Nested Backward Compensation of Unmatched Effects of Unknown Inputs Based on HOSM Identification

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***Abstract*— The paper considers the regulation problem of linear time invariant systems with unmatched perturbations. The proposed methodology exploits a high order sliding mode observer, which guarantees theoretically exact state and perturbation estimation. It is introduced a controller with a compensation strategy based on the identified perturbation values. When the system satisfies quite restrictive assumptions, the method ensures exact regulation of the unmatched states. In order to deal with the general case it is proposed a nested backward strategy to design the sliding surface, which allows to compensate the unmatched uncertainties and to stabilize some of the non-actuated state components, while all the remaining states are maintained bounded.**

I. INTRODUCTION

Motivation. Control under heavy uncertainties is one of the main problems of modern control theory. One of the most prospering control strategies insensitive w.r.t. uncertainties is sliding mode control (SMC) (see, e.g., [2]). This robust technique is well known for its ability to withstand external disturbances and model uncertainties which satisfy the matching condition. This condition is met when the perturbation or parameters variations are implicit at the input channels, for example in the case of completely actuated systems.

The SMC design methodology involves two stages: the design of a switching function which provides desirable system performance in the sliding mode and the design of the control law ensuring that the system states are driven to the sliding manifold and thus the desired performance is attained and maintained in spite of the matched uncertainties. Nevertheless, there are some disadvantages: the necessity to measure the whole state and the lack of robustness against unmatched uncertainties of the resulting controller.

In order to address the issue of robustness against unmatched perturbations, the main solution has been the combination of sliding mode technique with other robust strategies. In order to reduce the effects of unmatched uncertainties, a method that combines H_∞ and integral sliding mode control is proposed in [5]. The main idea is to choose such a projection matrix, ensuring not only that unmatched perturbations are not amplified, but even more, that its effects are minimized. In [9] the linear time-varying system with unmatched disturbances is replaced by a finite set of dynamic models such that each one describes a particular uncertain case then, applying a min-max SMC they develop an optimal robust sliding-surface design. A control scheme based on block control and quasi-continuous HOSM techniques is proposed in [10] for control of nonlinear systems with unmatched perturbations; this method assures exact finite time tracking.

The sliding surface design for systems with unmatched uncertainties considering only output information has been considered in [24], [14], [25]. In [14] a linear matrix inequalities (LMI) based method for designing an output feedback variable structure control system is presented. The author proposes an LMI based sliding surface design considering H_2 performance. Another possible solution to overcome the full state requirement is to use an observer to estimate the state. In [24] an output robust stabilization problem for a class of systems with matched and mismatched uncertainties using sliding mode techniques is considered. The idea is to use an asymptotic nonlinear observer to estimate system states, then a variable structure controller is proposed to stabilize the system. In [25] a integral sliding surface is designed, once the system is steered to the sliding surface a full order compensator is designed for the unmatched disturbance attenuation.

Contribution. In this paper a robust output control law is designed to reject the unmatched uncertainties and stabilize the underactuated dynamics using a high order sliding mode observer to reconstruct the states and perturbations in finite time.

The proposed control law exactly compensates (theoretically) the unmatched perturbation, stabilizing the underactuated states while ensuring the boundedness of the remained states. In order to achieve this:

- A sliding manifold is designed such that the system's motion along the manifold meets the specified performance: the regulation of the non-actuated states and the rejection of unmatched uncertainties.
- A nested backward strategy to design the sliding surface is proposed to compensate the unknown unmatched inputs stabilizing some of the non-actuate states.
- A sliding mode control law is designed such that the system's state is driven towards the manifold and stays there for all future time, regardless of disturbances and uncertainties.

Paper Structure. In Section II the problem formulation and

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control challenge are presented. The high order sliding mode observer algorithm is described in Section III as well as the procedure to identify the perturbations and its derivatives.The compensation of the unmatched perturbations through the sliding surface is outlined in Section IV. Section V presents a backward nested compensation strategy. A simulation example illustrates the performance of the robust exact unmatched uncertainties compensation controller in Section VI.

