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Results on stability, stabilization and passivity of fractional linear systems

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***Abstract*— In this paper based in new definitions of passivity, some facts on preservation of stability and stabilization are inferred for fractional-order linear dynamical systems, using the substitution of the variable s by the function s^{α} with $0 < \alpha < 1$, for commensurate frectional order systems. The main results are on preservation of "quasi-passivity", "quasi-strictly passivity", $\mathcal{H}_{\infty}$-norm and coprime factorizations.**

Keywords: Linear systems, fractional-order derivatives, stabilization passivity.

I. Introduction

In (Petráš, 1999)—and reference therein—it is mention that "real objects" are generally fractional in order (Bagley y Torvik, 1984), however, for many of them the fractionality is very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy RC line or diffusion of the heat into a semi-infinite solid, where heat flow is equal to the half-derivative of temperature (Podlubny *et al.*, 1995), also it has been proposed in (Metzler y Compte, 1999) a model for the description of the anomalous diffusion in amorphous materials by a fractional-order differential equation. Recently, there has been an effort to take into account the real order of dynamic systems (Podlubny, 1999). It is also mention in (Petráš, 1999), that performance improvements with respect to standard PID controlles have been found when implementations of PID controllers with a fractional-order derivative and integral part are used (Oustaloup, 1995). As we can see this are has begun to prove a rich and interesting field of research.

Generally, there are three main advantages for introducing fractional-order controllers to control design: i) more adequate modeling of dynamic systems, ii) more clear-cut robust control design, and iii) reasonable implementation by approximation (Ma y Hori, 2004). Real systems are more or less affected by non-integer orders. Systems such as materials having memory and hereditary effects and dynamical processes including mass diffusion and heat conduction, etc (Wang y Li, 2005), need fractional-order models to obtain more precise mathematical models and control performances.

In this paper based in the definition of stability of (Matignon, 1996; Matignon, 1998), and new basic definitions of passive and bounded systems some facts on preservation of stability and stabilization are inferred for fractional-order linear dynamical systems. Starting with a single-input single-output (SISO) linear systems and after the substitution of variable s by the function s^{α} with $0 < \alpha < 1$, several new properties for fractional-order linear dynamical systems are presented. The main results are on preservation of "quasi-passivity", "quasi-strictly passivity", or equivalently "quasi-positive real functions", and "quasi-strictly positive real functions" respectively, $\mathcal{H}_{\infty}-$norm and coprime factorizations. As far as we know, there is not exist results on these problems and "quasi-passivity", "quasi-strictly passivity" are new definitions of an specific class of passivity for fractional-order systems.

II. Preliminaries

We present the notation, definitions and technical lemmas use through out the presentation which can be found in (Lozano *et al.*, 2000; Matignon, 1996; Matignon, 1998; Podlubny, 1999).

Let $\mathbb{R}$ be the field of real numbers; $\mathbb{C}$ the complex plane, $\operatorname{Im}\mathbb{C}$ the imaginary axis and $\mathbb{C}^{+}$ the open right-half-plane of the complex; $R[s]$ the ring of real polynomials and $R(s)$ the field of real rational functions. Let $\mathcal{L}_{\infty}$ be a Banach space of functions that are essentially bounded on $\operatorname{Im}\mathbb{C}$. Let $\mathcal{H}_{\infty}$ be the closed subspace of $\mathcal{L}_{\infty}$ which consists of all functions analytic and bounded in $\mathbb{C}^{+}$ and $\mathcal{RH}_{\infty}$ is the subspace of $\mathcal{H}_{\infty}$ which consists of all proper and real rational stable transfer functions. Let $\mathcal{RH}_{\infty}(s^{\alpha})$ be the space of transfer functions from $\mathcal{RH}_{\infty}$ evaluated in s^{α} where $0 < \alpha < 1$.

The fractional calculus is a generalization of integration and derivation to non-integer order operators. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibnitz and

L'Hospital in 1695, (Oldham y Spanier, 2002). The first step is to present a generalization for the differential and integral operators, that is, a fundamental operator ${}_aD_t^\alpha$ such that:

Definition 1

1) *The fractional-order differential arithmetic operator is*

$$ {}_aD_t^\alpha \triangleq \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \operatorname{Re}\alpha > 0 \\ 1 & \operatorname{Re}\alpha = 0 \\ \displaystyle\int_a^t (d\tau)^{-\alpha} & \operatorname{Re}\alpha < 0 \end{cases} $$

where α is a complex number and is considered as real number in this paper.

