d)\epsilon + \mathcal{K}\int_0^t \epsilon(\tau)d\tau = 0 \tag{9}$$

Hence, considering the positive definite function

$$V = \frac{1}{2}\epsilon^T\mathcal{D}\epsilon + \frac{1}{2}\left[\int_0^t \epsilon^T(\tau)d\tau\right]\mathcal{K}\left[\int_0^t \epsilon(\tau)d\tau\right]$$

and its time derivative along the trajectories of (9) given by $\dot{V} = -\epsilon^T K_d \epsilon$, it is possible to conclude, using standard arguments, that $\epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and from (9) that $\dot{\epsilon} \in \mathcal{L}_\infty$. Therefore $\lim_{t\to\infty} \epsilon = 0$.

In order to prove that convergence of the control error to zero implies achievement of the consensus objective, it can be followed the ideas presented in [12] and [18] in the sense that, under the definitions of ϵ_1 and e_1, it is possible to get that

$$\begin{aligned} \dot{\bar{q}}_1 &= \epsilon_1 - \lambda_1 e_1 \\ &= -\lambda_1(\mathbf{L}\otimes I_n)\bar{q}_1 + \epsilon_1 \end{aligned} \tag{10}$$

Hence, since matrix $\mathbf{L}\otimes I_n$ is not Hurwitz, it can be defined $y_1 = (Q\otimes I_n)\bar{q}_1 \in \mathbb{R}^{(N-1)n}$, which satisfies the differential equation

$$\begin{aligned} \dot{y}_1 &= (Q\otimes I_n)\dot{\bar{q}}_1 \\ &= -\lambda_1(Q\mathcal{L}Q^T\otimes I_n)y_1 + (Q\otimes I_n)\epsilon_1 \end{aligned} \tag{11}$$

if the matrix $Q \in \mathbb{R}^{(N-1)\times N}$ satisfies in its turn: i) $QQ^T = I_{N-1}$, ii) $Q^TQ = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$, iii) $Q\mathbf{1}_N = 0$, iv) $(Q\otimes I_n)\bar{q}_1 = 0 \Leftrightarrow \bar{q}_1 = (\mathbf{1}_N\otimes q_c)$ for some $q_c \in \mathbb{R}^n$. In this way, since matrix $-Q\mathcal{L}Q^T$, and consequently matrix $-(Q\mathcal{L}Q^T\otimes I_n)$, is Hurwitz, it can be proved that, considering the convergence to zero of ϵ_1, that $|y_1| \to 0$ implying that $(Q\otimes I_n)\bar{q}_1 = 0$, expression that allows to conclude, under property (iv) that $\bar{q}_1 \to (\mathbf{1}_N\otimes q_c)$ for some q_c. The final part of the proof, i.e. to show that $|\dot{\bar{q}}_1| \to 0$, is completed by noting that $\bar{q}_1 \to (\mathbf{1}_N\otimes q_c)$ implies that $e_1 = (\mathbf{L}\otimes I_n)(\mathbf{1}_N\otimes q_c)$, which exploiting the properties of the Kronecker product, can be written as $e_1 = (\mathbf{L}\mathbf{1}_N\otimes I_n q_c)$ showing, due to the fact that $\mathbf{L}\mathbf{1}_N = 0$, that $|e_1| \to 0$. From this result and the convergence of ϵ_1 to zero, the desired result is obtained from (10). $\nabla\nabla\nabla$