II. PROBLEM STATEMENT

Let us consider a linear time invariant system with unknown inputs

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, and $y(t) \in \Re^p$ $(1 \leq p < n)$ are the state vector, the control and the output of the system, respectively. The vector $z(t) \in \Re^m$ represents unmatched states which we wan to stabilize. The unknown inputs are represented by the vector $w(t) \in \Re^q$, and $\mathtt{rank}C = p$ and $\mathtt{rank}B = m$.

The following assumptions are assumed to hold about the system:

A1. The (A, B) pair is assumed to be controllable.

A2. For $u = 0$, the system is strongly observable, or equivalently (A, C, D) has no invariant zeros.

A3. $w(t)$ has successive derivatives up to order α bounded by the same constant w^+, i.e. $\left\| w^{(\alpha+1)}(t) \right\| \leq w^+$ for all $t \geq 0$.

Here $\|\cdot\|$ is understood as the vector Euclidean norm.

A. LTISUI in regular form

Let us transform the system into a suitable regular form, [18], such that the system is decomposed into two connected subsystems. By assumption $\mathtt{rank}(B) = m$, there exists an invertible matrix of elementary row operations $T \in \Re^{m \times n}$

$$T = \begin{bmatrix} B^{\perp} \\ B^{+} \end{bmatrix}, \; B^{\perp}B = 0, \; B^{+} = \left(B^T B\right)^{-1} B^T \quad (3)$$

such that

$$TB = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \quad (4)$$

where $I_m \in \Re^{m \times m}$. Applying the coordinate transformation $Tx = \begin{bmatrix} x_1^T & \eta_1^T \end{bmatrix}^T$ to system (2) yields

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 \eta_1(t) + D_1 w(t) \quad (5)$$
$$\dot{\eta}_1(t) = E_1 x_1(t) + F_1 \eta_1(t) + H_1 w(t) + u(t) \quad (6)$$

where $x_1 \in \Re^{n-m}$, $\eta_1 \in \Re^m$, $D_1 \in \Re^{(n-m) \times q}$, $H_1 \in \Re^{m \times q}$.

The control aim is to design a controller, which allows to regulate the perturbed non-actuated subsystem (5) and which is based on the measurement of the state and the identification of the unmatched perturbation.

A first solution is proposed, then it is considered the case when the condition $D_1 \in \mathtt{span}(B_1)$ does not hold. In this more general situation the design of the compensator exploits a nested backward strategy, implemented via a suitably chosen sliding surface, which allows to compensate the unmatched effects of the unknown inputs and to stabilize some of the non-actuated state components, while all the remaining states are maintained bounded.

The unmatched inputs compensator relies on the availability, in a finite time, of an exact estimation of the state and the identification the unknown inputs and its derivatives.

To this end we introduce a high order sliding mode observer (HOSMO), [8]. The HOSMO provides the exact value of the state vector and the unknown inputs identification in a finite time, thereby achieving the best possible observer precision [4].

III. High Order Sliding Mode Observer (HOSMO)

The HOSMO provides the exact value of the state vector and the unknown inputs identification in a finite time. Basically, the observer works in two stages: first, a linear observer is used to maintain bounded the estimation error between a linear observer and the original state; then, by means of a differentiation scheme, the state vector and unknown inputs identification vector are found. Below a general description of the observer is given.

Stage 1: Let us design a linear observer in order to bound the observation error,

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + L(y(t) - \tilde{y}(t)),$$
$$\tilde{y}(t) = C\tilde{x}$$

where L must be designed such that the matrix $\tilde{A} := (A - LC)$ is Hurwitz. Defining $e(t) := x(t) - \tilde{x}(t)$

$$\dot{e}(t) = \tilde{A}e(t) + Dw(t) \quad (7)$$

Thus, $e(t)$ has a bounded norm, i.e., there exist a known constant e^+ and a finite time t_e, such that

$$\|e(t)\| \leq e^+, \text{ for all } t > t_e \quad (8)$$

Stage 2: In order to reconstruct the state, we rely on an algorithm, which allows to decouple the unknown inputs from the successive derivatives of the output of the linear estimation error system, $y_e := y - \tilde{y}$. The steps are the following.

