Pinning control of dynamical networks with different nodes

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Abstract: We investigate the stabilization of dynamical networks of different nodes for the case where the nodes, although different, can be make passive by feedback. The so-called V-stability characterization of the network allows for a simple set of conditions for the stabilization of the network. In particular, using the pinning control strategy to stabilized the network, the V-stability characterization proves advantageous as the stabilization condition reduces to the design of feedback gains that make a matrix negative definitive. Further, we extend these results for the tracking problem in networks, by imposing a reference trajectory to the network of different nodes by pinning some nodes to the desired solution. We illustrate our results with numerical simulation of well-known benchmark systems.

Keywords: Control of Networks, Passivity, Different nodes, Pinning control.

1. INTRODUCTION

Many complex systems of interest can be modeled as networks, including the Internet, WWW, genetic networks, social networks, and many others [Newman (2010); Bornhold and Schuster (2003)]. The dynamical analysis of the different behaviors that can occur in a dynamical network have become of great interest in recent years [Wang y Chen (2003); Bocalletti et al. (2006); Arenas et al. (2008); Stefánski (2009); Su and Wang (2014)]. The dynamical analysis of dynamical networks differs from general dynamical systems in the fact that its behavior is determine by two components: The rules governing the evolution of the states of its nodes; and the information flows traveling along its links, that is isolated dynamics and network topology [Wu1995 (1995); Wang (2002); Wang2002a (2002); Wu (2007)]. This is further complicated in the case of networks with different node dynamics.

Conventionally the first thing one does when analyzing the dynamics of a system is to determine its stability. Therefore, we start by analyzing the stability properties of the nodes in isolation, then the effect of the structural characteristics of its connections are determine. To this end, we use a basic concept in nonlinear dynamical systems, *e.g.*, the energy of the nodes dynamics, then the amount of feedback control necessary for the node to be passive is determine. Then, the coupling is requiere to preserve the overall dissipative nature of the nodes once they are interconnected. Briefly, we look for a common Lyapunov function $V(x)$ for all nodes in the network, which is constructed such that for each node one can determine a passivity degree, that is, a scalar parameter indicating the extent of the effort needed to stabilize the node by feedback, which makes the derivation of $V(x)$ negative. Characterizing the nodes of the network is this way we have the so-called V-stability description of the network [Xiang and Chen (2007)]. Then, the effect of the topology can be derived from the eigenspectrum of its Laplacian matrix. In particular, for nonidentical nodes, the V-stability characterization of the network has the advantage that replaces the actual node dynamics by its degree of passivity independently of the description of the node's dynamics, under mild conditions.

Despite the conservativeness associated with Lyapunov stability analysis, using the V-stability characterization of the network conditions for synchronization and stabilization of a dynamical network can be derived. Furthermore, for networks where the nodes have a common stationary solution, the control objective can be achieved even when only a small fraction of the nodes in the network are controlled. That is, it can be controlled by pinning [Li et al. (2004); Sorrentino et al. (2007); Sorrentino (2007); Xiang et al. (2007)]. In this contribution we show that using this approach a pinning strategy can stabilize a network of non identical nodes, and even synchronize them to a desired solution.

The remainder of this paper is organized as follows: In Section 2, we define with detail the dynamical network with different nodes that will be studied and its V-

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stability characterization. The stabilization problem under pinning control is described in Section 3. While in Section 4, the results are extended to the imposition of a desired trajectory in the network. A numerical illustration of the approach proposed is presented in Section 5. Finally, the contribution is concluded with final comments in Section 6.

2. NETWORK DESCRIPTION

Consider a network of N dynamical systems, linearly and diffusively coupled given by

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + \sum_{j=1, j\neq i}^{N} c_{ij}a_{ij}\Gamma(x_j - x_i) + B_i u_i \\ y_i &= \Gamma x_i \end{aligned} \tag{1}$$

for $i = 1, 2, ..., N$, where $x_i \in \mathbf{R}^n$ is the state variable, $y_i \in \mathbf{R}^n$ is the output, and $u_i \in \mathbf{R}^m$ $(m \leq n)$ is the control input to the i-th node, respectively. The input matrix for the i-th node is $B_i \in \mathbf{R}^{n\times m}$. The inner coupling matrix $\Gamma \in \mathbf{R}^{n\times n}$ describes which states of the i and j-th nodes are coupled. $f_i(.) : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a locally Lipschitz nonlinear vector field describing the dynamics of the i-th node in isolation. $A = \{a_{ij}\} \in \mathbf{R}^{N\times N}$ and $C = \{c_{ij}\} \in \mathbf{R}^{N\times N}$ describe the connection structure and connection strength of the network ($c_{ij} \geq 0$), respectively. The connection is bidirectional, therefore $a_{ij} = a_{ji} = 1$ if the i and j-th nodes are connected, otherwise $a_{ij} = a_{ji} = 0$ $(i \neq j)$. Since the connection topology is diffusive the row and column sums $\sum_{j=1}^{N} c_{ij}a_{ij} = \sum_{j=1}^{N} c_{ji}a_{ji} = 0$ are null. Therefore, the network (1) can be rewritten as:

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j + B_i u_i, \text{ for } i = 1, 2, ..., N \\ y_i &= \Gamma x_i \end{aligned} \tag{2}$$

We assume the following about the dynamics of an each isolated node without control ($u_i = 0$):

Assumption 1: There is a common equilibrium state, $\bar{x} \in \mathbf{R}^n$, satisfying

$$f_i(\bar{x}) = 0, \text{ for } i = 1, 2, ..., N \tag{3}$$

Assumption 2: There is a continuously differentiable Lyapunov function $V(x) : D \subseteq \mathbf{R}^n \mapsto \mathbf{R}_+$ satisfying $V(\bar{x}) = 0$ with $D = \bigcup_{i=1}^{N} D_i$, $D_i = \{x_i : \|x_i - \bar{x}_i\| < \alpha\}$, $\alpha > 0$ and $\bar{x} \in D$. Such that for each node function $f_i(x_i)$, there is a scalar θ_i guaranteeing

$$\frac{\partial V(x_i)}{\partial x_i}(f_i(x_i) - \theta_i\Gamma(\bar{x} - x_i)) < 0 \tag{4}$$

for all $x_i \in D_i$, $x_i \neq \bar{x}$, $i = 1, 2, ..., N$.
The value θ_i is called the passivity degree of node i.

In following section we use these assumptions to determine conditions to stabilize a network of different nodes.

3. STABILIZATION OF DYNAMICAL NETWORKS

Under *Assumption 1*, a stationary state for the entire network is

$$x_1 = x_2 = ... = x_N = \bar{x} \tag{5}$$

To stabilize the network at $\bar{X} = [\bar{x}^\top, ..., \bar{x}^\top]^\top \in \mathbf{R}^{nN}$, we can use *Assumption 2* to define a Lyapunov function for the entire network

$$V_{all}(X) = \sum_{i=1}^{N} V(x_i) \tag{6}$$

where $X = [x_1^\top, x_2^\top, ..., x_N^\top]^\top \in \mathbf{R}^{nN}$.

The time derivative of $V_{all}(X)$ along the trajectories of (2) is given by

$$\begin{aligned} \dot{V}_{all}(X) = &\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} f_i(x_i) + \\ &\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j \end{aligned} \tag{7}$$

With out lost of generality, $\bar{X} = 0$. From *Assumption 1* one has that $V_{all}(\bar{X}) = 0$ and $\dot{V}_{all}(\bar{X}) = 0$. Then, for $X \neq \bar{X} = 0$, using (4) from *Assumption 2*, the following inequality is found

$$\begin{aligned} \dot{V}_{all}(X) < &-\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \theta_i\Gamma x_i + \\ &\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j \end{aligned} \tag{8}$$

That is, $\dot{V}_{all}(X) < M(X)$ with

$$M(X) = \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \left(-\theta_i\Gamma x_i + \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j \right) \tag{9}$$

Then, the stability of the stationary solution of the network (2) is asymptotically stable about its equilibrium point if $M(X) \leq 0$ for all $x \in \mathcal{D}$, with $\mathcal{D} = D_1 \times D_2 \times ... \times D_N \subseteq \mathbf{R}^{nN}$. Moreover, if the matrix is positive definitive the network is locally exponentially stable about its equilibrium point if, that is, $M(X) \leq -\mu_1\|X\|^2$, $\mu_2\|X\|^2 \leq \dot{V}(x) \leq \mu_3\|X\|^2$ for some constants μ_1, μ_2, $\mu_3 > 0$ for all $X \in \mathcal{D}$. The region of attraction is given by

$$\Omega = \{X : \dot{V}_{all}(X) < r\} \tag{10}$$

with $r = \inf_{X\in\mathcal{D}} \dot{V}_{all}(X)$. In the case of $\mathcal{D} = \mathbf{R}^{nN}$, the result becomes global Xiang and Chen (2007).

