Stabilization of the ball on the beam system by means of the Inverse Lyapunov Approach

Abstract— **In this work we used the novel Inverse Lyapunov Approach in conjunction with the energy shaping technique is applied to derive a stabilizing controller for the ball on the beam system. The strategy consists of shaping a candidate Lyapunov function as if it was an inverse stability problem. To this end, we fix a suitable dissipation function of the unknown energy function, with the property that the selected dissipation divides the corresponding time derivative of the candidate Lyapunov function. Then the stabilizing controller is directly obtained from the already shaped Lyapunov function. The stability analysis of the closed-loop system is carried out by using the invariance theorem of LaSalle. To test the effectiveness of the obtained controller numerical simulations are presented.**

I. INTRODUCTION

Despite that the ball and the beam system (**BBS**) (see Figure 1) is a popular and important nonlinear system due to its simplicity and easiness to understand it and implement it in the laboratory, it is unstable. Due to this fact it has been widely used as a test bed for the effectiveness of control design techniques offered by modern control theory [15], [5], and to avoid the danger that usually accompanies real unstable systems when brought to the laboratory. In fact, the dynamics of this system are very similar to those found in aerospace systems.

Because the **BBS** does not have a well defined relative degree at the origin, the exact input-output linearization approach cannot be directly applied to stabilize it around to the origin. That is, this system is not feedback linearizable by means of static or dynamic state feedback. This obstacle makes it difficult to design either a stabilizable or a tracking controller [5], [14]. Fortunately, the system is locally controllable around to the origin. Hence, it is possible to control it, if it is initialized close enough to the origin by using the direct pole placement method.

As controlling the **BBS** is a challenging and important problem, several works devoted to its solution can be found in the literature. A control strategy based on an approximate feedback linearization was proposed by Hauser *et al.* in [5]. The main idea consists of discarding certain terms to avoid singularities. The drawback of this strategy is that the closed-loop system behaves properly in a small region, but it fails in a large one. In the same spirit, combined with suitable intelligent switches, we mention the works of [2], [23]. In the first work the authors present a control scheme that switches between exact and approximate input-output linearization control laws; in the other work the use of exact input-output linearization in combination with fuzzy dynamic control is proposed. A constructive approach based on the Lyapunov theory was developed in [20], where a numerical approximation for solving one **PDE** was considered. In the similar works of [9], [10], [6], [7], energy matching conditions were used for the stabilization of the **BBS**. They also used some numerical approximations in order to solve approximately two matching conditions required to derive the candidate Lyapunov function. A major contribution, rather similar to the matching-energy-based approach, was considered in [21], [22]. In these works, the authors solved the two matching conditions related with the potential and kinetic energies of the closed-loop system. In [18], a nested saturation design was proposed in order to bring the ball and the beam to the unstable equilibrium position. Following the same idea, a global asymptotic stabilization was developed with state-dependent saturation levels [19]. A novel work based on a modified nonlinear **PD** control strategy, tested in the laboratory, was presented in [24]. Finally, many control strategies for the stabilization of the **BBS** can be found in the literature, but most of them manage the physical model by introducing some nonlinear approximations or switching through singularities (see [15] and [14]).

Here we introduce a novel inverse Lyapunov based procedure combined with with the energy shaping method to stabilize the **BBS**. Broadly speaking the strategy consists of finding the Lyapunov function as if it were an inverse stability problem. That is, we first choose the dissipation rate function of the time derivative of the unknown candidate Lyapunov function. For that purpose, we shape a suitable candidate Lyapunov function, which is locally strictly positive definite inside of an admissible set of attraction. Having our Lyapunov function, the control is proposed in such a way that the time derivative of the obtained Lyapunov function is forced to be equal to the proposed dissipation rate function. The proposed Lyapunov function is formed by adding a kinetic energy function and a particular function, which can be considered as the corresponding potential energy function. To carry this out, we found two restriction equations related to the potential and kinetic energies. The main characteristic of our control strategy is that we do not need to force the closed-loop system to follow another stable Euler-Lagrange or Hamiltonian system, contrary to what was previously proposed in [9], [10], [6], [7], [21], [1], [8].

