Trajectory Tracking Inverse Optimal Control for Discrete-Time Nonlinear Systems with Block Controllable Form

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***Abstract*— This paper presents an inverse optimal control approach for trajectory tracking of discrete-time nonlinear systems, avoiding to solve the associated Hamilton-Jacobi-Bellman (HJB) equation, and minimizing a meaningful cost function. This stabilizing on the reference optimal controller is based on a discrete-time control Lyapunov function. The applicability of the proposed approach is illustrated via simulations by trajectory tracking of an unstable system with the block controllable form.**
Index Terms: Trajectory Tracking, Inverse optimal control, control Lyapunov function, Block controllable form systems.

I. INTRODUCTION

In optimal nonlinear control, we deal with the problem of finding a stabilizing control law for a given system such that a criterion, which is a function of the state variables and the control inputs, is minimized; the major drawback is the requirement to solve the associated HJB equation (Sepulchre *et al.*, 1997; Krstić and Deng, 1998). Actually, the HJB equation has so far rarely proved useful except for linear regulator problem, to which it seems particularly well suited (Anderson and Moore, 1990).

In this paper, the inverse optimal control approach proposed initially by Kalman (Kalman, 1964) for linear systems and quadratic cost functions, it is treated for the discrete-time nonlinear systems case. The aim of the inverse optimal control is to avoid the solution of the HJB equation (Freeman and Kokotović, 1995). In the inverse approach, a stabilizing feedback control law, based on a priory knowledge of a control Lyapunov function (CLF), is designed first, and then it is established that this control law optimizes a meaningful cost functional. Finally, the proposed inverse optimal controller is applied to nonlinear systems with the block controllable form, which after an error transformation, the trajectory tracking problem is solved as a stabilization problem for the referred transformed system.

The main characteristic of the inverse problem is that the meaningful cost function is a posteriori determined for the stabilizing feedback control law. For continuous-time inverse optimal control applicability, we refer to the results presented in (Krstić and Deng, 1998; Anderson and Moore, 1990; Freeman and Kokotović, 1995; Moylan and Anderson, 1973; Willems and Voorde, 1977; Magni and Sepulchre, 1997; Freeman and Kokotović, 1996*b*). To the best of our knowledge, there are few results on discrete-time nonlinear inverse optimal control (Ahmed-Ali *et al.*, 1999). In (Ornelas *et al.*, 2010), an inverse optimal control scheme is proposed based on passivity approach, where a storage function is used as Lyapunov function and the output feedback is used as stabilizing control law. In this paper, we directly propose a CLF to establish the stabilizing control law and to minimize a cost functional.

Although stability margins do not guarantee robustness, they do characterize basic robustness properties that well designed feedback systems must posses. Optimality is thus a discriminating measure by which to select from among the entire set of stabilizing control laws those with desirable properties (Freeman and Kokotović, 1996*a*).

Systematic techniques for finding CLFs do not exist for general nonlinear systems; however, this approach has been applied successfully to classes of systems for which CLFs can be found such as: feedback linearizable, strict feedback and feed-forward systems, etc. (Primbs *et al.*, 1999; Freeman and Primbs, 1996). Moreover, by using a CLF, it is not required the system to be stable for zero input ($u_k = 0$). The applicability of the proposed approach is illustrated via simulations by trajectory tracking of an unstable system with the block controllable form.

II. MATHEMATICAL PRELIMINARIES

A. OPTIMAL CONTROL

Although the main goal of the paper is to design of an inverse optimal discrete-time control, this section is devoted to briefly discuss the optimal control methodology and their limitations. Consider the nonlinear discrete-time affine system

$$x_{k+1} = f(x_k) + g(x_k)\, u_k, \qquad x_0 = x(0) \tag{1}$$

with the associated meaningful cost functional

$$V(x_k) = \sum_{n=k}^{\infty} l(x_n) + u_n^T\, R(x_n)\, u_n \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time $k \in \mathcal{N}$. $\mathcal{N}$ denotes the set of nonnegative integers. $u \in \mathbb{R}^m$ is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$

