Generating Self-Excited Oscillations with a Second Order Sliding Mode Controller

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Abstract: Generation of self-excited oscillations of desired amplitude and frequency in a linear system in closed loop with a continuous modification of the twisting controller is under study. Existence and stability conditions for the periodic solutions is shown using approximate frequency domain methods. Global stability is addressed through Lyapunov stability theory. A simulation example is used to validate the proposed approach.

Keywords: Self-excited oscillations; Limit cycles; Second-order sliding modes; Frequency domain methods; Stability analysis.

1. INTRODUCTION

High order sliding mode control algorithms (Fridman and Levant, 1996) have been considered as a promising technique that allows chattering attenuation while preserving the good properties of classical sliding modes (Bartolini et al., 1998). The technique consists in placing the discontinuous control action in high order (second order, in this case) derivatives of the sliding function, thus the discontinuous control is not applied directly to the system.

It has been a popular notion that second order sliding mode controllers eliminate chattering completely; however, it has been shown that twisting controllers cause chattering in the presence of parasitic dynamics (Boiko et al., 2004). The same chattering-free property has been attributed to continuous second order sliding mode controllers based, solely, on their continuity property; however, parasitic dynamics are also excited by the infinite gain terms inside the controller and chattering also appears (Boiko et al., 2005).

A self-excited system exhibits the property to generate steady state oscillations without an external modulated driving signal (Jenkins, 2013), (Chatterjee, 2011). Classical examples of such systems include the Van der Pol Oscillator (Khalil, 1996) and predator and prey models (Hou, 2012). Although it has been believed that limit cycles may degrade the performance of mechanical systems, applications have been found recently for such type of systems (see (Chatterjee, 2011), (Aoustin et al., 2010), (Jenkins, 2013) and references therein). Recent works regarding self-generation of oscillation of systems with sliding mode controllers are presented next.

In Aguilar et al. (2006), the property of continuous sliding mode controllers to generate self-excited oscillations in underactuated systems, namely the power fractional and super-twisting controller, is studied. The describing function and harmonic balance methods are used to analyze the closed loop system. Furthermore, in Boiko et al. (2004) it is shown that the same property that allows to generate self-excited oscillations is what causes chattering; thus, proving that chattering may still be present in second order sliding mode controllers.

A similar approach is followed in Aguilar et al. (2009b) to induce self-excited oscillations in a linear system with a twisting controller. The frequency and amplitude of the oscillations are defined by the gains of the twisting controller. The framework is applied to induce periodic motions of the passive link of a Furuta's pendulum (Xu et al., 2001) at the unstable orientation. The same approach is followed in Aguilar et al. (2009a) to induce self-generated oscillations in an exact linearization of the inertia wheel pendulum.

In the present work, a second order linear system, with a continuous modification of the twisting controller is under study. The conditions in which the closed loop system generates stable self-excited oscillations are determined. The contributions of this paper are the following: a mathematical expression is derived to show the existence conditions of periodic oscillations at the output of the system for determined controller gains. It is shown that the existence condition is a generalization of the twisting controller's conditions from Aguilar et al. (2009b). Stability theory is used to show global asymptotic stability and uniqueness of the oscillations. Furthermore, the proposed methodology allows to design the controller gains such that the output exhibits a limit cycle of desired amplitude and frequency.

The paper is organized as follows: in section 2 the problem is defined, in section 3 the describing function analysis for the closed loop system is carried out, the existence of periodic solutions is shown in section 4 using the harmonic balance method and the Poincaré-Bendixson's theorem. Local and global stability of the oscillations are addressed in section 5. A simulation example is shown in section 6 to validate the derived mathematical expressions. A discussion and where further work might be directed is presented in section 7, and the conclusions in section 8.