The rest of this work is organized as follows. In Section

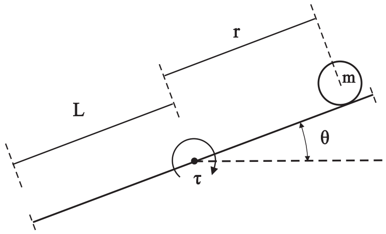

Figura 1. The ball and beam system.

2 we present the control model of the **BBS.** In Section 3 we briefly introduce the inverse Lyapunov method for solving the stabilization of the **BBS;** we also discusses the asymptotically convergence of the closed-loop system. In Section 4 we present some numerical simulations to assess the effectiveness of our control strategy. In Section 5 some conclusions are given.

II. SYSTEM DYNAMICS

Consider the **BBS** shown in Figure 1. Its non-linear model is described by the following set of differential equations (see [5], [15]):

$$\begin{array}{c}\left(m+\frac{J_B}{R^3}\right)\ddot{r}-mr\dot{\theta}^2+mg\sin\theta+\beta\dot{r}=0\\ \left(mr^2+J_B+J\right)\ddot{\theta}+2mr\dot{r}\dot{\theta}+mgr\cos\theta=\tau\end{array} \tag{1}$$

where r is the ball position along the beam, θ is the beam angle, J is the moment of inertia of the beam around the rotating pivot, J_B is the moment of inertia of the ball with respect to its center, R is the radius ball, m is the ball mass, $\beta>0$ is the friction coefficient and τ is the torque of the system. After applying the following feedback :

$$\tau=u\left(mr^2+J_B+J\right)+2mr\dot{r}\dot{\theta}+mgr\cos\theta \tag{2}$$

into system (1), it can be re-written as:

$$\begin{array}{l}\ddot{r}=dr\dot{\theta}^2-\overline{n}\sin\theta-b\dot{r}\\ \ddot{\theta}=u,\end{array} \tag{3}$$

where

$$b=\frac{\beta}{m+\frac{J_B}{R^3}}\quad d=\frac{m}{m+\frac{J_B}{R^3}}\quad \overline{n}=\frac{mg}{m+\frac{J_B}{R^3}} \tag{4}$$

Note that u can be seen as a virtual controller that acts directly on the actuated coordinate θ. Naturally, the latter system equations can be written, as:

$$\ddot{q}=S(x)+Fu; \tag{5}$$

where $q^T=(r,\theta)$ and $x^T=(q,\dot{q})$.[1]

[1]Evidently,

$$S(x)=\left[\begin{array}{cc}\delta r\dot{\theta}^2-n\sin\theta-b\dot{r} & 0\end{array}\right]^T\quad F=\left[\begin{array}{cc}0 & 1\end{array}\right]^T$$

.

III. CONTROL STRATEGY

The control objective is to bring all the states of the system (3) to the unstable equilibrium point $x=0$; restricting, both, the beam angle and the ball position being inside of the admissible set $Q\in R^2$, defined by

$$Q=\{q=(r,\theta):|r|\leq L\wedge|\theta|\leq\overline{\theta}<\pi/2\}, \tag{6}$$

where the positive constants L and $\overline{\theta}$ are given a priory. To this end, a suitable candidate Lyapunov function is constructed by using the *Inverse Lyapunov Approach*. In other words, the candidate Lyapunov function is obtained as an inverse stability problem. The framework of this work if described below.

A. Inverse Lyapunov approach

First of all we need to propose a candidate Lyapunov function[2] for the closed-loop system energy function, of the form:

$$V(x)=\frac{1}{2}\dot{q}^TK_c(r)\dot{q}+V_p(q), \tag{7}$$

where the closed-loop inertia matrix $K_c(r)=K_c^T(r)>0$, and the closed-loop potential energy functions $V_p(q)>0$, will be defined in the forthcoming developments. A straight-forward calculation shows that, along of the solutions of (5), $\dot{V}$ is given by

$$\dot{V}(x)=\left(\nabla_qV\right)^T\dot{q}+(\nabla_{\dot{q}}V^T)(S(x)+Fu(x)); \tag{8}$$