are smooth mappings; $f(0) = 0$ and $g(x_k) \neq 0$ for all $x_k \neq 0$; $V : \mathbb{R}^n \to \mathbb{R}_+$; $l : \mathbb{R}^n \to \mathbb{R}_+$ is a positive semidefinite[1] function and $R : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite[2] weighting matrix. Meaningful cost functional (2) is a performance measure (Kirk, 1970). The entries of R can be functions of the system state in order to vary the weighting on control effort according to the state value (Kirk, 1970). Considering the state feedback control design problem, we assume that the full state x_k is available.

Equation (2) can be rewritten as

$$\begin{aligned} V(x_k) &= l(x_k) + u_k^T R(x_k)\, u_k \\ &\quad + \sum_{n=k+1}^{\infty} l(x_n) + u_n^T R(x_n)\, u_n \\ &= l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) \end{aligned} \tag{3}$$

where we require the boundary condition $V(0) = 0$ so that $V(x_k)$ becomes a Lyapunov function.

From Bellman's optimality principle ((Lewis and Syrmos, 1995; Basar and Olsder, 1995)), it is known that, for the infinite horizon optimization case, the value function $V(x_k)$ becomes time invariant and satisfies the discrete-time Hamilton-Jacobi-Bellman (DT HJB) equation (Basar and Olsder, 1995; Al-Tamimi and Lewis, 2008; Ohsawa *et al.*, 2009)

$$V(x_k) = \min_{u_k} \left\{ l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) \right\} \tag{4}$$

where $V(x_{k+1})$ depends on both x_k and u_k by means of x_{k+1} in (1). Note that the DT HJB equation is solved backward in time (Al-Tamimi and Lewis, 2008).

In order to establish the conditions which optimal control must satisfy, we define the discrete-time (DT) Hamiltonian $\mathcal{H}$ ((Haddad *et al.*, 1998), pages 830–832) as

$$\mathcal{H}(x_k, u_k) = l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) - V(x_k). \tag{5}$$

A necessary condition the optimal control law should satisfy is $\frac{\partial \mathcal{H}}{\partial u_k} = 0$ (Kirk, 1970), which is equivalent to calculate the gradient of (4) right-hand side with respect to u_k, then

$$\begin{aligned} 0 &= 2R(x_k)\, u_k + \frac{\partial V(x_{k+1})}{\partial u_k} \\ &= 2R(x_k)\, u_k + g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}}. \end{aligned} \tag{6}$$

Therefore, the optimal control law is formulated as

$$u_k = -\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \tag{7}$$

with the boundary condition $V(0) = 0$.

[1]A function $l(z)$ is positive semidefinite (or nonnegative definite) function if for all vectors z, $l(z) \geq 0$. In other words, there are some vectors z for which $l(z) = 0$, and for all others z, $l(z) > 0$ (Kirk, 1970).

[2]A real symmetric matrix R is positive definite if $z^T R z > 0$ for all $z \neq 0$ (Kirk, 1970).

Moreover, $\mathcal{H}$ has a quadratic form in u_k and $R(x_k) > 0$, then

$$\frac{\partial^2 \mathcal{H}}{\partial u_k^2} > 0$$

holds as a sufficient condition such that optimal control law (7) (globally (Kirk, 1970)) minimizes $\mathcal{H}$ and the performance index (2) (Lewis and Syrmos, 1995).

