A Fractional Approach to Car-following Pipe's Model

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Abstract: Car-following models are representations of two driving cars in a single lane and most of them are expressed as ordinary differential equations. A model suggested by L. A. Pipes is a first degree differential equation and it solves for the change in the velocity of the car that is behind. The velocity of the car in front works as a disturbance for its follower. In this paper we compare this classical model with a fractional order modification of the derivative that appears on it. Real data obtained directly of instrumented vehicles are used in order to test the level of approach of both models by means of simulations. The results indicate that there is an advantage when using the fractional order modification.

Keywords: Car-following Models; Pipes' Model; Conformal Fractional Derivative.

1. INTRODUCTION

Car-following models are a group of expressions that represent a system of two vehicles that move in a single lane, assuming that overpassing does not happen (Olstam & Tapani, 2004). This implies that there is a vehicle in front, usually called leader (L), and a vehicle behind, usually identified as follower (f). The leader car is free to accelerate or decelerate, but the follower must respond to these changes in order to drive as fast as it can, but having the caution needed for not reaching and hitting the leader.

Such models are known since more than sixty years (Ioannou et al., 2008). They can be divided into three main families (van Wageningen-Kessels et al., 2015):

- Safe-distance models
- Stimulus-response models
- Action-point models

Pipes' model (Pipes, 1953) is consider one of those models that pertains to the first family. It is also known as the oldest of the car-following models. It works pretty well in most of the cases. It represents very well real situations in spite of its simplicity (Rosas-Jaimes et al., 2013). Of course, in order to achieve a better approach to the whole range of behaviors relating these phenomena, other researchers have proposed other models that have increased precision and fitness to reality (Helly, 1959; Gazis et al., 1961; Bando et al., 1995; Treiber et al. , 2000). But, as expected, they also have taken them to higher levels of complexity, involving nonlinear dynamics, cellular automata and other approaches, trying to reflect reality in a more accurate way (Olstam & Tapani, 2004; Ioannou et al., 2008; van Wageningen-Kessels et al., 2015).

Pipes' model is as simple as the differential equation of first order (1).

$$\frac{dv_f(t)}{dt} = \lambda[v_L(t) - v_f(t)] \tag{1}$$

where $v_L(t)$ is the leader car's velocity, $v_f(t)$ stands for the follower car's velocity, and λ is a sensitivity parameter that relates the mood or driving habits of the driver in the follower car and that must be calibrated in such a way that values near to 0 reflect a less reactive driver, while values of λ tending to 1 reflects a more reactive one (Kesting & Treiber, 2008).

Like many dynamical phenomena in which time t is the independent variable, differential equations like (1) are traditionally expected to describe the way in which natural and artificial systems behave. The main feature of those equations in most of the situations is that they have an integer order. In this present case, Pipes' model has a 1-order differential equation. If α is identified with the order of a differential equation, then for Pipes' model $\alpha = 1$. In the case of a simple pendulum, for example, it is well known that a 2-order differential equation describes its dynamics, and then $\alpha = 2$. Many complex dynamical systems are expressed through an n-order differential equation, or in their equivalent forms, that are directly related with integer-order equations (Teschl, 2012).

However, differential equations with a fractional order have been considered since many years and in a more frequent quantity (Uchaikin , 2013a). This type of equations deals with $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{R}$ instead of $\alpha \in \mathbb{N}$ as their order (Kilbas, 2006).

"Fractional calculus is a topic being more than 300 years old. The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695 where half-order derivative was mentioned. In a correspondence between Johann Bernoulli and Leibniz in 1695, Leibniz mentioned the derivative of general order. In 1730 the subject of fractional calculus did not escape Euler's attention. J. L. Lagrange in 1772 contributed to fractional calculus indirectly, when he developed the law of exponents for differential operators. In 1812, P. S. Laplace defined the fractional derivative by means of integral and in 1819 S. F. Lacroix mentioned a derivative of arbitrary order in his 700-page long text, followed by J. B. J. Fourier in 1822, who mentioned the derivative of arbitrary order. The first use of fractional operations was made by N. H. Abel in 1823 in the solution of tautochrome problem. J. Liouville made the first major study of fractional calculus in 1832, where he applied his definitions to problems in theory. In 1867, A. K. Grünwald worked on the fractional operations. G. F. B. Riemann developed the theory of fractional integration during his school days and published his paper in 1892. A. V. Letnikov wrote several papers on this topic from 1868 to 1872. Oliver Heaviside published a collection of papers in 1892, where he showed the so-called Heaviside operational calculus concerned with linear generalized operators. In the period of 1900 to 1970 the principal contributors to the subject of fractional calculus were, for example, H. H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, etc. From 1970 to the present, they are for instance J. Spanier, K. B. Oldham, B. Ross, K. Nishimoto, O. Marichev, A. Kilbas, H. M. Srivastava, R. Bagley, K. S. Miller, M. Caputo, I. Podlubny, and many others" (Miller & Ross, 1993; Cafagna, 2007) as cited by (Petráš, 2011).