Comment 1: *In fact, $V_p(q)$ is selected such that $\nabla_xV_p(x)|_{x=0}$ and $\nabla_x^2V_p(x)\big|_{x=0}>0$. That is, we require that V_p be strictly locally convex around to the origin.*

Having described the form of the candidate Lyapunov function and fixed the following auxiliary variable as $\eta(x)=\dot{\theta}+\alpha(r)\dot{r}$, with $\alpha(r)\neq0,\forall r\in Q$ (that is, it is given in advance), we desired to find $u(x)\in R$, $V_p(q)\in R^+$ and $K_c(r)>0$; with $x\in D\subset Q\times R^2$;[3] such that, $\dot{V}$ can be rewritten, as

$$\dot{V}(x)=\eta(x)(\beta(x)+u(x))+R(x); \tag{9}$$

where $\beta(x)$ and $R(x)$, are continuous functions. Then, we propose the control law as

$$u(x)=-k_d\eta(x)-\beta(x), \tag{10}$$

for some $k_d>0$, which evidently leads to

$$\dot{V}(x)=-k_d\eta^2(x)+R(x). \tag{11}$$

In order to guarantees that $\dot{V}$ be semi-definite negative, we require that exists a k_d, such that

$$-k_d\eta^2(x)+R(x)\leq0. \tag{12}$$

[2]The form of this candidate Lyapunov function corresponds to an stable Euler-Lagrange system.

[3]The set D is related with the region of attraction of the closed-loop system.

Physically, we are choosing a convenient dissipation function, $\eta(x)$, of the unknown closed-loop energy function $V(x)$; with the property that η divides $(\dot{V} - R)(x)$. We must underscore that the fixed η relies on the unactuated coordinate r, in agreement with the structure of the closed-loop energy function.

Comment 2: *$R(x)$ corresponds to the work developed by the system internal friction forces, which cannot be compensated because they act over the unactuated coordinate. Evidently, when $R(x) = 0$, inequity (12) always holds. Conversely, if $R(x) \neq 0$, then we can only increase the gain k_d in order fulfils (12).*

Closed-loop system stability: Notice that if we are able to shape the candidate Lyapunov function (7), such that its corresponding time derivative, along the trajectories of system (3), can be expressed as (9), under the assumption that (12) holds. Then V qualifies as a Lyapunov function, because is a non-increasing and positive definite function in the neighborhood of the origin and proper on its sub-level (that is, there exists a $c > 0$, such that, $V(x) \leq c$ defines a compact set, with closed level curves). Consequently, x is stable in the Lyapunov sense.

*B. Solving the **BBS** stabilization problem by applying the Inverse Lyapunov approach*

In this section we explain how to take the original expression of $\dot{V}$, defined in (8), to the desired form (9) for the particular case of the **BBS**.

Defining K_c, as [4]

$$K_c = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}, \tag{13}$$

and according with (8), we have that $\dot{V}$ can be expressed, as

$$\dot{V}(x) = \dot{q}^T \left(\upsilon_p(q) + \upsilon_d(x) + K_c F u\right) + R_\upsilon(x), \tag{14}$$

where

$$\upsilon_d(x) = \{\upsilon_{d_i}\}_{i=1}^2 = \frac{1}{2}\nabla_q\left(\dot{q}^T K_c \dot{q}\right) + K_c \begin{bmatrix} dr\dot{\theta}^2 \\ 0 \end{bmatrix}, \tag{15}$$

$$\upsilon_p(q) = \{\upsilon_{p_i}\}_{i=1}^2 = \nabla_q V_p(q) + K_c \begin{bmatrix} -\overline{n}\sin\theta \\ 0 \end{bmatrix}. \tag{16}$$

Equating equation (14) with (8) we obtain, after some simple algebraic manipulations, the following

$$\begin{aligned} &\overbrace{R_\upsilon(x) - R(x)}^{T_0} + \overbrace{\left(\dot{q}^T K_c F - \eta(x)\right)}^{T_1} u \\ &+ \overbrace{\dot{q}^T \left(\upsilon_p(q) + \upsilon_d(x)\right) - \eta(x)\beta(x)}^{T_2} = 0; \end{aligned} \tag{17}$$

where η is given a priory.