Substituting (7) in (4), it becomes

$$\begin{aligned} V(x_k) &= l(x_k) + \left(-\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \right)^T \\ &\quad \times R \left(-\frac{1}{2} R^{-1}(x_k)\, g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \right) \\ &\quad + V(x_{k+1}) \\ &= l(x_k) + V(x_{k+1}) + \frac{1}{4} \frac{\partial V^T(x_{k+1})}{\partial x_{k+1}} g(x_k) \times \\ &\quad R^{-1}(x_k) g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \end{aligned} \tag{8}$$

which can be rewritten as

$$\begin{aligned} l(x_k) + V(x_{k+1}) - V(x_k) + \frac{1}{4} \frac{\partial V^T(x_{k+1})}{\partial x_{k+1}} g(x_k) \times \\ R^{-1}(x_k) g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} = 0. \end{aligned} \tag{9}$$

Nevertheless, solving the partial differential equation (9) is not simple. Thus, to solve the above HJB equation for $V(x_k)$ constitutes an important disadvantage in discrete-time optimal control for nonlinear systems.

B. LYAPUNOV STABILITY

Due to the fact that the inverse optimal control is based on a Lyapunov function, we establish the following definitions.

A function $V(x_k)$ satisfying the condition $V(x_k) \to \infty$ as $\|x_k\| \to \infty$ is said to be *radially unbounded* (Khalil, 1996).

Definition 1: (Amicucci *et al.*, 1997) Let $V(x_k)$ be a radially unbounded, positive definite function, with $V(x_k) > 0$, $\forall x_k \neq 0$ and $V(0) = 0$. If for any $x_k \in \mathbb{R}^n$, there exist real values u_k such that

$$\Delta V(x_k, u_k) < 0$$

where the Lyapunov difference $\Delta V(x_k, u_k)$ is defined as $V(x_{k+1}) - V(x_k) = V(f(x_k) + g(x_k)\, u_k) - V(x_k)$. Then $V(\cdot)$ is said to be a "discrete-time control Lyapunov function" (CLF) for system (1).

Theorem 1: (Exponential stability (Vidyasagar, 1993)) Suppose that there exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and constants $c_1,\ c_2,\ c_3 > 0$ and $p > 1$ such that

$$c_1 \|x\|^p \leq V(x_k) \leq c_2 \|x\|^p \tag{10}$$

$$\Delta V(x_k, u_k) \leq -c_3 \|x\|^p, \quad \forall k \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Then $x_k = 0$ is an exponentially stable equilibrium of system (1).

III. INVERSE OPTIMAL CONTROL

For the inverse approach, a stabilizing feedback control law is first developed, and then it is established that this control law optimizes a meaningful cost functional. When we want to emphasize that u_k is optimal, we use u_k^*. We establish the following assumptions and definitions which allow the inverse optimal control solution.

In the next definition, we establish the discrete-time *inverse optimal* control problem

Definition 2: The control law

$$u_k^* = -\frac{1}{2}R^{-1}(x_k)g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \quad (11)$$

is inverse optimal (globally) stabilizing if

(i) it achieves (global) asymptotic stability of $x = 0$ for system (1);

(ii) $V(x_k)$ is (radially unbounded) positive definite function such that inequality

$$\overline{V} := V(x_{k+1}) - V(x_k) + u_k^{*T}\, R(x_k)\, u_k^* \leq 0 \quad (12)$$

is fulfill. When we select $l(x_k) := -\overline{V}$, then $V(x_k)$ is a solution for (9).

As established in Definition 2, inverse optimal control problem is based on the knowledge of $V(x_k)$; thus, we propose a CLF $V(x_k)$ such that (i) and (ii) can be guaranteed.

For the control law (11), let us consider a twice differentiable positive ($\mathcal{C}^2$) definite function

$$V(x_k) = \frac{1}{2}x_k^T P\, x_k \quad (13)$$

as a CLF, where $P \in \mathbb{R}^{n\times n}$ is assumed to be positive definite ($P > 0$) and symmetric ($P = P^T$) matrix. Considering one step ahead for (13) and evaluating (11), we obtain

$$\begin{aligned} u_k^* &= -\frac{1}{2}R^{-1}(x_k)\, g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \\ &= -\frac{1}{2}R^{-1}(x_k)\, g^T(x_k)(P\, x_{k+1}) \\ &= -\frac{1}{2}R^{-1}(x_k)\, g^T(x_k)(P\, f(x_k) + P\, g(x_k)\, u_k^*). \end{aligned}$$