2. PROBLEM STATEMENT

Consider a linear second order system with asymptotically stable dynamics given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -h_1 x_1 - h_2 x_2 + u, \\ y &= x_1; \end{aligned} \tag{1}$$

where $x_1, x_2 \in \mathbb{R}$ are the states of the system, $y \in \mathbb{R}$ is the output, $h_1, h_2 \in \mathbb{R}^+$. The control signal is defined as

$$u = -k_1 |x_1|^{\frac{\alpha}{2-\alpha}} \operatorname{sgn} x_1 - k_2 |x_2|^{\alpha} \operatorname{sgn} x_2; \tag{2}$$

where $\alpha \in [0, 1)$. Such state feedback represents a family of continuous finite time stabilizing controllers (Orlov et al., 2011), (Haimo, 1986), (Bhat and Bernstein, 1997); which is a generalization of the well known twisting controller. Note that for $\alpha = 0$, the right hand side of the system (1) - (2) is discontinuous; thus, the solutions of the system are defined in the sense of Filippov (1960). For $\alpha \in (0, 1)$, the system is continuous and thus, the solutions are defined in the classical sense. A mathematical expression is to be derived to define the controller gains $k_1, k_2 \in \mathbb{R}$ that guarantee the existence of a periodic solution in x_1 of desired amplitude A and frequency ω for a given α.

3. DESCRIBING FUNCTION ANALYSIS

Let us assume that the output of system (1) with control (2) is periodic of the form

$$y \approx A \sin \omega t$$

for certain gains k_1, k_2 and α; $A \in \mathbb{R}^+$ represents the amplitude and $\omega \in \mathbb{R}^+$ stands for frequency. Note that the output's derivative $\dot{y} = x_2 \approx \omega A \cos \omega t$ also enters the controller. The controller output can be expressed as

$$\begin{aligned} u(t) = &-k_1 |A \sin \omega t|^{\frac{\alpha}{2-\alpha}} \operatorname{sgn}(\sin \omega t) \\ &-k_2 |\cos \omega t|^{\alpha} \operatorname{sgn}(\cos \omega t). \end{aligned}$$

The controller is made of the parallel connection of two terms with the form

$$u_g = |z|^r \operatorname{sgn} z, \tag{3}$$

where $r = \{\alpha/(2-\alpha), \alpha\}$ and $z = \{x_1, x_2\}$. Thus, the periodic controller signal

$$u_g(t) = |A \sin \omega t|^r \operatorname{sgn}(A \sin \omega t) \tag{4}$$

can be represented by its Fourier series

$$u_g(t) = \frac{1}{2} a_0 + \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx.$$

The definition of the describing function of a nonlinear element is the input/output magnitude relation of the nonlinear element (Khalil, 1996):

$$N(A, \omega) = \frac{\sqrt{a_1^2 + b_1^2}}{A}, \tag{5}$$

where a_1, b_1 are the coefficients of the first order harmonics of the periodic control signal. Note that the transfer function of system (1)

$$G(s) = \frac{1}{s^2 + h_2 s + h_1} \tag{6}$$

is a low pass filter; i.e. its frequency response $|G(j\omega)|$ vanishes as $\omega \to \infty$. This is known as the low pass filter hypothesis. Based on this fact, the assumption that high order harmonics can be neglected is valid, as they are sufficiently attenuated by the system's dynamics. Therefore, the output is close to a sinusoidal signal.

The Fourier coefficients can be found from the following expression

$$a_1 = \int_{-\pi}^{\pi} u(t) \cos \omega t \, \mathrm{d}(\omega t), \quad b_1 = \int_{-\pi}^{\pi} u(t) \sin \omega t \, \mathrm{d}(\omega t). \tag{7}$$