Now, to solve the equation avobe, we first take $T_0 = 0$, which can be justified by defining $R(x) \widehat{=} R_\upsilon(x)$. Next, $T_1 = 0$ can be hold by making

$$\dot{q}^T K_c F = \eta(x). \tag{18}$$

Evidently, the parameters k_2 and k_3 are directly obtained from the expression above. Then, $T_2 = 0$ implies that

$$\dot{q}^T \left(\upsilon_p(q) + \upsilon_d(x)\right) = \eta(x)\beta(x). \tag{19}$$

Note that the equation above has two unknown parameters, given by k_1 and V_p. Hence, in order to solve it we can require that

$$\dot{q}^T \upsilon_p(q) = \zeta_p(q)\eta(x); \quad \dot{q}^T \upsilon_d(x) = \zeta_d(x)\eta(x). \tag{20}$$

Where the continuous functions ζ_p and ζ_d will be computed later (by using simple polynomials factorization). Consequently $\beta(x)$ is directly computed by:

$$\beta(x) = \zeta_p(q) + \zeta_d(x). \tag{21}$$

Finally, ζ_p and ζ_d are obtained accordingly to the following remark.

Remark 1: *Notice that $\dot{q}^T \upsilon_p(q)$ and $\dot{q}^T \upsilon_d(q)$ are polynomials with respect to variables $(\dot{r}, \dot{\theta})$. Consequently, the following equalities*

$$\dot{q}^T \upsilon_p(q)\Big|_{\dot{\theta} = -\alpha(r)\dot{r}} = 0; \tag{22}$$

$$\dot{q}^T \upsilon_d(x)\Big|_{\dot{\theta} = -\alpha(r)\dot{r}} = 0, \tag{23}$$

imply that functions $\zeta_p(q)$ and $\zeta_d(x)$ satisfy the restrictions in (20). In other words, the selected η must divide the two scalar functions $\dot{q}^T \upsilon_p(q)$ and $\dot{q}^T \upsilon_d(q)$.

1) Computing the needed candidate Lyapunov function:

In this section we obtain the unknown control variables K_c and V_p. We begin by solving the restriction equation (18). For simplicity, we set $\alpha(r) = 1$ and $\eta = -\dot{r} + \dot{\theta}$. Therefore, from (18) and (13) we evidently have that

$$\dot{q}^T K_c F = k_2 \dot{r} + k_3 \dot{\theta} = -\dot{r} + \dot{\theta}, \tag{24}$$

which leads to $k_2 = -1$ and $k_3 = 1$. Now, substituting the fixed values k_2 and k_3 into the equation (15), we have that

$$\begin{bmatrix} \upsilon_{d_1} \\ \upsilon_{d_2} \end{bmatrix} = \frac{1}{2}\nabla_q\left(\dot{q}^T K_c \dot{q}\right) + K_c \begin{bmatrix} dr\dot{\theta}^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{k_1'}{2}\dot{r}^2 + dk_1 r\dot{\theta}^2 \\ -dr\dot{\theta}^2 \end{bmatrix}. \tag{25}$$

Next, substituting the above υ_{d_1} and υ_{d_2} in (22), we obtain

$$\upsilon_{d_2}(x) + \upsilon_{d_1}(x)|_{\dot{\theta} = \dot{r}} = \dot{r}^2\left(\frac{k_1'}{2} + dr(k_1 - 1)\right) = 0,$$

which produces the following equation: $k_1' = -2rd(k_1 - 1)$ and whose solution is given by $k_1 = 1 + \overline{k}_1 e^{-dr^2}$; where $\overline{k}_1 > 0$. Hence, matrix K_c can be taken as

$$K_c = \begin{bmatrix} 1 + \overline{k}_1 e^{-dr^2} & -1 \\ -1 & 1 \end{bmatrix}. \tag{26}$$

[4] For simplicity, we use $K_c = K(r)$, $k_i = k_i(r)$, $k_i' = \frac{d}{dr}k_i(r)$, for $i = \{1, 2, 3\}$.