Thus,

$$\left(I + \frac{1}{2}R^{-1}(x_k)g^T(x_k)P\, g(x_k)\right)u_k^* = -\frac{1}{2}R^{-1}(x_k)\, g^T(x_k)\, P\, f(x_k). \quad (14)$$

Multiplying by $R(x_k)$, (14) becomes

$$\left(R(x_k) + \frac{1}{2}g^T(x_k)P\, g(x_k)\right)u_k^* = -\frac{1}{2}\, g^T(x_k)\, P\, f(x_k) \quad (15)$$

which results in the following state feedback control law:

$$u_k^* = \alpha(x_k) = -\frac{1}{2}\left(R(x_k) + P_2(x_k)\right)^{-1}\, P_1(x_k) \quad (16)$$

where $P_1(x_k) = g^T(x_k)\, P\, f(x_k)$ and $P_2(x_k) = \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)$. Note that $P_2(x_k)$ is positive definite and symmetric matrix, which ensures that the inverse matrix in (16) exists.

Once we have proposed a CLF for solve the inverse optimal control in accordance with Definition 2, the main contribution is presented.

Theorem 2: Consider the affine discrete-time nonlinear system (1). If there exists a matrix $P = P^T > 0$ such that the following inequality holds:

$$V_f(x_k) - \frac{1}{4}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1}\, P_1(x_k) \leq -\zeta_Q\, \|x_k\|^2 \quad (17)$$

where $V_f(x_k) = \frac{1}{2}\left[V(f(x_k)) - V(x_k)\right]$, with $V(f(x_k)) = f^T(x_k)\, P\, f(x_k)$ and $\zeta_Q > 0$; $P_1(x_k)$ and $P_2(x_k)$ as defined in (16); then, the equilibrium point $x_k = 0$ of system (1) is globally exponentially stabilized by the control law (16), with the CLF (13).

Moreover, with (13) as a CLF, this control law is inverse optimal in the sense that it minimizes the meaningful functional given by

$$\mathcal{J} = \sum_{k=0}^{\infty}\left(l(x_k) + u_k^T\, R(x_k)\, u_k\right) \quad (18)$$

with

$$l(x_k) = -\overline{V}\big|_{u_k^*=\alpha(x_k)}. \quad (19)$$

Proof: First, we analyze stability. Global stability for the equilibrium point $x_k = 0$ of system (1) with (16) as input, is achieved if function $\overline{V}$ in (12), is satisfied. Thus, $\overline{V}$ results in

$$\begin{aligned} \overline{V} &= V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &= \frac{f^T(x_k)\, P\, f(x_k) + 2f^T(x_k)\, P\, g(x_k)\, \alpha(x_k)}{2} \\ &\quad + \frac{\alpha^T(x_k)\, g^T(x_k)\, P\, g(x_k)\, \alpha(x_k) - x_k^T\, P\, x_k}{2} + \\ &\quad \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &= V_f(x_k) - \frac{1}{2}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1}\, P_1(x_k) \\ &\quad + \frac{1}{4}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1}\, P_1(x_k) \\ &= V_f(x_k) - \frac{1}{4}P_1^T(x_k)\left(R(x_k) + \right. \\ &\quad \left. P_2(x_k)\right)^{-1} P_1(x_k). \end{aligned} \quad (20)$$

Selecting P such that $\overline{V} \leq 0$, stability of $x_k = 0$ is guaranteed. Furthermore, by means of P, we can achieve a desired negativity amount (Freeman and Primbs, 1996) for the closed-loop function $\overline{V}$ in (20). This negativity amount can be bounded using a positive definite matrix Q as follows:

$$\begin{aligned} \overline{V} &= V_f(x_k) - \frac{1}{4}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1}\, P_1(x_k) \\ &\leq -x_k^T\, Q\, x_k \\ &\leq -\lambda_{min}(Q)\, \|x_k\|^2 \\ &= -\zeta_Q\, \|x_k\|^2, \qquad \zeta_Q = \lambda_{min}(Q) \end{aligned} \quad (21)$$

where $\|\cdot\|$ stands the Euclidean norm and $\zeta_Q > 0$ denotes the minimum eigenvalue of matrix Q ($\lambda_{min}(Q)$). Thus, from (21) follows condition (17).