0. Define $M_1 := C$.
1. Differentiate a linear combination of the output $y_e = y - \tilde{y}$, ensuring that the derivative of this combination is unaffected by the uncertainties, i.e.,

$$\frac{d}{dt}(M_1 D)^{\perp} y_e(t) = (M_1 D)^{\perp} C\tilde{A}e(t)$$

Construct the extended vector

$$\begin{bmatrix} \frac{d}{dt}(M_1 D)^{\perp} y_e(t) \\ y_e(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (M_1 D)^{\perp} C\tilde{A} \\ C \end{bmatrix}}_{M_2} e(t)$$

$J_1 = (M_1 D)^{\perp}$ and $I_p \in \Re^p$

$$M_2 e(t) = \frac{d}{dt} \begin{bmatrix} J_1 & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_e(t) \\ \int y_e(\tau) d\tau \end{bmatrix}$$

There is a $k \le n$ such that $rank(M_k) = n$, [21].

j. Differentiate a linear combination of the entries of the vector $M_{k-1}e(t)$ that are unaffected by the uncertainties

$$\begin{bmatrix} \frac{d}{dt}\left(M_{j-1}D\right)^{\perp} M_{j-1}e(t) \\ y_e(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \left(M_{j-1}D\right)^{\perp} M_{j-1}\tilde{A} \\ C \end{bmatrix}}_{M_j} e(t)$$

$$M_j e(t) = \frac{d^{j-1}}{dt^{j-1}} \begin{bmatrix} J_{j-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[j-1]} \tag{9}$$

where $J_{j-1} = \left(M_{j-1}D\right)^{\perp} \begin{bmatrix} J_{j-2} & 0 \\ 0 & I_p \end{bmatrix}$ and

$$Y^{[j-1]} = \begin{bmatrix} y_e(t) \\ \int_0^t y_e(\tau)\,d\tau \\ \vdots \\ \int_0^t \cdots \int_0^{\tau_{j-1}} y_e\left(\tau_{j-1}\right) d\tau_{j-1} \cdots d\tau_2 d\tau_1 \end{bmatrix}.$$

Under A2 there exists a matrix M_k $(k \le n)$, generated recursively by (9), that satisfies the condition $\texttt{rank}\ M_k = n$ (see, e.g., [21]). This means that the algebraic equation

$$M_k e(t) = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]}$$

has a unique solution for $e(t)$. Such solution could be found by means of the pre-multiplication of both sides of the previous equation by $M_k^+ := \left(M_k^T M_k\right)^{-1} M_k^T$. That is

$$e(t) = \frac{d^{k-1}}{dt^{k-1}} \underbrace{M_k^+ \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]}}_{\Theta(t)} \tag{10}$$

The Assumption A3 allows realizing an $(\alpha + k - 1) - th$ order sliding mode differentiator, which is the highest order we can construct for this case. The HOSM differentiator is given by

$$\begin{aligned} \dot{v}_0 &= \lambda_0 \Lambda^{\frac{1}{i+1}} \left|v_0 - \Theta(t)\right|^{\frac{i}{i+1}} sign\left(v_0 - \Theta(t)\right) + v_1 \\ \dot{v}_1 &= \lambda_1 \Lambda^{\frac{1}{i}} \left|v_1 - \dot{v}_0\right|^{\frac{i-1}{i-2}} sign\left(v_1 - \dot{v}_0\right) + v_2 \\ &\vdots \\ \dot{v}_{i-1} &= \lambda_{i-1} \Lambda^{\frac{1}{2}} \left|v_{i-1} - \dot{v}_{i-2}\right|^{\frac{1}{2}} sign\left(v_{i-1} - \dot{v}_{i-2}\right) + v_i \\ \dot{v}_i &= \lambda_i \Lambda sign\left(v_i - \dot{v}_{i-1}\right) \end{aligned} \tag{11}$$

The observer order is $i = \alpha + k - 1$, the values of the λ_i can be calculated as is shown in [20], Λ is a Lipschitz constant of $\Theta^{(\alpha+k)}(t)$, which for our case can be calculated in the following way: starting from the fact that $v^{k-1} = e(t)$ remains bounded by (8), the next derivative $v^k = \dot{e}(t)$, will be also bounded $\|\dot{e}(t)\| \le \|A - LC\| e^+ + \|B\| w^+$. In general $e^{\alpha}(t)$ can be represented as a linear combination of $\left\{e^k,\ e^{k+1}, \ldots,\ e^{\alpha-1},\ \dot{w}, \ldots,\ w^{\alpha}\right\}$ and it can be verified that

$$\Lambda \ge \|A - LC\|^{\alpha} e^+ + \sum_{j=0}^{\alpha-1} \|A - LC\|^j \|B\| w^+ \tag{12}$$