Remark. It is interesting to note that the proposed controller exhibits some robustness properties with respect to uncertainty in the initial conditions of the robots. Indeed, if instead of using the real values in the controller (8), it is used some estimate, then the control error dynamics will take the form

$$\mathcal{D}(\bar{q}_1)\dot{\epsilon} + (\mathcal{C}(\bar{q}_1, \dot{\bar{q}}_1) + K_d)\epsilon + \mathcal{K}\int_0^t \epsilon(\tau)d\tau = \mathcal{K}\tilde{q}_0$$

where $\tilde{q}_0$ denotes the error in the initial conditions. Under this scenario, it is possible to show that still $\epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ but now whenever the condition

$$\|\epsilon\| \geq \frac{\|\mathcal{K}\|\|\tilde{q}_0\|}{\theta\lambda_{min}\{K_d\}};\ \theta < 1$$

is satisfied, i.e. with respect to a ball centered at the origin and with radii defined by the initial condition error. Under this condition, the input of the LTI system (10) still guarantees that $|y_1| \to 0$ achieving the consensus objective. These properties will be illustrated under the numerical evaluation carried out in Section V.

IV. Case with unknown delays

As explained in [18] dealing with the existence of delays in the communication channels can not accomplished using the representation (10). Instead of, the consensus and auxiliary errors are expressed for the $i-th$ robot as

$$e_{i1} = \sum_{r\in N_i} [q_{i1} - q_{r1}(t - T_{ir})] \tag{12}$$

$$\epsilon_{i1} = \dot{q}_{i1} + \lambda_1 e_{i1} \tag{13}$$

$$\epsilon_{i2} = \dot{\tilde{q}}_{i2} + \Lambda_2\tilde{q}_{i2} \tag{14}$$

leading to the control error $\epsilon_i = \dot{\tilde{q}}_i + \lambda_1 E_i$ where $\tilde{q}_{i2} = q_{i2} - q_{i2d} \in \mathbb{R}^n$, $\epsilon_i = \text{col}(\epsilon_{i1}, \epsilon_{i2}) \in \mathbb{R}^{2Nn}$ and

$$\dot{\tilde{q}}_i = \begin{bmatrix} \dot{q}_{i1} \\ \dot{q}_{i2} - \dot{q}_{i2d} \end{bmatrix};\ \ E_i = \begin{bmatrix} e_{i1} \\ \tilde{q}_{i2} \end{bmatrix}$$

On the other hand and following the notation introduced in Section II, consider the $k-th$ coordinate of the $i-th$ robot given by $q_{ijk} \in \mathbb{R}$ and define $q_1^k = \text{col}(q_{11k}, q_{21k}, \ldots, q_{N1k}) \in \mathbb{R}^N$ the vector composed by the

$k-th$ under–actuated coordinates of all the N robots. Under this definition, equations (12) and (13) can be re–written as

$$e_1^k = \mathcal{L}q_1^k \quad (15)$$
$$\epsilon_1^k = \dot{q}_1^k + \lambda_1 e_1^k \quad (16)$$

Following the same ordering procedure than [18], introduce $m = \sum_{i=1}^{N} card\, \mathcal{N}_i$ and $\tau_k = T_{ir}$, where $k \in \bar{m} = \{1,\ldots,m\}$ in order to obtain the expression

$$e_1^k = A_0 q_1^k - \sum_{r=1}^{m} A_r q_1^k(t-T_r) \quad (17)$$

with $A_0 = \text{diag}\{l_{ii}\} \in \mathbb{R}^{N\times N}$ and the matrices $A_r \in \mathbb{R}^{N\times N}$ having all the elements equal to zero, except one off–diagonal element equal to one. If this last equation is substituted in (16) yields

$$\dot{q}_1^k = -\lambda A_0 q_1^k + \lambda_1 \sum_{r=1}^{m} A_r q_1^k(t-\tau_k) + \epsilon_1^k \quad (18)$$

which stands for delay–differential equation that describes the dynamic behavior of the q_1^k coordinate.

With the different expressions developed above, it is possible to state the next

Proposition 2.