Considering (20)-(21), if $\overline{V} = V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\,R(x_k)\,\alpha(x_k) \leq -\zeta_Q\,\|x_k\|^2$, then $\Delta V = V(x_{k+1}) - V(x_k) \leq -\zeta_Q\,\|x_k\|^2$. Moreover, as $V(x_k)$ is a radially unbounded function, then the solution $x_k = 0$ of the closed-loop system (1) with (16) as input, is globally exponentially stable according to Theorem 1.

When function $-l(x_k)$ is set to be the (21) right-hand side, that is

$$
\begin{aligned}
l(x_k) &:= -\overline{V}\big|_{u_k^*=\alpha(x_k)} \\
&= -V_f(x_k) + \frac{1}{4}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k)
\end{aligned}
$$

then $V(x_k)$ as proposed in (13), is a solution of the DT HJB equation (9).

In order to establish optimality, considering that (16) stabilizes (1), and substituting $l(x_k)$ in (18), we obtain

$$
\begin{aligned}
\mathcal{J} &= \sum_{k=0}^{\infty}\left(l(x_k) + u_k^T\,R(x_k)\,u_k\right) \\
&= \sum_{k=0}^{\infty}\left(-\overline{V} + u_k^T\,R(x_k)\,u_k\right) \\
&= -\sum_{k=0}^{\infty}\Big[V_f(x_k) - \frac{1}{4}P_1^T(x_k)\,(R(x_k)+ \\
&\qquad P_2(x_k))^{-1}\,P_1(x_k)\Big] + \sum_{k=0}^{\infty}u_k^T\,R(x_k)\,u_k. \quad (22)
\end{aligned}
$$

Now, factorizing (22) and then adding the identity matrix $I_m \in \mathbb{R}^{m\times m}$ presented as $I_m = (R(x_k)+P_2(x_k))\,(R(x_k)+P_2(x_k))^{-1}$, we obtain

$$
\begin{aligned}
\mathcal{J} &= -\sum_{k=0}^{\infty}\Big[V_f(x_k) - \frac{1}{2}P_1^T(x_k) \\
&\qquad (R(x_k)+P_2(x_k))^{-1}\,P_1(x_k) + \\
&\qquad \frac{1}{4}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\times \\
&\qquad P_2(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k) \\
&\qquad +\frac{1}{4}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\times \\
&\qquad R(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k)\Big] + \\
&\qquad \sum_{k=0}^{\infty}u_k^T\,R(x_k)\,u_k. \quad (23)
\end{aligned}
$$

Being $\alpha(x_k) = -\frac{1}{2}\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k)$, then (23) becomes

$$
\begin{aligned}
\mathcal{J} &= -\sum_{k=0}^{\infty}\Big[V_f(x_k) + P_1^T(x_k)\,\alpha(x_k) + \alpha^T(x_k)\times \\
&\qquad P_2(x_k)\,\alpha(x_k)\Big] + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k \\
&\qquad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -\sum_{k=0}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k \\
&\qquad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big]. \quad (24)
\end{aligned}
$$

After evaluating the summation for $k = 0$, then (24) can be written as

$$
\begin{aligned}
\mathcal{J} &= -\sum_{k=1}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] - V(x_1) + V(x_0) \\
&\qquad +\sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -\sum_{k=2}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] - V(x_2) + V(x_1) \\
&\qquad -V(x_1) + V(x_0) + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k \\
&\qquad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big]. \quad (25)
\end{aligned}
$$