Fractional-order derivatives, and by extension, fractional-order differential equations are gaining an interesting status in an exponential manner (Sabatier et al., 2007). For the present communication, we have modified the integer-order Pipes' model into a fractional-order one, with the intention to study its behavior and some of the possible consequences.

However, there are more than one fractional-order derivative definition, and by consequence, there are lots of possibilities to achieve such a modification. Further, not all those classical properties that follows the integer-order derivative are respected by most of the fractional-order derivatives (Katugampola, 2011).

In order to achieve a valid and precise comparisson between classical and fractional-order approaches, we have selected the *conformal fractional derivative* proposed by Khalil et al. (2014), which follows all of the main classical properties: derivative of a constant value, power rule, product rule, quotient rule, chain rule and linearity. To perform proper simulations, we are using Euler's numerical method (Mathews & Fink, 2001) taking into account the integer-order derivative included in our Pipes' modification, and an analogous method that takes into account the fractional modification.

We have organized this paper in the following way: next section is dedicated to present some useful definitions for our purposes, including the modification made to Pipes' model in order to write it down as a fractional-order differential equation. Because Pipes' model is capable to be used with real data, Section 3 describes the acquirement of measurements from instrumented cars used in a set of car-following experiments. These results will be used in Section 4 where some simulations are shown, comparing Pipes' classical model with Pipes' fractional-order model. At the end, some conclusions are listed about our findings.

2. DEFINITIONS

Definition 2.1. Let $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. Thus the definition (Khalil et al., 2014) of the derivative of f at t is

$$\frac{df}{dt} = \lim_{\epsilon \to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon} \tag{2}$$

Regarding this, it is possible to notice that

$$\frac{dt^p}{dt} = pt^{p-1} \tag{3}$$

with $p \in \mathbb{N}$ as an integer power of t.

It is evident that Definition 2.1 is that which corresponds to the pretty well known 1-order derivative. This derivative can be treated as an operator D^n, where $n \in \mathbb{N}$ is the order of the derivative.

It is also known that this derivative complies with the following properties for $n = 1$, $f = f(t)$ and $g = g(t)$:

(1) Derivative of a constant value: $D^1(\lambda) = 0$, with $f(t) = \lambda$
(2) Power rule: $D^1(t^p) = pt^{p-1}$
(3) Product rule: $D^1(fg) = fD^1(g) + gD^1(f)$
(4) Quotient rule: $D^1(\frac{f}{g}) = \frac{gD^1(f) - fD^1(g)}{g^2}$
(5) Chain rule: $D^1(f \circ g) = D^1(f)(g(t))D^1(g(t))$
(6) Linearity: $D^1(af + bg) = aD^1(f) + bD^1(g)$ for all $a, b \in \mathbb{R}$

When defining a fractional-order derivative, the main intention is to modify $n \in \mathbb{N}$ to become $n = \alpha \in \mathbb{Q}$ or $n = \alpha \in \mathbb{R}$.

Khalil et al. (2014) define and apply a fractional-derivative that also complies with these same properties, in opposition to most of the other fractional-derivatives defined until now, through an expression like that in Definition 2.2:

Definition 2.2. Let $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the *conformable fractional derivative* (Khalil et al., 2014) of f of order α is defined by:

$$D^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{4}$$

for $\alpha \in (0, 1)$.

Sometimes, it is convenient to write $f^{(\alpha)}(t)$ for $D^\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then it is possible to say that f is α-differentiable (Khalil et al., 2014).

This α-order derivative has the following properties (Khalil et al., 2014) for $n = \alpha$, $f = f(t)$ and $g = g(t)$:

(1) Derivative of a constant value: $D^\alpha(\lambda) = 0$, with $f(t) = \lambda$

(2) Power rule: $D^\alpha(t^p) = pt^{p-\alpha}$

(3) Product rule: $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$

(4) Quotient rule: $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$

(5) Chain rule: $D^\alpha(f \circ g) = D^\alpha(f)(g(t))D^\alpha(g(t))$

(6) Linearity: $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$ for all $a, b \in \mathbb{R}$

By looking at these properties, it is possible to notice an important and main feature of the conformal fractional derivative, i.e. if α is equaled to 1, then all the properties of the classical derivative are obatined.