According with (26), we have that det$(K_c) = \overline{k}_1 e^{-dr^2} > 0$. That is, $K_c > 0$, when r is finite.

Now, to obtain $V_p(q)$ we proceed to substitute the obtained K_c into the relation (16), having

$$\begin{bmatrix} \upsilon_{p_1} \\ \upsilon_{p_2} \end{bmatrix} = \nabla_q V_p + K_c \begin{bmatrix} -\overline{n}\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -(1+\overline{k}_1 e^{-dr^2})\overline{n}\sin\theta + \frac{\partial V_p}{\partial r} \\ \overline{n}\sin\theta + \frac{\partial V_p}{\partial \theta} \end{bmatrix}. \tag{27}$$

From (23), we have that

$$0 = \upsilon_{p_2}(q) + \upsilon_{p_1}(q) = -\overline{n}\overline{k}_1 e^{-dr^2}\sin\theta + \frac{\partial V_p}{\partial r} + \frac{\partial V_p}{\partial \theta},$$

whose solution is given by

$$V_p(q) = \overline{n}\overline{k}_1 \int_0^r \sin(\theta - r + s)\, e^{-ds^2} ds + \Omega(r-\theta),$$

where $\Omega(*)$ must be selected such that V_p has a local minimum at the origin $q = 0$. To assure this condition it is enough to define $\Omega(s) = k_p s^2/2$, where $k_p > \overline{n}\overline{k}_1$. Therefore, V_p reads as

$$V_p(q) = \overline{n}\overline{k}_1 \int_0^r \sin(\theta - r + s)\, e^{-ds^2} ds + \frac{k_p}{2}(r-\theta)^2, \tag{28}$$

where the integral term can be exactly computed as

$$\begin{array}{c} \int_0^r \sin(\theta - r + s)\, e^{-ds^2} ds = \\ \cos(\theta - r)\, I_{\sin r} + \sin(\theta - r)\, I_{\cos r}, \end{array} \tag{29}$$

where

$$\begin{array}{l} I_{\cos r} = \int_0^r \cos(s)\, e^{-ds^2} ds = \alpha_1 \phi_s(r); \\ I_{\sin r} = \int_0^r \sin(s)\, e^{-ds^2} ds = \alpha_1(\alpha_0 + \phi_c(r)), \end{array} \tag{30}$$

and[5]

$$\begin{array}{ll} \alpha_0 = 2Im\left[erf\left(\frac{i}{2\sqrt{d}}\right)\right]; & \alpha_1 = \frac{\sqrt{\pi}\exp\left(\frac{-1}{4d}\right)}{4\sqrt{d}}; \\ \phi_s(r) = 2Re\left[erf\left(\frac{i+2dr}{2\sqrt{d}}\right)\right]; & \phi_c(r) = -2Im\left[erf\left(\frac{i+2dr}{2\sqrt{d}}\right)\right]. \end{array} \tag{31}$$

Remark 2: *Notice that the relation (12) can be rewritten as* $R_d(x) = \dot{q}^T H \dot{q}$*; where*

$$H = \begin{bmatrix} -k_d & k_d + bk_1/2 \\ k_d + bk_1/2 & -bk_1 - k_d \end{bmatrix} \dot{q}. \tag{32}$$

So that $\dot{q}^T H \dot{q} < 0$*; if the parameter* k_d *is selected such that* -$b + 4(k_1 - 1)k_d > 0$*; recall that* $k_1 > 1$

$$\zeta_p = n\sin\theta + \tfrac{\partial V_p}{\partial \theta}; \quad \zeta_d = -dr\dot{\theta}^2 + d\overline{k}_1 r\dot{r}(\dot{r}+\dot{\theta})e^{-dr^2}.$$

. Hence, the needed controller (10) is given by:[6]

$$\begin{array}{c} u = -k_d(-\dot{r}+\dot{\theta}) - \left(\overline{n}\sin\theta + \frac{\partial V_p}{\partial\theta}\right) \\ -\left(-dr\dot{\theta}^2 + d\overline{k}_1 r\dot{r}(\dot{r}+\dot{\theta})e^{-dr^2}\right), \end{array}$$

where $\overline{k}_1 > 0$ and $k_d > 0$.