2) *The Riemann-Liouville (RL) definition for general fractional differintegral operator is given as*

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

where $0 \leq n-1 < \alpha < n$ and Γ is defined by

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. $$

The formula for the Laplace transform of the RL-fractional derivative has the form:

$$ \int_0^\infty e^{-st}\, {}_0D_t^\alpha f(t) dt = $$
$$ s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \; {}_0D_t^{\alpha-k-1} f(t)\big|_{t=0}. $$

Definition 2

1) *A function $G(s)$ (rational or irrational) of complex variable $s = \sigma + j\omega$ is positive real (PR) if*
   a) *$G(s)$ is analytic in $\operatorname{Re} s > 0$ (critically stable),*
   b) *$G(s)$ is real for s real,*
   c) *$\operatorname{Re} G(s) \geq 0$ for all $\operatorname{Re} s > 0$.*
2) *If a rational or irrational complex function $G(s)$ satisfies subitems 1a) and 1c), it is called a quasi-positive real function (q-PR).*
3) *A rational complex function $G(s)$, with $s = \sigma + j\omega$, is strictly positive real of zero relative degree (SPR0) if*
   a) *$G(s)$ is analytic in $\operatorname{Re} s \geq 0$ (stable),*
   b) *$\operatorname{Re} G(jw) > 0$ for all $w \in R$.*
4) *A complex function with variable $s = \sigma + j\omega$ defined as*

$$ G(s^\alpha) = \frac{Q(s^\alpha)}{P(s^\alpha)} = \frac{\sum_{k=0}^m a_k (s^\alpha)^k}{\sum_{k=0}^n b_k (s^\alpha)^k} $$

is a quasi-strictly positive real of zero relative degree ($m = n$) function (q-SPR0), if this satisfies the item 3).

In Definition 2 items 2) and 4), we introduce new definitions based in the classical concepts of passivity for systems of order-fractional in a first approximation.

To test Definition 2 subitem 1a), we note that in the case of commensurate order systems it holds that, $\alpha_k = \alpha k$, $\beta_k = \beta k$, $(0 < \alpha < 1)$, for all $k \in \mathbb{Z}$, and the transfer function has the following form:

$$ G(s^\alpha) = \frac{\sum_{k=0}^m a_k (s^{\alpha_k})}{\sum_{k=0}^n b_k (s^{\alpha_k})} = \frac{\sum_{k=0}^m a_k (s^\alpha)^k}{\sum_{k=0}^n b_k (s^\alpha)^k} = \frac{Q(s^\alpha)}{P(s^\alpha)} $$

When $n \geq m$ the function $G(s)$ becomes a proper rational function in the complex variable s^α. For this class of systems the stability condition can be expressed as follows (Matignon, 1996; Matignon, 1998):

Lemma 3 *A commensurate order system described by a transfer function $G(s^\alpha) = \frac{Q(s^\alpha)}{P(s^\alpha)}$, where $0 < \alpha < 1$, is stable (bounded-input bounded-output (BIBO) in the sense of (Matignon, 1998)), if $|\arg(u_i)| > \alpha\frac{\pi}{2}$, for each i-th root of $P(u)$ denoted by u_i.*

The following facts, presented without their proofs, are immediate consequence of the Lemma 3. However, as far as we know they have not been reported in the literature.

Fact 4 If $G(s) = \frac{Q(s)}{P(s)}$ is a rational stable transfer function, then $G(s^\alpha)$ is BIBO stable in the sense of (Matignon, 1998), for $0 < \alpha < 1$.

Fact 5 If the controller $C(s)$ stabilizes to the system $G(s)$, then the controller $C(s^\alpha)$ stabilizes to the system $G(s^\alpha)$ in the sense of (Matignon, 1998), for $0 < \alpha < 1$.

Fact 6 If $G(s)$ has some positive real root, then $G(s^\alpha)$ is not BIBO stable in the sense of (Matignon, 1998), for $0 < \alpha < 1$.