Consider a network composed by N flexible–joint robots of the form (4). Assume **A.2** *and*

A.3 *The robots are interconnected through a connected graph with constant delays T_{ij} between each connected pair (i,j).*

Under these conditions, the control law

$$\begin{aligned} u_i = &-K_{id2}\epsilon_{i2} + J_i(\ddot{q}_{i2d} - \Lambda_{i2}\dot{\tilde{q}}_{i2}) + \\ &K_i\left[\int_0^t(\lambda_1 e_{i1} - \Lambda_{i2}\tilde{q}_{i2})d\tau + q_{i2d}\right] - \\ &K_i(q_{i10} - q_{i20}) - b_{i2}\dot{\tilde{q}}_{i2} \\ q_{i2d} = &\, p(p+\lambda)^{-1}\left\{K_i^{-1}u_{ir} - \int_0^t(\lambda_1 e_{i1} - \Lambda_{i2}q_{i2})d\tau\right\} + \\ &p(p+\lambda)^{-1}(q_{i10} - q_{i20}) \\ u_{ir} = &-D_i\lambda\dot{e}_{i1} - C_i\lambda e_{i1} + G_i - K_{id1}\epsilon_{i1} - b_{i1}\dot{e}_{i1} \end{aligned} \quad (19)$$

with e_{i1}, ϵ_{i1} and ϵ_{i2} defined above, and $p = \frac{d}{dt}$, $K_{id} = K_{id}^T = diag(K_{id1}, K_{id2}) \geq 0$ and $B_i = B_i^T = diag\{b_{i1}, b_{i2}\} \geq 0$ achieves the consensus objective.

Proof of Proposition 2.

Due to space constraints the attention of this proof will be concentrated on proving that the control error tends to zero and only a sketch of how the consensus objective will be given, since this part of the proof exactly follows the presented in [18]. The control error dynamics for each robot, under the control scheme (19), is given by

$$\mathcal{D}_i(q_{i1})\dot{\epsilon}_i + (\mathcal{C}_i(q_{i1},\dot{q}_{i1}) + K_{di})\epsilon_i + \mathcal{K}_i\int_0^t \epsilon_i(\tau)d\tau + B_i\dot{E}_i = 0 \quad (20)$$

Now, consider the functional

$$W_i(\epsilon_i, e_i) = \tfrac{1}{2}\epsilon_i^T\mathcal{D}_i\epsilon_i + \tfrac{1}{2}\left[\int_0^t \epsilon_i^T(\tau)d\tau\right]\mathcal{K}_i\left[\int_0^t \epsilon_i(\tau)d\tau\right] + \tfrac{1}{2}\lambda E_i^T B_i E_i + \tfrac{b_{i1}}{2}\sum_{r\in\mathcal{N}_i}\int_{t-T_{ir}}^t |\dot{q}_{r1}(\sigma)|^2 d\sigma$$

whose time derivative along the trajectories of (20) satisfies

$$\dot{W}_i = -\epsilon_i^T K_{di}\epsilon_i - b_{i2}\left|\dot{\tilde{q}}_{i2}\right|^2 - \tfrac{b_{i1}}{2}\left[\sum_{r\in N_i}|\dot{q}_{i1} - \dot{q}_{r1}(t-T_{ir})|^2 + |\dot{q}_{i1}|^2 - |\dot{q}_{r1}|^2\right]$$

Defining $W = \sum_{i=1}^{N}\gamma_i W_i$ it is possible to show that

$$\dot{W} \leq -\underline{b_1}\sum_{i=1}^{N}\gamma_i\left[-\tfrac{1}{\underline{b_1}}\epsilon_i^T K_{di}\epsilon_i - \tfrac{b_{i2}}{\underline{b_1}}\left|\dot{\tilde{q}}_{i2}\right|^2\right] - \underline{b_1}\sum_{i=1}^{N}\gamma_i\left[-\tfrac{1}{2}\sum_{r\in N_i}|\dot{q}_{i1} - \dot{q}_{r1}(t-T_{ir})|^2\right] + \underline{b_1}\gamma^T\mathbf{L}F_1$$

with $\underline{b_1} = \min\{b_{i1}\}$ and $F_1 = \text{col}(|\dot{q}_{11}|^2, \ldots, |\dot{q}_{N1}|^2) \in \mathbb{R}^N$.