The result can be further simplify by restricting our attention to the case where *Assumption 2* is satisfied for all nodes in the network by a common quadratic monomial Lyapunov function

$$V(x) = x^\top Q x \tag{11}$$

with $Q \in \mathbf{R}^{n\times n}$ a symmetric and positive definite matrix.

From *Assumption 1* and network (2), we can write $\Theta = diag(\theta_1, \theta_2, ..., \theta_N) \in \mathbf{R}^{N\times N}$ and $G = \{g_{ij}\} = \{(c_{ij}a_{ij})\} \in \mathbf{R}^{N\times N}$. Then, the time derivative of the Lyapunov function (11) along the network dynamics results in

$$M(X) = \sum_{i=1}^{N} x_i^\top \left(-\theta_i Q\Gamma x_i + \sum_{j=1}^{N} c_{ij}a_{ij}Q\Gamma x_j \right)$$

which can be rewritten using the Kronecker product as

$$M(X) = X^\top(-\Theta + G) \otimes Q\Gamma X$$

which is negative definitive if the following inequalities hold [Xiang and Chen (2007)]:

$$\begin{array}{ll} Q\Gamma + \Gamma^\top Q & \geq 0 \\ -\Theta + G & \leq 0 \end{array} \tag{12}$$

To control de dynamics of the network one can use the pinning strategy [Li et al. (2004)], where only a small fraction ρN of the nodes in the network are controlled ($\rho \ll 1$) [Su and Wang (2014)]. In the context of V-stability characterization of the network the idea es to let $u_i \neq 0$ for a few nodes, then identify the effect of the introduced controllers on the passivity of the network. Explicitly, the controllers need to be chosen sufficiently many and sufficiently strong as to make the entire network asymptotically stable. That is, chose the controls such that the restrictions of (12) are satisfied.

In what follows, we restrict our attention to linear feedback controllers of the form:

$$u_i = -K_i x_i \tag{13}$$

where $K_i \in \mathbf{R}^{m\times n}$ is the control gain of node i-th, which is to be designed.

There are two type of nodes in the network: uncontrolled and controlled nodes. We assume that, as defined in *Assumption 2*, the passivity degree of each uncontrolled node is θ_i. For the nodes controlled by (13), in isolation we have:

$$\dot{x}_i = f_i(x_i) - B_i K_i x_i \tag{14}$$

The passivity degree for the controlled node is obtained as in (4), to be:

$$\frac{\partial V(x_i)}{\partial x_i}\left(f_i(x_i) - B_iK_ix_i + \theta_i\Gamma x_i + \kappa_i\Gamma x_i\right) < 0 \tag{15}$$

for all $x_i \in \mathcal{D}_i \subseteq \mathcal{D}$, $x_i \neq 0$, where the constants $\kappa_i \geq 0$ represent the effect of the feedback controller on the passivity of the node. A positive value of κ implies that control energy is necessity of make the node passive, while a negative κ implies that the node is already stable.

Assuming that *Assumption 2* holds and the common $V_{all}(X)$ is a quadratic as in (11), then

$$\begin{aligned} \dot{V}_{all}(X) = & \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} f_i(x_i) - B_iK_ix_i \\ & \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j \end{aligned}$$

$$\begin{aligned} \dot{V}_{all}(X) < & -\sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i}\theta_i\Gamma x_i + \mathcal{K} \\ & \sum_{i=1}^{N} \frac{\partial V(x_i)}{\partial x_i} \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma x_j \end{aligned}$$

with the stability of the controlled network determine by the matrix

$$\mathcal{C} = -\Theta + G - \mathcal{K} \tag{16}$$

where $\mathcal{K} \in \mathbf{R}^{N\times N}$ is a diagonal matrix with ρN elements κ_i, $i = 1, 2, ..., l$ different from cero, and its remaining $(1-\rho)N$ elements zeros.

To achieve the control objective by pining control, we need to determine which and with how much control gain to pin the nodes of the network, such that the matrix $\mathcal{C}$ is negative definite. This is an optimization problem subject to constrains in the number of nodes and the size of the energy of the controller. However, in general, when a sufficiently large number of nodes are controlled with sufficiently large gains, the stabilization of the network is achievable [Xiang and Chen (2007)].

In the following section, we consider that the objective is to impose a reference trajectory to the network.

4. IMPOSING A REFERENCE ON DYNAMICAL NETWORKS

Suppose that the objective is that the dynamical network of nonidentical nodes (2) follows the reference dynamics:

$$\dot{s} = f_r(s) \tag{17}$$

where $s \in \mathbf{R}^n$ is the reference state, with $f_r(.) : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a locally Lipschitz nonlinear vector field describing the reference dynamics.