A. State variables observation

In [20] it was shown that with the proper choice of the constants λ_i, there is a finite time t_σ such that the identity $v_j(t) = \frac{d^j}{dt^j}\Theta(t)$ is achieved for every $j = 0, \ldots, \alpha + k - 1$. The vector $e(t)$ can be reconstructed from the $(k-1) - th$ order sliding dynamics. Thus, we achieve the identity $v_{k-1}(t) = e(t)$, and consequently

$$\hat{x}(t) := v_{k-1}(t) + \tilde{x}(t) \text{ for all } t \ge t_\sigma$$

where $\hat{x}$ represents the estimated value of x. Therefore, the identity $\hat{x}(t) \equiv x(t)$, for all $t \ge t_\sigma$ is achieved.

B. Uncertainties Identification

Now, considering the system error dynamics (7). We can recover $\dot{e}(t)$ from the HOSM differentiator (11) in finite time, the equality $v_k(t) = \dot{e}(t)$ is achieved for all $t \ge t_\sigma$ and the next equation holds

$$\hat{w}(t) = -B^+ \left[(A - LC)\, v^{k-1}(t) - v^k(t)\right] \tag{13}$$

The next derivatives can be obtained following a similar procedure, $\dot{\hat{w}} = -B^+ \left[(A - LC)\, v^k(t) - v^{k+1}(t)\right]$, under Assumption A3 it could be identified the successive $\dot{\hat{w}}, \ldots, \hat{w}^{\alpha}$ derivatives.

C. Precision of the observation and identification processes

Suppose that we would like to realize the HOSM observer with a sampling step δ. Then, as follows from Theorem 7 of [20] , the error caused by the sampling time δ in the absence of noise for an $(\alpha + k - 1) - th$ order HOSM differentiator, is

$$\left\|\Theta^{(j)}(t) - v_j(t)\right\| \le O(\delta^{\alpha+k-j}) \text{ for } j = 0, \ldots, \alpha + k - 1 \tag{14}$$

For recovering the estimated state, $(k-1)$ differentiations are needed. From expression (14) follows that the observation error provoked by the sampling time δ is $O(\delta^{\alpha+1})$.

Now, equation (13 shows that k differentiations are needed in order to recover the estimated unknown input $\hat{w}$. Therefore, from (14) the sampling step identification error will be $O(\delta^{\alpha})$ and, for the succesive unkonwn inputs derivatives identification, i.e. $\dot{\hat{w}}, \ddot{\hat{w}}, \ldots, \hat{w}^{(\alpha)}$ the error will be, respectively, $O(\delta^{\alpha-1}), O(\delta^{\alpha-2}), \ldots, O(\delta)$.

IV. COMPENSATOR OF THE UNMATCHED EFFECTS OF THE UNKNOWN INPUTS

Let us introduce the following sliding output

$$s(t) = Kx_1(t) + \eta_1(t) + g(\hat{w}), \tag{15}$$

where $s \in \Re^m$, the matrix $K \in \Re^{m \times (n-m)}$ will be suitably chosen such that, on the sliding manifold $s = 0$, the behavior of the reduced-order system is the desired one. The term $g(\hat{w})$ is added to compensate unmatched uncertainties and it will be specified in the sequel.

The following control law is applied to enforce the sliding motion on $s=0$

$$u(t) = -\rho(x)\frac{s(t)}{\|s(t)\|}, \quad (16)$$

where $\rho(x) > \|\Phi\|\|x\| + \phi + \zeta$, $\Phi := [KA_1 + E_1 \quad KB_1 + F_1]$, $\phi := \|KD_1 + H_1\| w^+ + \|\Gamma\| w^+$, $\zeta > 0$, [7]. Γ will be specified later.

On the sliding surface $s=0$, it holds

$$\begin{aligned} \eta_1 &= -Kx_1 - g(\hat{w}), & (17)\\ \dot{x}_1 &= (A_1 - B_1K)x_1 - B_1 g(\hat{w}) + D_1 w. & (18)\end{aligned}$$

Since the pair (A,B) is controllable (Assumption A1), it is well known, [15], that also the pair (A_1, B_1) is controllable. Then it is possible to design a matrix K_1 such that matrix $A_s \triangleq (A_1 - B_1K_1)$ has stable eigenvalues.