Fact 7 If $G(s)$ has some poles s_i such that $0 < \operatorname{Re}[s_i]$, but any other pole with $\operatorname{Re}[s_i] = 0$. Then there always exists $\lambda \in (0,1)$ such that $G(s^\lambda)$ is BIBO stable in the sense of (Matignon, 1998).

III. Results

The results on preservation of passivity, properties of passive systems, preservation of $\mathcal{H}_\infty-$norm and coprime factorizations are given in this section.

Proposition 8

1) *If $G(s)$ is a PR rational transfer function, then $G(s^\alpha)$ is q-PR transfer function for $0 < \alpha < 1$.*
2) *If $G(s)$ is a SPR0 rational transfer function, then $G(s^\alpha)$ is q-SPR0 transfer function for $0 < \alpha < 1$.*

The Proposition 8 is a result on preservation of passivity of rational transfer functions in RH_∞ to fractional linear transfer functions in $RH_\infty(s^\alpha)$.

Corollary 9 *For $0<\alpha<1$.*

1) *If $G(s^\alpha)$ is either q-SPR0 or q-PR transfer function, then $G^{-1}(s^\alpha)$ is also either q-SPR0 or q-PR transfer function, respectively.*
2) *If $G_1(s^\alpha)$ and $G_2(s^\alpha)$ are q-SPR0 or q-PR transfer functions, then $\lambda p_1(s^\alpha)+\mu p_2(s^\alpha)$ is either q-SPR0 or q-PR transfer function, respectively for $\lambda,\mu\geq 0$, eliminating the case $\lambda=\mu=0$.*
3) *If $G_1(s^\alpha)$ and $G_2(s^\alpha)$ are q-SPR0 or q-PR transfer functions, then*
$$\frac{G_1(s^\alpha)}{1+G_1(s^\alpha)G_2(s^\alpha)}$$
is either q-SPR0 or q-PR transfer function, respectively.

Corollary 10 *If $G_1(s^\alpha)$ is q-PR transfer function and $G_2(s^\alpha)$ is q-SPR0, then*
$$\frac{G_1(s^\alpha)}{1+G_1(s^\alpha)G_2(s^\alpha)}$$
is BIBO stable in the sense of (Matignon, 1998) or equivalently see Lemma 3, for $0<\alpha<1$.

The Corollary 9 and Corollary 10 are properties well-known for passive linear systems, that are valid for this class of fractional linear systems, also.

Proposition 11 *If $\|G(s)\|_\infty=\gamma$, then $\|G(s^\alpha)\|_\infty\leq\gamma$ for $0<\alpha<1$.*

Corollary 12 *The function $F:\mathcal{RH}_\infty\rightarrow\mathcal{RH}_\infty(s^\alpha)$ defined as $F(G(s))=G(s^\alpha)$ is an homomorphism non surjective, between subspaces of $\mathcal{H}_\infty$, where $\mathcal{RH}_\infty(s^\alpha)$ is the subspace of $\mathcal{H}_\infty$, which consists of all proper and real rational stable transfer functions in the variable s^α with $0<\alpha<1$.*

The Proposition 11 and the Corollary 12 are structural properties, the first preserves the $\mathcal{H}_\infty-$norm of $\mathcal{RH}_\infty$ to $\mathcal{RH}_\infty(s^\alpha)$. The second preserves the algebraic structure of $\mathcal{RH}_\infty$ to $\mathcal{RH}_\infty(s^\alpha)$.

Corollary 13 *Let $(N_G(s),D_G(s))$ be a coprime factorization of $G(s)\in\mathcal{RH}_\infty$, then $(N_G(s^\alpha),D_G(s^\alpha))$ is a coprime factorization of $G(s^\alpha)\in\mathcal{RH}_\infty(s^\alpha)$, for $0<\alpha<1$. Where $(N_G(s^\alpha),D_G(s^\alpha))$ is a coprime factorization of $G(s^\alpha)$, if $N_G(s^\alpha),D_G(s^\alpha)\in\mathcal{RH}_\infty(s^\alpha)$ and there exist $X_G(s^\alpha),Y_G(s^\alpha)\in\mathcal{RH}_\infty(s^\alpha)$ such that*
$$X_G(s^\alpha)N_G(s^\alpha)+Y_G(s^\alpha)D_G(s^\alpha)=u(s^\alpha),$$
where $u(s^\alpha)$ is an unit in $\mathcal{RH}_\infty(s^\alpha)$ i.e., there exists $u^{-1}(s^\alpha)\in\mathcal{RH}_\infty(s^\alpha)$ such that
$$u(s^\alpha)u^{-1}(s^\alpha)=u^{-1}(s^\alpha)u(s^\alpha)=1.$$

Corollary 13 preserves coprime factorizations of $\mathcal{RH}_\infty$ to $\mathcal{RH}_\infty(s^\alpha)$, and establishes a fundamental property to extend the approach for design of controllers via coprime factorizations in $\mathcal{RH}_\infty$ to design of controllers in $\mathcal{RH}_\infty(s^\alpha)$.