The dynamics of the error ($e_i = x_i - s$) between the network (2) and the reference (17) is given by

$$\dot{e}_i = \hat{f}_i(e_i) + \sum_{j=1}^{N} c_{ij}a_{ij}\Gamma e_j, \text{ for } i = 1, 2, ..., N \tag{18}$$

where $\hat{f}_i(e_i) = f_i(x_i) - f_r(s)$.

Now, letting *Assumption 2* also be satisfied for the $\hat{f}_i(e_i)$ we have that there exist a Lyapunov function such that

$$\frac{\partial V(x_i)}{\partial x_i}(\hat{f}_i(e_i) - \hat{\theta}_i\Gamma e_i) < 0 \tag{19}$$

where the passivity degree of the error dynamics measures of how far the variational dynamics of node i-th is from being passive. Furthermore, since the stability of (18) is very unlikely, we use a pinning control strategy to stabilize the error dynamics to its zero fixed point. As before, we apply to a small fraction of the nodes (ρN) the local linear feedback controllers of the form:

$$v_i = -\hat{K}_i e_i \tag{20}$$

The closed-loop error dynamics of (18) becomes

$$\dot{e}_i = \hat{f}_i(e_i) - B_i v_i + \sum_{j=1}^{N} c_{ij} a_{ij} \Gamma e_j, \text{ for } i = 1, 2, ..., N \quad (21)$$

with B_i the input matrix of each node. Then, using an overall quadratic Lyapunov function $V_{all}(X)$ in the same way as before, the stability of the error dynamics is guarantied if the matrix

$$\mathcal{C} = -\hat{\Theta} + G - \hat{\mathcal{K}} \quad (22)$$

is negative definite, with $\hat{\mathcal{K}} \in \mathbf{R}^{N\times N}$ is a diagonal matrix with ρN elements $\hat{\kappa}_i$, $i = 1, 2, ..., l$ different from cero, and its remaining $(1-\rho)N$ elements zeros.

In the following section we use numerical simulations to illustrate the results presented in this contribution.

5. NUMERICAL SIMULATIONS

Consider a network of nodes in the form of Lorenz (23), Chen (24), and Chua's circuit (25) equations

$$\begin{aligned} \dot{x}_1 &= a_L(x_2 - x_1) \\ \dot{x}_2 &= c_L x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - b_L x_3 \end{aligned} \quad (23)$$

$$\begin{aligned} \dot{x}_1 &= a_C(x_2 - x_1) \\ \dot{x}_2 &= (c_C - a_C)x_1 + c_C x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - b_C x_3 \end{aligned} \quad (24)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{cases} A_1[x_1, x_2, x_3]^\top + b_1, \; if \; x_1 > 1 \\ A_2[x_1, x_2, x_3]^\top + b_2, \; if \; |x_1| \le 1 \\ A_3[x_1, x_2, x_3]^\top + b_3, \; if \; x_1 < -1 \end{cases} \quad (25)$$

where $A_1 = A_3 = \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} -\alpha(1+m_1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

$b_1 = [-\alpha(m_1 - m_0), 0, 0]^\top$, $b_2 = [0, 0, 0]^\top$, and $b_3 = [\alpha(m_1 - m_0), 0, 0]^\top$.

With the parameter sets $a_L = 10$, $b_L = \frac{8}{3}$, and $c_L = 28$ for Lorenz; $a_C = 35$, $b_C = 3$, and $c_C = 28$ for Chen; and $\alpha = 9$, $\beta = \frac{100}{7}$, $m_0 = -\frac{5}{7}$, and $m_1 = -\frac{8}{7}$. As such, all systems are in their chaotic state.

A common Lyapunov function is given by:

$$V(x) = x^\top Q x$$

with $Q = diag(1, 1, 1)$. Then, we investigate which are adequate values for the passivity degree θ_i in each case. For the Lorenz equation, we have:

$$\frac{\partial V(x_i)}{\partial x_i}(f_i(x_i) - \theta_i \Gamma(\bar{x} - x_i)) =$$

$$\frac{1}{2}[x_1, x_2, x_3](\begin{bmatrix} a_L(x_2 - x_1) \\ c_L x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - b_L x_3 \end{bmatrix} + \theta\Gamma \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) < 0 \quad (26)$$

Letting $\Gamma = I_n$ be the identity matrix of dimension n and for the parameters above, the inequality becomes:

$$38x_1x_2 - (10x_1^2 - \theta x_1^2 + x_2^2 - \theta x_2^2 + \frac{8}{3}x_3^2 - \theta x_3^2) < 0$$