Since for some $\gamma = \text{col}(\gamma_1,\ldots,\gamma_N) \in \mathbb{R}^N$, with $\gamma_i > 0$, it is satisfied that $\gamma^T\mathbf{L} = 0$, then $\dot{W} \leq 0$, which implies that $\epsilon_i \in L_2 \cap L_\infty$ and $E_i \in L_\infty$. Hence e_{i1}, $\tilde{q}_{i2} \in L_\infty$ leading, from (13), to the fact that $\dot{q}_{i1} \in L_\infty$ and, using (12), (14), $\dot{e}_{i1} \in L_\infty$, $\dot{\tilde{q}}_{i2} \in L_\infty$, respectively, implying that $\dot{E}_i \in L_\infty$. Finally, considering (20) it is possible to conclude $\dot{\epsilon}_i \in L_\infty$ and therefore that $|\epsilon_i| \to 0$.

The proof is completed by showing that $\left|\dot{q}_1^k\right| \to 0$ using the Laplace transform of (18) and exploiting the fact that $|\epsilon_i| \to 0$ and proving that under these conditions the consensus objective is achieved. ∇∇∇

Remark. Although it seems feasible to state the same robustness properties of controller (19) with respect to uncertainty on the initial conditions, the authors have not been able to succeed about this goal. Current research is carried out in this sense. However, as will be shown in Section V, the consensus objective is achieved even when the controller is implemented without using the real value of the initial conditions.

V. Simulation results

The usefulness of the proposed controller was validated through numerical simulations. It was considered a network of three flexible joints robots, where each one has a single degree of freedom, i.e. two generalized coordinates. The communication channels among the agents were represented by a topology shown in Figure 1. This figure shows a network with $N = 3$ nodes. The digraph in this figure have 0-1 weights and it is only connected.

The considered robot parameters are shown in table I and the initial conditions for the robots were $q_{10} = 0.4363$ (rad), $q_{20} = 0.2618$ (rad), $q_{30} = 0.0873$ (rad) while, in order to stress the robustness properties, the considered initial conditions in the controller were set to zero. The control gains were $\lambda_1 = 1.5$, $\Lambda_2 = diag(3,3,0.5)$, $K_{d1} =$

$diag(0.25, 0.25, 0.25)$, $K_{d2} = diag(0.5, 0.5, 0.5)$ and $b_{i1} = 0.5$.

The first set of simulations presents the results for the case without delays. Figure 4 shows the non actuated state trajectories of the network, while Figure 3 presents the consensus and tracking error signals.

On the other hand Figures 4, 5 and 6 depict the achieved performance for the consensus problem with different communication time delay for the three robots network.

VI. Concluding remarks

In this paper a solution to the control of networks composed by under–actuated Lagrangian systems was proposed. It was approached the particular case of agents defined by Flexible–joint robots and it was considered the existence of unknown delays in the communications channels. The presented results are extension of a previously reported controlled for fully–actuated Lagrangian systems and the performance achieved, even under uncertainty in the initial conditions of the robots (required by the control scheme), is remarkable. Current research is carried out with the aim to enlarge the class of under–actuated systems that could be controller under the same approach.

References

[1] P. Moyland and David Hill. Stability Criteria for Large-Scale Systems. *IEEE Trans. on Automatic Control*, VOL. AC-23, No. 2, April 1978.

[2] Blekhman, I. Synchronization in Science and Technology. *ASME Press*, New York, 1988.

[3] J. A. Fax and R. M. Murray. Graph laplacians and stabilization of vehicle formations. *Proceeings of 15th IFAC Conference*, pages 283-288, 2002.

[4] A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. on Automatic Control*, 2004.