We end this section introducing the following important remark.

Remark 3: Notice that we can always compute

$$\widetilde{c} = \max_{c>0} q \in Q : V_p(q) = c\}; \text{ such that } V_p(q) = \widetilde{c} \tag{33}$$

is a closed curve.

To illustrated the geometrical estimation of the bound "$\widetilde{c}$*", we fixed the parameter values* $k_p = 2.5$ *and* $k_p = 1.3$*; the others physical parameter values were set as* $\overline{n}\overline{k}_1 = 1$ *and* $d = 0.015$*; and the admissible restricted set was choused as* $|r| \leq 0.6$*[m] and* $|\theta| \leq 0.5$*[rad]. This set-up allows us to give an estimated c, which evidently guaranties that the level curves of* V_p *are close. By a thoroughly numerical inspection, we found that* $\widetilde{c} \cong 0.108$ *for* $k_p = 2.5$ *and* $\widetilde{c} = 0.05$ *for* $k_p = 1.3$*. Figure 2 shows the corresponding level curves. As we can see, constant,* k_p*, modifies the rotation angle of the level curve.*

C. Asymptotic convergence of the closed-loop system

Since the obtained V *is a non-increasing and positive definite function, in some neighborhood that contains the origin, then the closed-loop system is, at least, locally stable in the Lyapunov sense. To assure that the trajectories of the closed-loop system asymptotically converge to the origin, restricted to* $q(t) \in Q$*, for* $t > 0$*, we must define the set* $\Omega_{\widetilde{c}} \in R^4$*, where*

$$\Omega_{\widetilde{c}} = \left\{(q,\dot{q}) : q \in Q \wedge V(q,\dot{q}) < \widetilde{c}\right\}. \tag{34}$$

The set $\Omega_{\widetilde{c}}$ *defines a compact invariant set because, for any initial conditions* $x_0 = (q_0, \dot{q}_0)$*; with* $q_0 \in Q$*, provided that* $V(x_0) < \widetilde{c}$*, then* $V(x) < \widetilde{c}$*; with* $q \in Q$*.*

The rest of the stability proof is based on the LaSalle's invariant theorem ([16] and [11]). To apply this theorem we need to define a compact (closed and bounded) set $\Omega_{\widetilde{c}}$*, which must satisfies that every solution of the system (3), in closed-loop with (III-B.1), starting in* $\Omega_{\widetilde{c}}$ *remains in* $\Omega_{\widetilde{c}}$*, for all future time. Then, we define the following invariant set* S*, as:*

$$\begin{array}{c} S = \left\{x \in \Omega_{\widetilde{c}} : \dot{V}(x) = 0\right\} \\ = \left\{(q,\dot{q}) \in \Omega_{\widetilde{c}} : R_d(x) = \dot{q}^T H \dot{q} = 0\right\}, \end{array}$$

where $H < 0$*. Let* M *be the largest invariant set in* S*. Because the Theorem of LaSalle claims that every solution starting in a compact set* $\Omega_{\widetilde{c}}$ *approaches to* M*, as* $t \to \infty$

[5]Symbol erf stands for the Gauss error function, defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds.$$

[6]After some simple algebraic manipulations it is easy to show that

$$\zeta_p = n\sin\theta + \tfrac{\partial V_p}{\partial \theta}; \quad \zeta_d = -dr\dot{\theta}^2 + d\overline{k}_1 r\dot{r}(\dot{r}+\dot{\theta})e^{-dr^2}.$$

, we compute the largest invariant set $M \subset S$. Clearly, we have that $\dot{r} = 0$ and $\dot{\theta} = 0$, on the set S. Therefore, we must have that $\ddot{r} = 0$ and $\ddot{\theta} = 0$, on the set S. Similarly, we must have that $r = r_*$ and $\theta = \theta_*$, with r_* and θ_* being constants. Hence, on the set S, the first equation of (3) is written as 0$= -n \sin\theta_*$, then $\theta_* = k\pi$; where k is an integer. However, $\theta_* \in (-\pi/2, \pi/2)$ because $(q, \dot{q}) \in \Omega_{\tilde{c}}$, consequently $\theta = 0$, on the set S. In a similar way, we can show that $r_* = 0$. Then, on the set S, we have that $q = 0$ and $\dot{q} = 0$. Therefore, the largest invariant set M contained inside of set S is given by the single point $x = (q = 0, \dot{q} = 0)$. Thus, according to the Theorem of LaSalle [16] all the trajectories starting in $\Omega_{\tilde{c}}$ asymptotically converge towards to the largest invariant set $M \subset S$, which is the equilibrium point $x = 0$.