Definition 2.3. Consider the following fractional differential equation:

$$y^\alpha(t) = f(t, y) \tag{5}$$

with $y_0 = y(0)$

Let's substitute the conformal fractional derivative definition in (5) to obtain the following expression:

$$\lim_{\epsilon \to 0} \frac{y(t + \epsilon t^{1-\alpha}) - y(t)}{\epsilon} = f(t, y) \tag{6}$$

If we consider ϵ only small enough as for doing without the limit notation then

$$\frac{y(t + \epsilon t^{1-\alpha}) - y(t)}{\epsilon} = f(t, y) \tag{7}$$

Let $h = \epsilon t^{1-\alpha}$. Then $\epsilon = h/(t^{1-\alpha}) = ht^{\alpha-1}$ and then Equation (7) can be written as in (8)

$$\frac{y(t + h) - y(t)}{ht^{\alpha-1}} = f(t, y) \tag{8}$$

Solving for $y(t + h)$ from this last expression results in

$$y(t + h) = y(t) + f(t, y)ht^{\alpha-1} \tag{9}$$

Hence, Equation (9) is Euler's method to solve numerically (5) in close reference to classical Euler's method to solve 1-order differential equations (Mathews & Fink, 2001).

Definition 2.4. From Equation (1) and from Definition 2.3, it is possible to obtain a fractional-order equation as follows

$$D^\alpha v_f(t) = \lambda[v_L(t) - v_f(t)] \tag{10}$$

called the fractional-order Pipes' model or Pipes' fractional model.

3. EXPERIMENTAL DATA

In order to conduct a series of car-following experiments as attached as possible to the reality, six cars were used to run in a close circuit by pairs (see Figure 1).

Table 1 shows information about those cars. As can be seen, the pairs have been formed taken into account those cars with the closest features, as the car company, brand and year. Another important feature to notice not shown in Table 1 is that all these cars had manual gear transmission.

Fig. 1. Two-car Experiments

Table 1. Vehicles driven and drivers data

Vehicle				Driver	
Car Company	Brand	Vehicle	Year	Gender	Age
Volkswagen	Jetta	1	2003	Male	52
		2	2004	Male	23
Nissan	Sentra	3	2008	Male	21
		4	2013	Female	43
Toyota	Yaris	5	2010	Male	40
		6	2010	Male	52

These pairs of cars performed runnings in a circuit of about 2.5 km made up by urban streets surrounding an area that was selected for containing traffic lights and bumps, but also for the confort and safety reasons for those participants in these experiment (See Figure 2).

Of the main objectives of such experimental planification was to diminish the influence of the specifications of those cars coupling with a similar vehicle, and to increase the influence of individual drivers. In this manner, vehicles 1 and 2 were paired in such a way that one took the roll of leader and the other took the roll of follower, running in the described circuit. At the end of such a running, they changed those rolls for a second experiment.

Suitable OBD (On-Board Diagnostics) hardware and software were utilized to obtain velocity from the on-board computer of every involved car.

Four of these trials (made by vehicles 1-4 as indicated in Table 1) were performed on a Saturday morning, where moderate traffic influence in the surrounding streets was taken as part of the conditions of the experiment. Another pair of trials with vehicles 5 and 6 were performed during Sunday light traffic.

All of the drivers were asked to drive as they used to, with only two restrictions:

(1) Followers should not pass leaders.
(2) Followers should not permit any external car to be located between them and the leaders, except if safety was at risk.

Six runnings were performed, which trials were as those presented in Table 2, where the identification numbers given to cars in Table 1 indicate the role as leader or as follower.

Fig. 2. Circuit selected to be used in the set of experiments with cars

The data obtained had to be processed and refined, i.e. a synchronization-type treatment had to be performed.

Table 2. Organization of the trials

Trial	Vehicle Identification	Car Number	
		Leader	Follower
A	Jetta	1	2
B		2	1
C	Sentra	4	3
D		3	4
E	Yaris	6	5
F		5	6

A copilot was in the same car as the driver to manage the software in each computer where data had been captured. This implies the starting times to record the data differ in each pair of cars as do the stopping times. By monitoring convenient and similar times between data sets, it is possible to establish analogous time series for all the pairs of vehicles for each trial (Rosas-Jaimes et al., 2014).