The compensator term $g(\hat{w})$ should be designed in order to compensate the unmatched uncertainties.

A. Compensation of all the unmatched effects of the unknown inputs

Let us consider the case when

$$D_1 \in span(B_1). \quad (19)$$

Then, there exists a matrix $\Gamma \in \Re^{m\times p}$ such that

$$B_1\Gamma_1 = D_1. \quad (20)$$

In (18) the unmatched effects of the unknown inputs D_1w result to be matched with respect to the state vector η_1.

The compensator term is designed as

$$g(\hat{w}) = \Gamma_1 \hat{w}. \quad (21)$$

Substituting (21) in equation (18) yields

$$\dot{x}_1(t) = (A_1 - B_1K)x_1(t) + D_1(w - \hat{w}). \quad (22)$$

In the ideal case of a exact identification of the unknown inputs, when $\hat{w} = w$, we obtain

$$\dot{x}_1(t) = A_s x_1(t). \quad (23)$$

Since the eigenvalues of A_s have negative real part, equation (23) is exponentially stable. The unmatched uncertainties are compensated and x_1 is stabilized. The trajectories of the state x_1 will converge to a bounded region, i.e. there exist some constants a_1, $a_2 > 0$ such that

$$\|x_1(t)\| \le a_1 \|x_1(0)\| \exp^{-a_2 t}, \quad \forall t > t_\sigma.$$

Furthermore, η_1 is bounded as well indeed during sliding motion. Taking the norm of equation (17) we have

$$\|\eta_1(t)\| \le \|K\|\|x_1(t)\| + \|\Gamma\| w^+, \quad \forall t > t_\sigma. \quad (24)$$

In particular, when $\texttt{rank}(B_1) = n-m$, matrix $\Gamma = B_1^+ D_1$, where B_1^+ is understood as the right inverse of B_1, that is $B_1^+ = B_1^T\left(B_1B_1^T\right)^{-1}$.

V. NESTED BACKWARD COMPENSATOR OF THE UNMATCHED EFFECTS OF THE UNKNOWN INPUTS

In order to deal with more general systems, since assumption (19) is quite restrictive, we consider the case when

$$D_1 \notin \texttt{span}(B_1). \quad (25)$$

Let us transform the subsystem (5) into two connected subsystems. Assuming $\texttt{rank}(B_1) = m$, there exist an invertible transformation $T_1 \in \Re^{(n-2m)\times(n-m)}$ such that

$$T_1 B_1 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

Applying the coordinate transformation $T_1 x_1 = \begin{bmatrix} x_2^T & \eta_2^T \end{bmatrix}^T$ to subsystem (5), we have

$$\begin{aligned} \dot{x}_2(t) &= A_2x_2(t) + B_2\eta_2(t) + D_2 w(t) & (26)\\ \dot{\eta}_2(t) &= E_2x_2(t) + F_2\eta_2(t) + H_2 w(t) + \eta_1(t) & (27)\end{aligned}$$

where $x_2 \in \Re^{n-2m}$ and $\eta_2 \in \Re^m$. The pair (A_2, B_2) is controllable (Assumption A1).

If $D_2 \in span(B_2)$ there exists a matrix Γ_2 such that $B_2\Gamma_2 = D_2$.

The aim is to design a feedback control to stabilize subsystem (26). To this end η_2 can be exploited and regarded as an input.

Let us define $\xi_2 = \eta_2 + K_2x_2 + \Gamma_2\hat{w}$, then in the ideal case when $\hat{w} = w$ and $D_2(w - \hat{w}) = 0$, from (26) and (27)

$$\begin{aligned} \dot{x}_2 &= A_{2s}x_2 + B_2\xi_2 & (28)\\ \dot{\xi}_2 &= \overline{E}_2x_2 + \overline{F}_2\xi_2 + H_2w + \Gamma_2\dot{\hat{w}} + \eta_1 \end{aligned}$$

where $A_{2s} = A_2 - B_2K_2$, $\overline{E}_2 = E_2 - F_2K_2 - K_2A_{2s}$, $\overline{F}_2 = F_2 + K_2B_2$.