[5] R. Olfati-Saber and R. Murray. "Consensus problems in networks of agents with switching topology and time-delays". *IEEE Trans. on Automatic Control*, Sep. 2004.

[6] Luc Moreau. Stability of Multiagent SystemsWith Time-Dependent Communication Links. *IEEE Trans. on Automatic Control*, VOL. 50, NO. 2, February 2005.

[7] J. Zhao, D. Hill and T. Liu. Synchronization of complex dynamical networks with switching topology: A switched system point of view. *Automatica*, Vol. 45, No. 11, pp. 2502–2511, 2009.

[8] Xiwei Liu, Wenlian Lu and Tianping Chen. Consensus of multi-agent systems with unbounded time-varying delays. *IEEE Trans. on Automatic Control*, Vol. 55, No. 10, october 2010.

[9] A. Yu. Pogromsky. Passivity Based Design of Synchronizing Systems. *International Journal of Bifurcation and Chaos*, Vol. 8, No. 2 (1998) 295-319.

[10] Murat Arcak. "Passivity as a design tool for group coordination". *IEEE Transactions on Automatic Control*, 52(8):1380-1389, Agosto 2007.

[11] Nikhil Chopra and Mark W. Spong. "Output Synchronization of Nonlinear Systems with Time Delay in Communication". *Proceedings of the 45th IEEE Conference on Decision and Control. December 2006*

[12] L. Scardovi, M. Arcak, and E.D. Sontag. "Synchronization of interconnected systems with applications to biochemical networks: An input-output approach". *IEEE Trans. on Aut. Control*, 55(6):1367–1379, June 2010.

[13] Nikhil Chopra and Mark W. Spong. Passivity-Based Control of Multi-Agent Systems. I*nternational Symposium on Advanced Robotics and Machine Intelligence*, Beijing, China, October, 2006.

[14] Soon-Jo Chung and Jean-Jacques Slotine. "Cooperative Robot Control and Concurrent Synchronization of Lagrangian Systems". *IEEE Transactions on Robotics*, 25(3):686-700, June 2009.

[15] Wei Ren and Yongcan Cao. Distributed Coordination of Multi-agent Networks, Communications and Control Engineering. Springer 2011.

[16] A.J. Van der Schaft B.M. Maschke. Port-Hamiltonian Dynamics on Graphs: Consensus and Coordination Control Algorithms. *Proceedings of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems*, France, September 13-14, 2010.

[17] Sujit Nair and Naomi Ehrich Leonard. Stable Synchronization of Mechanical System Networks. *SIAM J. CONTROL OPTIM*, Vol. 47, No. 2, pp. 661–683, 2008.

[18] Emmanuel Nuño, Romeo Ortega, Luis Basañes and David Hill. "Synchronization of networks of nonidentical Euler–Lagrange systems with uncertain parameters and communication delays". *IEEE Trans. on Automatic Control. To appear 2011.*

[19] B. Brogliato, R. Ortega and R. Lozano. Global Tracking Controllers for Flexible-joint Manipulators: a Comparative Study. *Automatica*, Vol. 31, No 7, pp. 941-956, 1995.

[20] R. Ortega, A. Loria, P.J. Nicklasson, and H.J. Sira-Ramirez. Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications. Springer, 1st edition, 1998.

Fig. 1. Connected digraph

Robot 1	Robot 2	Robot 3
$I_1 = 0.05$	$I_2 = 0.031$	$I_3 = 0.062$
$B_1 = 0.0105$	$B_2 = 0.007$	$B_3 = 0.014$
$Mgl_1 = 1.2$	$Mgl_2 = 0.8$	$Mgl_3 = 1.6$
$D_1 = 0.046$	$D_2 = 0.031$	$D_3 = 0.062$
$J_1 = 0.006$	$J_2 = 0.004$	$J_3 = 0.008$
$K_1 = 46.5$	$K_2 = 31$	$K_3 = 62$