For $\theta < -15.5$, the inequality is satisfied on an basin of attraction of size $\{x : x < r = 10\}$

For Chen, using the same $V(x)$ and Γ, the inequality becomes:

$$28x_1x_2 + 28x_2^2 + \theta x_2^2 - 35x_1^2 + \theta x_1^2 - 3x_3^2 + \theta x_3^2 < 0$$

For $\theta < -28$, the inequality is satisfied on an basin of attraction of size $\{x : x < r = 10\}$

The passivity degree for Chua's circuit is found from the inequality

$$\frac{1}{2}[x_1, x_2, x_3]\left(A_j \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_j + \theta_i \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) < 0$$

for $j = 1, 2, 3$. Then, for $\theta = 5$ the inequality is satisfied.

Constructing a network form by Lorenz, Chen and Chua nodes, with a structure given by the matrix G, a regular fully conneted array. Then, G will be given by:

$$G = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$$

Adding a controller of the form (13) the stability of the network is given by

$$-\begin{bmatrix} -16 & 0 & 0 & 0 & 0 \\ 0 & -28 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & 0 & -16 \end{bmatrix} +$$

$$\begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} -$$

$$\begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where k_1 is designed such that we get all the eigenvalues negative. The results are shown in Figure 1.

Next, we use the dynamics of the Chen's equation as reference, and design a controller to stabilize the error dynamics of the network, as shown in Figure 2.

6. CONCLUSIONS

We use the V-stability characterization of a dynamical network with different nodes to establish conditions for

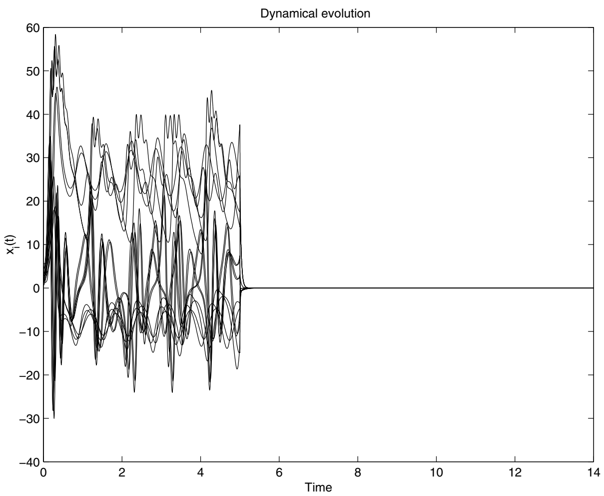

Fig. 1. Stabilization by pinning a node of the network

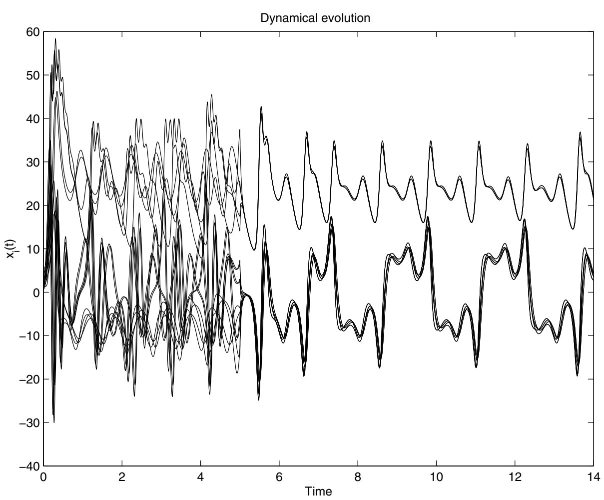

Fig. 2. Imposing a reference trajectory to the network of different nodes

pinning synchronization of the network. Further, we extend these results to the case of the tracking problem for networks by using this approach to impose a reference dynamics on the network. The proposed approach is based on the V-stability of the nodes of the network, the advantage of this approach is that by replacing the exact description of the node's dynamics by their passivity degree, the condition for stability is greatly simplified. However, the have the shortcoming that the nodes in the network must be made passive by linear feedback, this can be seen as a significant restriction, yet in the case of many benchmark chaotic systems this condition is satisfied. The conservativeness of the results are a consequence of the use of Lyapunov theory, however, they are very useful for the selection problem in pinning control of networks, as one can choose to focus the control action on the nodes with largest passivity index. Currently we are working on extensions of this approach for the case of networks with external perturbations, these results will be reported elsewhere.

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