IV. Examples

In this section some examples are presented.

Example 14 *Since the* PID_1 *given by*
$$C_1(s)=K_p+\frac{K_I}{s}+K_Ds$$
is a PR transfer function for $K_p,K_I,K_D>0$, then by Proposition 8, the $\mathrm{PI}^\alpha\mathrm{D}_1^\alpha$ *given by*
$$C_1(s^\alpha)=K_p+\frac{K_I}{s^\alpha}+K_Ds^\alpha$$
is a q-PR transfer function for $0<\alpha<1$.

Example 15 *Since the* PID_2 *given by*
$$C_2(s)=K_p+\frac{K_I}{1+Ts}+\frac{K_Ds}{1+Ts}$$
is a SPR0 transfer function for $K_p,K_I,K_D,T>0$, then by Proposition 8, the $\mathrm{PI}^\alpha\mathrm{D}_2^\alpha$ *given by*
$$C_2(s^\alpha)=K_p+\frac{K_I}{1+Ts^\alpha}+\frac{K_Ds^\alpha}{1+Ts^\alpha}$$
is a q-SPR0 transfer function for $0<\alpha<1$.

In the last two example, the passivity of the controllers $C_1(s)$ and $C_2(s)$ is preserved as quai-passivity in the fractional-order controllers $C_1(s^\alpha)$ and $C_2(s^\alpha)$, respectively.

Example 16 *The PI controller*
$$C(s)=\frac{5(s+1)}{s}$$
stabilizes the system
$$G(s)=\frac{2(s+1)}{s^2+2s-3}.$$
Then by Fact 5, the PI^α *given by*
$$C(s^\alpha)=\frac{5(s^\alpha+1)}{s^\alpha}$$
stabilizes the system
$$G(s^\alpha)=\frac{2(s^\alpha+1)}{s^{2\alpha}+2s^\alpha-3}$$
for $0<\alpha<1$.

Example 17 *It was presented in (Metzler y Compte, 1999) an example of a fractional Cattaneo's equation, which models the anomalous diffusion in amorphous materials by the following fractional-order transfer function*
$$F(s)=\frac{1}{1+(\tau s)^\gamma},\quad \gamma\in(0,1),\quad \tau>0.$$

It was shown in (Álvarez-Ramírez et al.*, to appear), that this fractional Cattaneo's equation is a q-PR transfer function for* $\gamma = 0.5$. *Now using the Proposition 8 and based on the fact that* $\frac{1}{1+\tau s}$ *is a PR transfer function, we can prove that* $F(s)$ *is a q-PR transfer function for* $\gamma \in (0,1)$. *In this form, it is apparent that the fractional order filter* $F(s)$ *induces weakly passive diffusion dynamics. These observations show that the fractional Cattaneos equation could also belong to the class of admissible diffusion relaxation dynamics [24].*

V. Conclusions

In this paper, considering the interest in order-fractional linear dynamical systems, we intent give a first step toward one theory on preservation of fundamental properties of the Euclidian domain $\mathcal{RH}_\infty$ for order-fractional linear systems, in particularly $\mathcal{RH}_\infty(s^\alpha)$. For instance, results on preservation of passivity, stability, stabilization, coprime factorizations and $\mathcal{H}_\infty-$norm were presented. Based in several examples, as the fractional Cattaneo's equation from Example 17, we believe there exist a great potential for possible applications, using preservation of fundamental properties in systems. In consequence, several techniques of the classical linear control can be extrapolated to specific families of fractional linear systems.

Based in several examples as the fractional Cattaneo's equation of the example 17, we believe there exists a great potential of possible applications, using preservation of fundamental properties in linear systems to fractional linear systems. As far as we know, these results are not reported in the literature.

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