For notation convenience in (25), the upper limit ∞ will treated as $N \to \infty$, and thus

$$
\begin{aligned}
\mathcal{J} &= -V(x_N) + V(x_{N-1}) - V(x_{N-1}) + V(x_0) \\
&\qquad +\sum_{k=0}^{N}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -V(x_N) + V(x_0) + \sum_{k=0}^{N}\Big[u_k^T\,R(x_k)\,u_k \\
&\qquad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big].
\end{aligned}
$$

Letting $N \to \infty$ and noting that $V(x_N) \to 0$ for all x_0, then

$$
\mathcal{J}(x_k) = V(x_0) + \sum_{k=0}^{\infty}\left[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\right]. \quad (26)
$$

Thus, the minimum value of (26) is reached with $u_k = \alpha(x_k)$. Hence, the control law (16) minimizes the cost functional (18). The optimal value function of (18) is $\mathcal{J}^*(x_0, \alpha(x_k)) = V(x_0)$ for all x_0. ∎

IV. TRAJECTORY TRACKING FOR BLOCK-CONTROLLABLE FORM SYSTEMS

Consider system (1) to be (globally) stabilized by inverse optimal control law $u_k = \alpha(x_k)$ as proposed in (16). Let consider system (1) can be presented (possibly after a nonlinear transformation) in the nonlinear block-controllable

(NBC) form (Loukianov and Utkin, 1981) consisting of r blocks as

$$\begin{aligned} x_{k+1}^{1} &= f^{1}\left(x_{k}^{1}\right)+B^{1}\left(x_{k}^{1}\right)x_{k}^{2} \\ &\vdots \\ x_{k+1}^{r-1} &= f^{r-1}\left(x_{k}^{1},x_{k}^{2},\ldots,x_{k}^{r-1}\right) \\ &\quad +B^{r-1}\left(x_{k}^{1},x_{k}^{2},\ldots,x_{k}^{r-1}\right)x_{k}^{r} \\ x_{k+1}^{r} &= f^{r}\left(x_{k}\right)+B^{r}\left(x_{k}\right)\alpha(x_{k}) \end{aligned} \tag{27}$$

where $x_k \in \mathbb{R}^n$, $x_k = \left[x_k^{1T}\ x_k^{2T}\ \ldots\ x_k^{rT}\right]^T$; $x^j \in \mathbb{R}^{n_j}$; $j = 1,\ldots,r$; n_j denotes the order of each r-th block; $x^j = \left[x^{j_1}\ x^{j_2}\ \ldots\ x^{j_{n_j}}\right]^T$; input $\alpha(x_k) \in R^m$; $f^j : \mathbb{R}^n \to \mathbb{R}^n$ and $B^j : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings. Without loss of generality, $x_k = 0$ is an equilibrium point for (27). We assume $f^j(0) = 0$, $rank\{B^j(x_k)\} = m_j\ \forall x_k \neq 0$ and $n = \sum_{j=1}^{r} n_j$.

For trajectory tracking of first block in (27), let define the tracking error as

$$z_k^1 = x_k^1 - x_{\delta,k}^1 \tag{28}$$

where $x_{\delta,k}^j$ is the desired trajectory signal.

Once defined the first new variable (28), we take one step ahead

$$z_{k+1}^1 = f^1\left(x_k^1\right) + B^1\left(x_k^1\right)x_k^2 - x_{\delta,k+1}^1. \tag{29}$$

Equation (29) is viewed as a block with state z_k^1 and the state x_k^2 is considered as a pseudo-control input, where desired dynamics can be imposed. This can be solved with the anticipation of the desired dynamics for (29) as follows:

$$\begin{aligned} z_{i,k+1}^1 &= f^1\left(x_k^1\right)+B^1\left(x_k^1\right)x_k^2 - x_{\delta,k+1}^1 \\ &= f^1\left(z_k^1\right)+B^1\left(x_k^1\right)z_k^2 \end{aligned} \tag{30}$$

Then, x_k^2 is calculated as

$$\begin{aligned} x_{\delta,k}^2 &= \Big(B^1\left(x_k^1\right)\Big)^{-1}\Big(x_{\delta,k+1}^1 - f^1\left(x_k^1\right) \\ &\quad + f^1\left(z_k^1\right)+B^1\left(x_k^1\right)z_k^2\Big). \end{aligned} \tag{31}$$

Note that the calculated value of state $x_{\delta,k}^2$ in (31) is not the real value of such state; instead of, it represents the desired behavior for x_k^2. Hence, to avoid confusions this desired value of x_k^2 is referred as $x_{\delta,k}^2$ in (31).