4. SIMULATIONS

Equation (1) can be solved numerically for $v_f(t)$ through Euler's classical numerical method. In addition, we use Euler's fractional numerical method expressed by (9) in order to solve Pipes' fractional model (10) for $v_f(t)$ too, in order to compare them with the measured data obtained from our experiments with real vehicles. As it will be seen sooner, this method results in a good approach to numerical solution of fractional α-order differential equations.

Figure 3 depicts those velocity profiles of the follower car obtained in such trials (blue line). To obtain the numerical results of this same variable, it is necessary to use the velocity profiles of the leader $v_L(t)$, not shown in this paper, to affect the behavior of the follower, which velocity from the integer-order Pipe's model is plotted in red.

On the other hand, when the follower's velocity is calculated for the fractional-order model (10) through the method (9), a very similar, though not identical, profile is obtained (black line).

It can be noticed that these plots in Figure 3 are very close in behavior, meaning that Pipes' classical equation and Pipes' fractional model are very look-alike. In order to measure the precision of these calculations respect to the measurements, the error has been also calculated. Equation (11) calculates this quantity between measured data of the follower velocities $v_{fm}(t)$ and the results of $v_{fc}(t)$ obtained by Euler's classical numerical method, during the complete duration T of the simulation.

$$e_c = \frac{\sum |v_{fm}(t) - v_{fc}(t)|}{T} \tag{11}$$

While Equation (12) obtaines the error between the measured velocities $v_{fm}(t)$ and those follower vehicle velocity $v_{ff}(t)$ values obtained from the fractional model.

$$e_f = \frac{\sum |v_{fm}(t) - v_{ff}(t)|}{T} \tag{12}$$

Table (3) shows results related to Equations (11) and (12). Column marked as *Error (c)* corresponds to those differences found for the classical Pipes' model, while column named as *Error (f)* relates to those acumulated errors due to fractional Pipes' model.

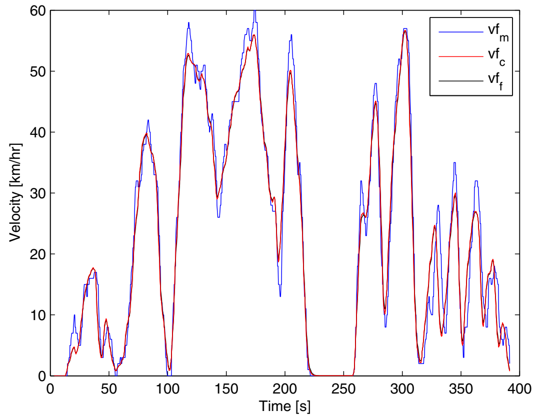
Table 3. Error (accumalted differences) for the velocity of the follower car

Trial	Vehicle Identification	α	Error (c)	Error (f)
A	Jetta	0.996	2.4032	2.3766
B		1.019	3.7287	3.4944
C	Sentra	0.980	2.6402	2.2132
D		0.996	2.0859	2.0742
E	Yaris	0.982	3.0803	2.6700
F		0.981	5.0009	4.4843

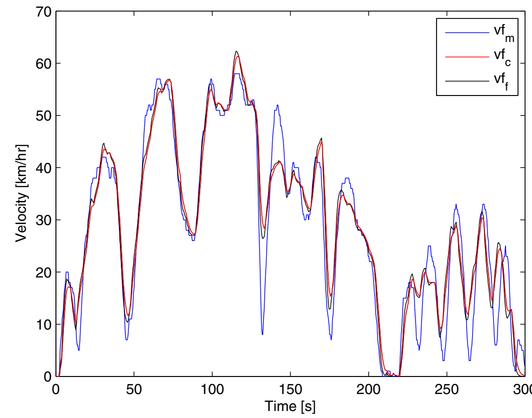
Column indicated as α is the order used in each case for the fractional model (of course, for Pipes' classical model $\alpha = 1 \in \mathbb{N}$). It is possible to note that this value is different for each case indentified in each row of Table 3. These values of α were adjusted for each simulation in order to found the minimum error as can be noticed in the last column of Table 3.

However, another important finding is that in all the cases, errors in the last column are lower than those in the column of the differences of the classical model. This is an important observation, because, at least for the results herein shown, it constitutes an evidence that fractional-order Pipes' model fits better than 1-order Pipes' model, at least for the circunstamces described in this paper, which can be added to other experimental results that find advantages in the use of fractional-order approaches, as particle dynamics (Hilfer, 2000) or even in physical devices (Petráš, 2009).