TABLE I
ROBOTS PARAMETERS

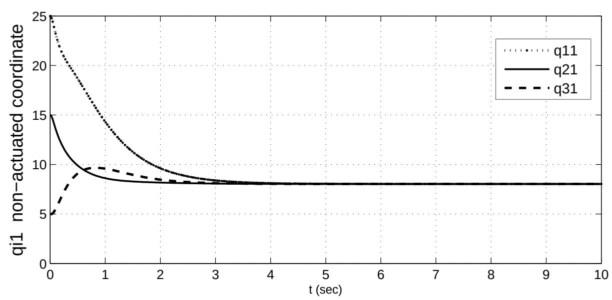

Fig. 2. Non actuated coordinates q_{i1} (in degrees) of the three robots network without time delay

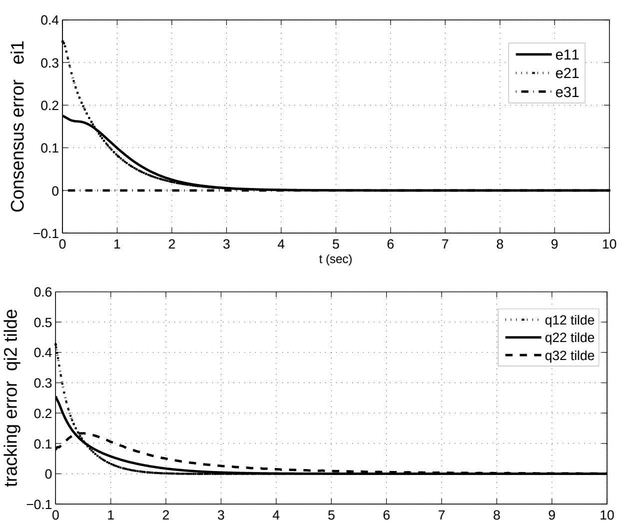

Fig. 3. Consensus and tracking error signals of the three robots network whitout time delay

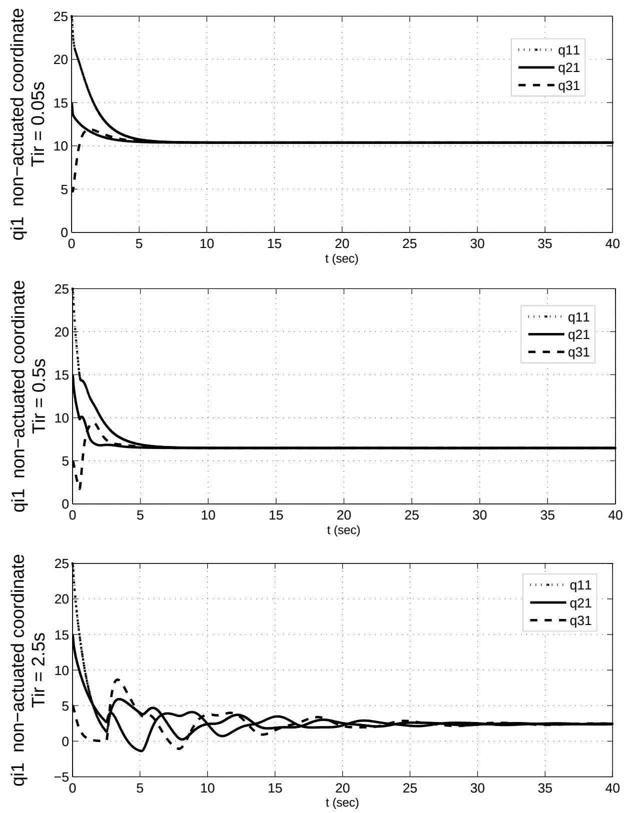

Fig. 4. Non actuated coordinates of the three robots network (q_{i1} in degrees) with time delays: a) $\tau_{ir} = 0.05$s, b) $\tau_{ir} = 0.5$s, c) $\tau_{ir} = 2.5$s