We finish this section by presenting the main proposition of this paper:

***Proposition 1**:* Consider the system (3) in closed-loop with (III-B.1), under conditions of the Remarks 2 and 3. Then the origin of the closed-loop system is locally asymptotically stable with the domain of attraction defined by (34).■

IV. NUMERICAL SIMULATIONS

To show the effectiveness of the proposed nonlinear control strategy we have carried out some numerical simulations by means of the Matlab program. The original system parameters, with their respectively physical restrictions, were set as:

$$
\begin{array}{ccc}
m = 0.1kg; & R = 0.015m & J_b = 2.25 \times 10^{-5} kg.m^2 \\
\overline{\theta} = 0.5rad & M = 0.2kg & L = 0.6m \\
J = 0.36kg.m^2 & \beta = 0.2 New.m/s &
\end{array}
$$

from the above, we have that $b = 0.029$, $d = 0.01477$ and $n = 0.1448$. The physical control parameters were fixed, as $k_p = 2.5$, $\overline{k}_1 = 1/\overline{n}$ and $k_d = 0.5$, while the initial conditions were fixed; as $x_0 = (0.55m; 0; 0.45rad; 0)$. Notice that the proposed set of parameters $\{d, k_p, \overline{k}_1, \overline{n}\}$ are in agreement with the computation of the restricted stability domain, which has been done in the previous section (see Figure 2-a); besides the initial conditions satisfies the inequality $V(x_0) < \tilde{c} = 1.08$[7].

*Figure 3 shows the corresponding response of the system (3) in closed-loop with (III-B.1), under the conditions in the **Remark 3**. From this figure can see that, both, the system position coordinates and the system torque asymptotically converge to the origin, assuring that $|\theta| \leq \overline{\theta}$ and $|r| \leq L$.*

*In order to provide an intuitive idea of how good is our nonlinear control strategy (**ACL**) in comparison with the control techniques proposed by Yu & Li in [24] and Hauser* et. al. *in [5], here respectively referred as (**YCL**) and (**HCL**), we carried out a second experiment using the same set-up as before and assuming that $\beta = 0$. The obtained characteristic polynomial of our control strategy of the linearized system is given by:*

$$p(s) = 0.0634 + 0.255s + 0.88s^2 + 1.72s^3 + s^4. \quad (35)$$

*The control parameters of the **YNC** and the **HNC** were selected, such that, their corresponding characteristic polynomials coincide with the polynomial (35). The initial conditions were fixed as $(0.3m; 0.18m/s; 0.35rad; 0.18rad/s)$. The simulation results are shown in Figure 4. As we can see, our control strategy outperforms the closed-loop responses of the **YNC** and **HNC** control strategies.*

***Comment 3**: A comparative study between our control strategy and other control strategies presented in the literature for solving the stabilization of the **BBS,** is beyond the scope of this work.*

V. CONCLUSIONS

*A novel procedure to stabilize the **BBS** by using the Inverse Lyapunov approach in conjunction with the Energy Shaping technique was proposed. It consists of finding the candidate Lyapunov function as if it were an inverse stability problem. To carried it out, we chosen a convenient dissipation function of the unknown closed-loop energy function. Then, we proceed to obtain the needed energy function, which is the addition of the positive potential energy and the positive kinetic energy. Afterwards, we directly derived the stabilizing controller from the already obtained time derivative of the Lyapunov function. The corresponding asymptotic convergence analysis was done by applying the Theorem of LaSalle. To assess the performance and effectiveness of the proposed control strategy, we carried out some numerical simulations. From these simulations we conclude that our strategy behaves quite well in comparison with other well known control strategies. It is worth to pointing out that as far as we know, the procedure used here to obtain the needed Lyapunov function has not been used before to control the **BBS**.*

[7]For this particular case, the condition of Remark 2 is satisfied, because $-b + 4(k_1 - 1)k_d = b + 2\exp(-d * L^2) = 13.7 > 0$.