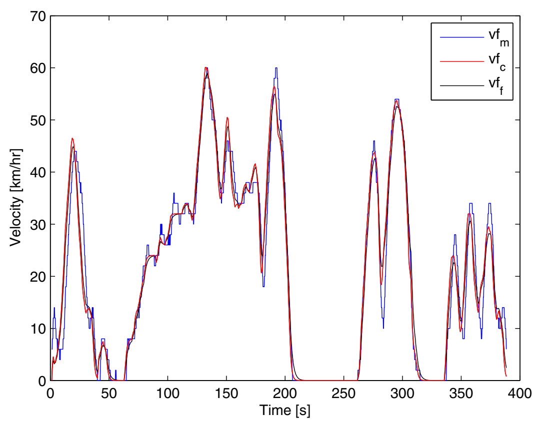
This advantageous results from fractional-order approaches can be due to the uncertainty directly related to the measurings, and not only in the nature of the formulation of the models. Our results are in close agreement with those related ones to that research which take into account, for example, fractality and other features that traditionally have been thought as part of the unpredictability nature of phenomena (Uchaikin , 2013b), as it is the case of the fine details in driving process.



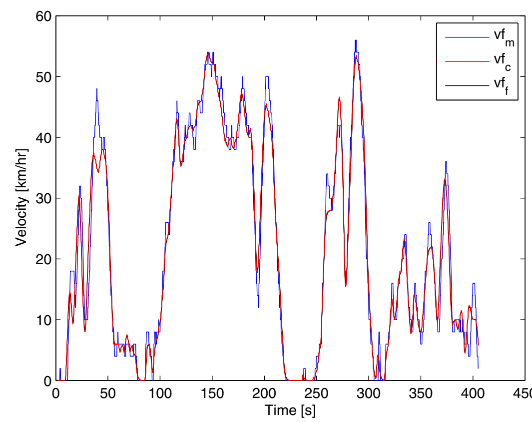
(a) Jetta trial A



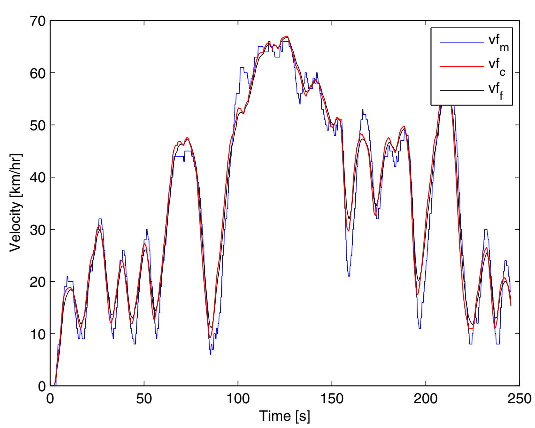
(b) Jetta trial B



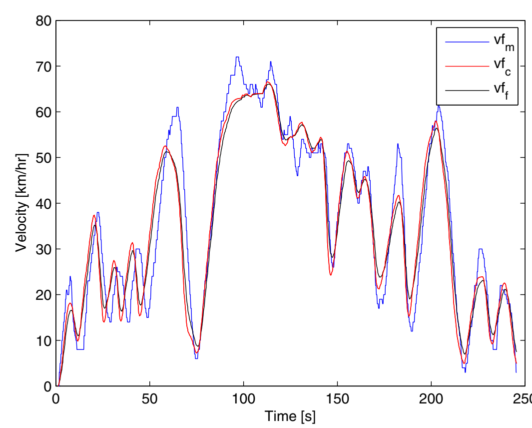
(c) Sentra trial C



(d) Sentra trial D



(e) Yaris trial E



(f) Yaris trial F

Fig. 3. Velocity Profiles obtained from experiemnts, and compared with the Classical and the Fractional Pipes' numerical solutions. Trials referred to those in Table 2

5. CONCLUSION

Pipes' model is an equation that describes very simplistically the behavior between two cars moving on a single lane. Mathematically. it is a 1-order differential equation that includes a variable $v_L(t)$ which can be treated as a disturbance to the system represented by that differential equation, affecting $v_f(t)$, the variable that is the solution of that same equation. When solved numerically, $v_f(t)$, from the classical model, is very closed in behavior if compared with the corresponding measured data of the analogous variable obtained from experiments.

However, at least from the results obtained in this work, there is evidence that if $v_f(t)$ is obtained from a fractional-order modification of the Pipes' model, those similarities increase, reducing the differences between those two sets of quantities, measured data and calculated values.

The fractional order α is not the same in each case and must be adjusted. This fact could be an indication that the differential equation under study is affected by the uncertainty of the measuring rather than the mathematical formulation of the model by itself.

Further study must be conducted in order to a better understanding of these observations herein reported.

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