Now, let us design a control η_1 to stabilize the auxiliary system (28).

A. Stabilization of x_2

Let us consider as Lyapunov function candidate $V = x_2^TPx_2 + \xi_2^TR\xi_2$, were P and R are two symmetric positive definite matrices satisfying $PA_{2s} + A_{2s}^TP = -I$ and $R\overline{F}_2 + \overline{F}_2^TR = -I$. It can be verified that for $\eta_2 = -R^{-1}B_2^TPx_2 - \eta - H_2\hat{w} - \Gamma_2\dot{\hat{w}}$, the derivative of Lyapunov function yields to $\dot{V} \le -\|x_2\|^2 - \|\xi_2\|^2$. In the ideal case, x_2, ξ_2 will be exponentially stable.

Since $\xi_2 = 0$, the next holds

$$\begin{aligned} \eta_2 &= -K_2x_2 - \Gamma_2\hat{w} \\ \dot{x}_2 &= (A_2 - B_2K_2)x_2 \end{aligned}$$

Matrix $(A_2 - B_2K_2)$ is designed Hurwitz. Finally, the coordinate x_2 is exponentially stable, i.e. there exist constants a_3, $a_4 > 0$ such that

$$\|x_2(t)\| \le a_3\|x_2(0)\| \exp^{-a_4 t} \quad \forall t > t_\sigma$$

The remaining trajectories will be bounded $\forall t > t_\sigma$

$$\begin{aligned} \|\eta_2(t)\| &\le \|K_1x_2(t)\| + \|\Gamma_2\| w^+ \\ \|\eta_1(t)\| &\le \|\varphi x_1\| + \|\Gamma_2 + H_2\| w^+ \end{aligned}$$

From the above analysis, the sliding surface will be

$$\begin{aligned} s &= \left(R^{-1}B_2^T P + K_2\right) x_2 + \ \eta_2 + \eta_1 + (H_2 + \Gamma_2)\,\hat{w} + \Gamma_2 \dot{\hat{w}} \\ &= \varphi x_1 + \eta_1 + (H_2 + \Gamma_2)\,\hat{w} + \Gamma_2 \dot{\hat{w}} \end{aligned}$$

where $\varphi = \left[\begin{array}{cc} \left(R^{-1}B_2^T P + K_2\right) & I \end{array} \right] T_1^{-1}$. The sliding mode dynamics yields

$$\dot{s} = \Phi x + (H_2 + \Gamma_2)\,\dot{w} + \Gamma_2 \ddot{\hat{w}} + u$$

where $\Phi = \left[\begin{array}{cc} \varphi A_1 + A_3 & \varphi A_2 + A_4 \end{array} \right] T^{-1}$. Choosing $V = \frac{s^T s}{2}$ and differentiating, it can be verified that

$$\begin{aligned} \dot{V}(s) &\leq -\|s\|\left(\rho(x) - \|\Phi\|\,\|x\| - \phi\right) \\ \rho(x) &> \|\Phi\|\,\|x\| + \phi + \zeta \end{aligned}$$

where $\zeta > 0$ y $\phi := \|H_2 + 2\Gamma_2\|\, w^+$.

The previous procedure can be iterated since, for some ℓ, x_ℓ can be stabilized, i.e. until the condition $D_\ell \in span\,(B_\ell)$ is satisfied. If $rank\left(B_{(\ell-1)}\right) = m$, there exists a nonsingular transformation $T_{(\ell-1)} \in \Re^{(n-\ell m)\times(n-(\ell-1)m)}$ such that

$$T_{(\ell-1)} B_{(\ell-1)} = \left[\begin{array}{c} 0 \\ I_m \end{array} \right].$$

Applying the coordinate transformation $T_{(\ell-1)} x_{(\ell-1)} = \left[\begin{array}{cc} x_\ell^T & \eta_\ell^T \end{array} \right]^T$ yields to

$$\begin{aligned} \dot{x}_\ell(t) &= A_\ell x_\ell(t) + B_\ell \eta_\ell(t) + D_\ell w(t) \\ \dot{\eta}_\ell(t) &= E_\ell x_\ell(t) + F_\ell \eta_\ell(t) + H_\ell w(t) + \eta_{(\ell-1)}(t) \end{aligned}$$

If $D_\ell \in span\,(B_\ell)$ there is a matrix Γ_ℓ such that $B_\ell \Gamma_\ell = D_\ell$. The vector x_ℓ will be stabilized by designing $\eta_{(\ell-1)}$ as a linear combination of $\left\{ x, \hat{w}, \ldots, \hat{w}^{(\ell)} \right\}$.