Proceeding in the same way as for the first block, a second variable in the new coordinates is defined as

$$z_k^2 = x_k^2 - x_{\delta,k}^2.$$

Taking one step ahead in z_k^2 yields

$$\begin{aligned} z_{k+1}^2 &= x_{k+1}^2 - x_{\delta,k+1}^2 \\ &= f^2\left(x_k^1,x_k^2\right)+B^2\left(x_k^1,x_k^2\right)x_k^3 - x_{\delta,k+2}^2. \end{aligned}$$

The desired dynamics for this block is imposed as

$$\begin{aligned} z_{k+1}^2 &= f^2\left(x_k^1,x_k^2\right)+B^2\left(x_k^1,x_k^2\right)x_k^3 - x_{\delta,k+2}^2 \\ &= f^1\left(z_k^1\right)+B^2\left(x_k^1,x_k^2\right)z_k^2 \end{aligned} \tag{32}$$

These steps are taken iteratively. At the last step, the known desired variable is $x_{\delta,k}^r$, and the last new variable is defined as

$$z_k^r = x_k^r - x_{\delta,k}^r.$$

As usually, taking one step ahead yields

$$z_{k+1}^r = f^r\left(x_k\right) + B^r\left(x_k\right)\alpha(x_k) - x_{\delta,k+1}^r \tag{33}$$

and the desired dynamics for this block is imposed by means of

$$\begin{aligned} \alpha(x_k) &= \Big(B^r\left(x_k\right)\Big)^{-1}\Big(x_{\delta,k+1}^r - f^r\left(x_k\right) \\ &\quad + f^r\left(z_k\right)+B^r\left(z_k\right)\alpha(z_k)\Big). \end{aligned} \tag{34}$$

Hence, system (27) can be presented in the new variables $z = \left[z^{1T} z^{2T} \cdots z^{rT}\right]$ of the form

$$\begin{aligned} z_{k+1}^{1} &= f^{1}\left(z_{k}^{1}\right)+B^{1}\left(x_{k}^{1}\right)z_{k}^{2} \\ &\vdots \\ z_{k+1}^{r-1} &= f^{r-1}\left(z_{k}^{1},z_{k}^{2},\ldots,z_{k}^{r-1}\right) \\ &\quad +B^{r-1}\left(z_{k}^{1},z_{k}^{2},\ldots,z_{k}^{r-1}\right)z_{k}^{r} \\ z_{k+1}^{r} &= f^{r}\left(z_{k}\right)+B^{r}\left(z_{k}\right)\alpha(z_{k}) \end{aligned} \tag{35}$$

which in a general form can be described by

$$z_{k+1} = f(z_k) + g(z_k)\,\alpha(z_k). \tag{36}$$

System (36) can be decomposed for x_{k+1} as $x_{k+1} - x_{\delta,k+1} = f(z_k) + g(z_k)\,\alpha(z_k)$, and thus

$$x_{k+1} = f(z_k) + g(z_k)\,\alpha(z_k) + x_{\delta,k+1}. \tag{37}$$

Theorem 3: Consider the equilibrium point $x_k = 0$ of system (27) to be (globally) asymptotically stabilized by inverse optimal control law $\alpha(x_k)$ (16), and therefore the Lyapunov difference becomes $V(x_{k+1}) - V(x_k) < 0$. Then, solution x_k of (27) with (34) as input is (globally) asymptotically stabilized along the desired trajectory $x_{\delta,k}$. Moreover, control law (34) minimizes the following cost functional:

$$\mathcal{J} = \sum_{k=0}^{\infty}\left(l(z_k) + \alpha(z_k)^T R(z_k)\,\alpha(z_k)\right) \tag{38}$$

with

$$l(z_k) = -\overline{V}(z_k) \geq 0. \tag{39}$$

Proof: Let system (27) to be described in a general form by (1) with $u_k = \alpha(x_k)$, and system (35) to be described by (36). Consider a candidate Lyapunov function as $V(z_k)$. Then, Lyapunov difference becomes

$$\begin{aligned} \Delta V : &= V(z_{k+1}) - V(z_k) \\ &= V(x_{k+1} - x_{\delta,k+1}) - V(z_k). \end{aligned} \tag{40}$$

Substituting (37) in (40) we have

$$\begin{aligned} \Delta V &= V(x_{k+1} - x_{\delta,k+1}) - V(z_k) \\ &= V(f(z_k) + g(z_k)\,\alpha(z_k)) - V(z_k). \end{aligned} \tag{41}$$

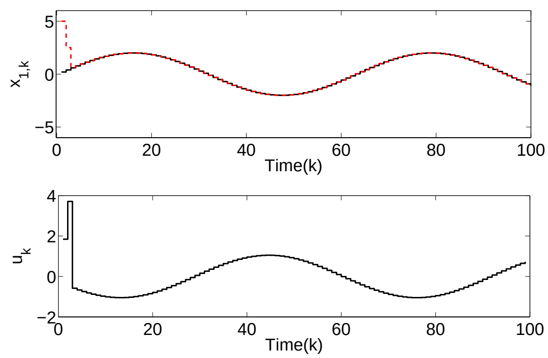

Fig. 1. Tracking performance of x_k. Solid line ($x_{\delta,k}$) is the reference signal and dashed line is the evolution of $x_{1,k}$. Control signal is also displayed.

Due to the fact $V(x_{k+1}) - V(x_k) = V(f(x_k) + g(x_k)\,\alpha(x_k)) - V(x_k) < 0$ for (1), then the Lyapunov difference for transformed system is $V(z_{k+1}) - V(z_k) = V(f(z_k) + g(z_k)\,\alpha(z_k)) - V(z_k) < 0$, and (global) asymptotic stability is guaranteed for transformed system (36).

The minimization of meaningful cost functional is established similarly as in Theorem 2, and hence it is omitted. ■

A. *EXAMPLE: Trajectory Tracking for an Unstable System*

In this section, we illustrate the applicability of the obtained results by means of an example for solving the trajectory tracking problem. By illustration easily and space limitation, we synthesize an trajectory tracking inverse optimal control law for a discrete-time second order system (unstable for $u_k = 0$) of the form (1) with:

$$f(x_k) = \begin{bmatrix} 1.5\,x_{1,k} + x_{2,k} \\ x_{1,k} + 2\,x_{2,k} \end{bmatrix} \tag{42}$$

and

$$g(\chi_k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{43}$$

In accordance with Section IV, control law (34) becomes

$$\begin{aligned} \alpha(x_k) \;\; = \;\; & x^1_{\delta,k+2} - 1.5(1.5\,x_{1,k} + x_{2,k}) \\ & +1.5(1.5\,z_{1,k} + z_{2,k}) + 2\,(1.5\,z_{1,k} \\ & +2\,z_{2,k}) + 2\alpha(z_k) - x1,k - 2\,z_{2,k}. \end{aligned} \tag{44}$$

Figure 1 presents the trajectory tracking for $x_{1,k}$ with

$$P = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

V. CONCLUSIONS

This paper has presented a discrete-time inverse optimal control, which achieve trajectory tracking and is inverse optimal in the sense that it, a posteriori, minimizes a meaningful cost functional. Simulation results illustrate that the required goal is achieved, i.e., the designed controller maintains stability on reference for the system.

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