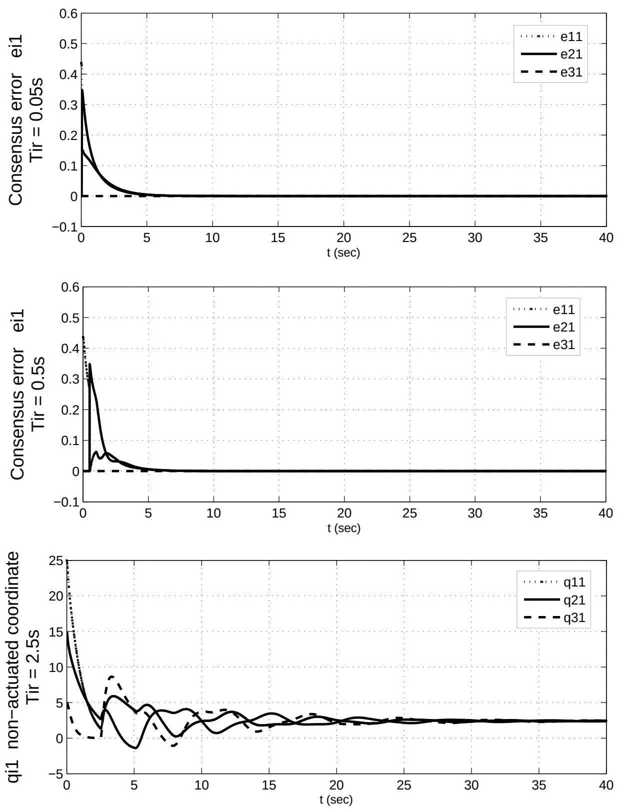

Fig. 5. Consensus error signals of the three robots network (e_{i1}) with time delays: a) $\tau_{ir} = 0.05$s, b) $\tau_{ir} = 0.5$s, c) $\tau_{ir} = 2.5$s

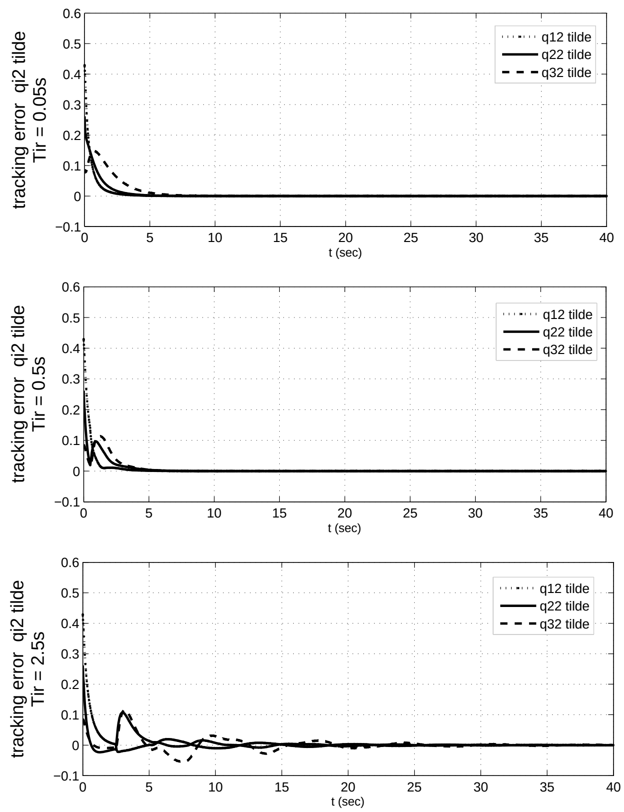

Fig. 6. Tracking error signals of the three robots network (e_{i1}) with time delays: a) $\tau_{ir} = 0.05$s, b) $\tau_{ir} = 0.5$s, c) $\tau_{ir} = 2.5$s