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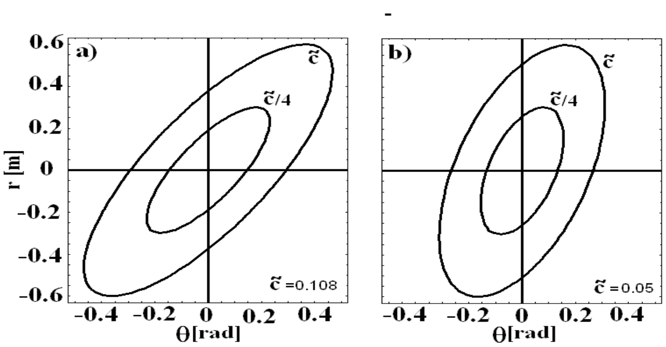

Figura 2. Level curves of the function $\Phi(q)$ around the origin, for two sets of values; $\overline{nk} = 1$, $\delta = 0.015$ and $k_p = 2.5$ (left) and $\overline{nk} = 1$, $\delta = 0.015$ and $k_p = 1.3$ (right). With restrictions on $|r| \leq 0.6$[m] and $|\theta| \leq 0.5$[rad].

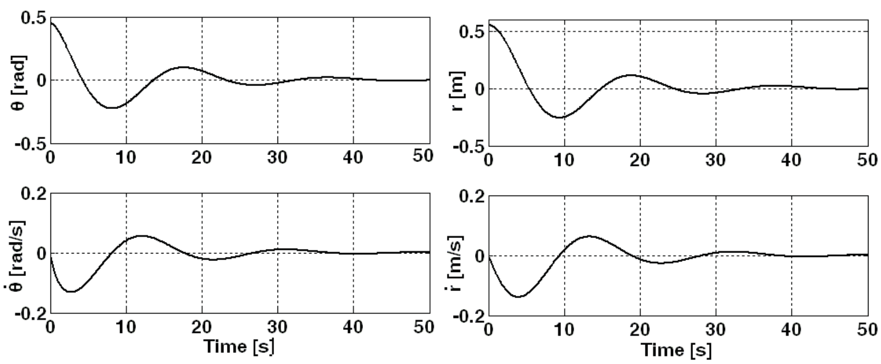

Figura 3. Closed-loop response of the **BBS** to the initial conditions: **a)** $x_0 = (0.55\text{m}; 0; 0.45\text{rad}; 0)$

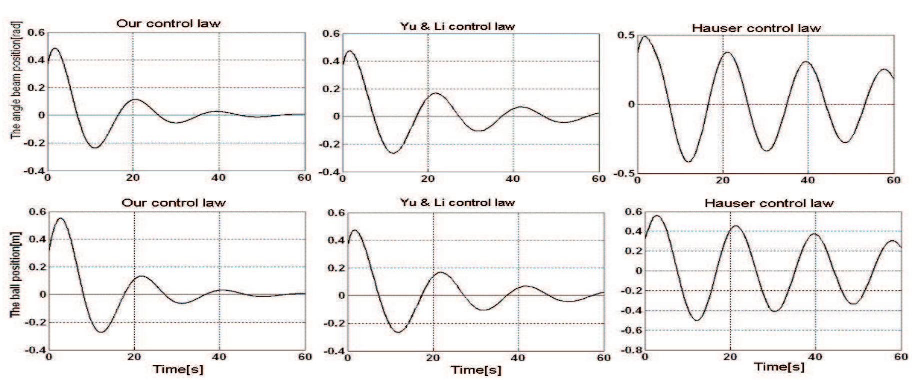

Figura 4. Closed-loop response of the **BBS** to the proposed **ANC** in comparison with **YNC** and **HNC.**