VI. Example

Consider the system

$$\begin{aligned} \dot{x} &= \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] x + \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] u + \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] w \\ y &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \end{aligned}$$

where the unknown input $w(t) = \sin 2t + 0.5$ is a smooth signal.

Observer Design. It could be verified that the triplet (A, D, C) is strongly observable. The observer parameters are $k = 2$, $M_1 = C$,

$$M_2 = \left[\begin{array}{ccc} 0.2072 & -12 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right], \quad J_2 = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right].$$

A $3th$ order HOSM differentiator is designed to recover $\left\{ x, \hat{w}, \dot{\hat{w}} \right\}$. The gains of the differentiator are $\Lambda = 1500$, $\left[\begin{array}{cccc} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{array} \right] = \left[\begin{array}{cccc} 5 & 3 & 1.5 & 1.1 \end{array} \right]$. Here, we point out that a $2th$ order HOSM would be enough to recover the variables $\left\{ x, \hat{w}, \dot{\hat{w}} \right\}$, nevertheless increasing the order of the HOSM will increase the accuracy of the estimated state and the identified variables [7].

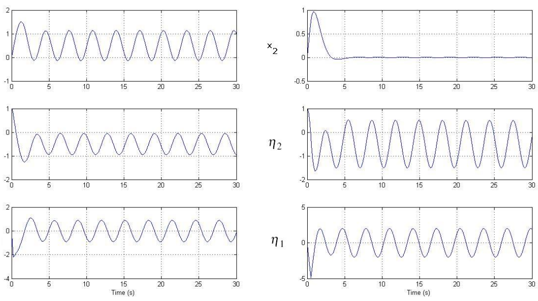

Fig. 1. Left column shows the system states without compensation $D_1 \notin \mathtt{span}B_1$. Right column shows the stabilization of the state x_2 when a nested backward strategy is applied.

Control Design. Transforming to regular form $Tx = \left[\begin{array}{cc} x_1 & \eta_1 \end{array} \right]^T$.

$$A_1 = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \quad B_1 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \quad D_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

The system do not satisfy $D_1 \in span\{B_1\}$. For the subsystem $x_1 = \left[\begin{array}{cc} x_2 & \eta_2 \end{array} \right]^T$

$$\begin{aligned} \dot{x}_2 &= \eta_2 + w \\ \dot{\eta}_2 &= \eta_1 \end{aligned}$$

Using η_2 as the control, with $\xi_2 = \eta_2 + \hat{w} + 2x_2$

$$\begin{aligned} \dot{x}_2 &= -2x_2 + \xi_2 \\ \dot{\xi}_2 &= -x_2 + \xi_2 + \dot{\hat{w}} + \eta_2 \end{aligned}$$

taking $V = \frac{x_2^2 + \xi_2^2}{2}$, with $\xi_2 = -4\xi_2 - x_2 - \dot{\hat{w}}$, $\dot{V} < 0$ can be guaranteed. The sliding surface

$$s = 5x_2 + 4\eta_2 + \eta_1 + 4\hat{w} + \dot{\hat{w}}$$

VII. Conclusions

The paper considers the regulation problem of linear time invariant systems with unmatched perturbations.

The proposed methodology exploits a high order sliding mode observer, which guarantees theoretically exact state and perturbation estimation. Based on the exact reconstruction of the unknown inputs, the unmatched uncertainties can be compensated through a sliding mode control. The compensation strategy of the proposed controller relies on the identified perturbation values.

The performed analysis shows that the possibility to compensate the total non-actuated states through the sliding surface depends on the system structure. When the system satisfies quite restrictive assumptions, the method ensures exact regulation of the unmatched states.

In order to deal with more general case it is proposed a nested backward strategy to design the sliding surface, which allows to compensate the unmatched uncertainties and to stabilize some of the non-actuated state components, while all the remaining states are maintained bounded.

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APPENDIX

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