The coefficient a_1 is zero, as it is an odd function integrated over the interval $[-\pi, \pi]$. From (5) the describing function of the general nonlinear element (3) is given by

$$N_g = \frac{2A^{r-1}}{\pi} \int_0^{\pi} \sin^{r+1} \omega t \, \mathrm{d}(\omega t) = \frac{2A^{r-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{r}{2}+1\right)}{\Gamma\left(\frac{r}{2}+1.5\right)}; \tag{8}$$

where $0 < r < 1$ and $\Gamma(\rho) = \int_0^{\infty} t^{\rho-1} \mathrm{e}^{-t} \, \mathrm{d}\, t$ is the gamma function (Abramowitz and Stegun, 1964). As mentioned before, the describing function of the nonlinear controller is the parallel connection between two nonlinear elements with the form (3); hence, the complete describing function equation for the control algorithm (2) is defined as

$$N = k_1 N_1 + s k_2 N_2,$$

where

$$N_1 = k_1 D_1 A^{\frac{2(\alpha-1)}{2-\alpha}}, \quad N_2 = k_2 D_2 \tilde{A}^{\alpha-1}.$$

The constant $\tilde{A}$ is the amplitude of the signal that enters the element N_2. A relation between A and $\tilde{A}$ can be derived from the definition of derivative in the Laplace domain as $\tilde{A} = \omega A$. Evaluating the describing function at $s = j\omega$ for control (2), it is readily available as

$$N(A, \omega) = k_1 D_1 A^{\frac{2(\alpha-1)}{2-\alpha}} + j k_2 D_2 \omega^{\alpha} A^{\alpha-1}, \tag{9}$$

where

$$D_1 = \frac{2\Gamma\left(\frac{\alpha}{2(2-\alpha)}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2(2-\alpha)}+1.5\right)}, \quad D_2 = \frac{2\Gamma\left(\frac{\alpha}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}+1.5\right)}.$$

From (8) and (9) it can be easily shown that for $\alpha = 0$ the describing function matches that of the twisting controller from Aguilar et al. (2009b).

4. EXISTENCE OF PERIODIC SOLUTIONS

To prove the existence of periodic solutions in system (1)-(2) we will make use of the method known as *harmonic balance*. It allow us to predict the existence of limit cycles but cannot guarantee it. To be certain if indeed there exist limit cycles, further investigation is needed. A stronger technique to show that a system has periodic solutions is the Poincaré-Bendixson's theorem, which guarantees existence; but can only be applied to second order systems (Vidyasagar, 2002).

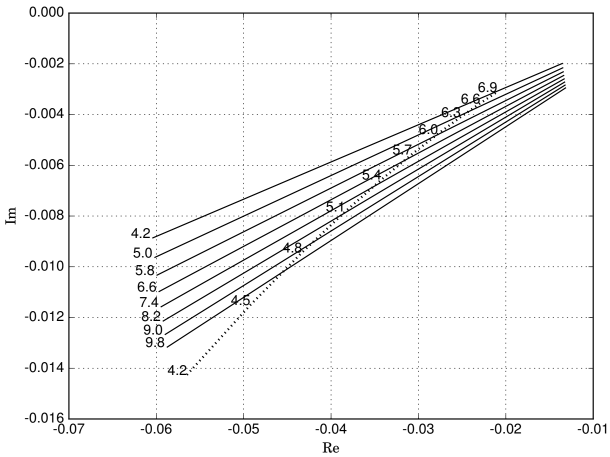

Fig. 1. Nyquist plot of $G(j\omega)$ (solid) and $N^{-1}(A,\omega)$ (dashed).

4.1 Harmonc Balance

The output of (1) and the input to controller (2) can be related as

$$y(t) = -e(t), \tag{10}$$

where e is the input to the controller. Following the feedback loop in the Laplace domain, the following expression is derived:

$$E(s)N(A,\omega)G(s) = Y(s),$$

where $E(s), Y(s)$ are the Laplace transformations of e and y, respectively. From this equation evaluated at $s = j\omega$, an expression to satisfy (10) is given as

$$G(j\omega)N(A,\omega) = -1. \tag{11}$$

This is known as the harmonic balance equation (Slotine and Li, 1991). Let us rewrite it substituting (9) as follows:

$$k_1 D_1 A^{\frac{2(\alpha-1)}{2-\alpha}} + jk_2 D_2 \omega^\alpha A^{\alpha-1} = G^{-1}(j\omega).$$

It is clear that this equation can be separated into its imaginary and real parts, this allow us to solve the harmonic balance equation for the controller gains:

$$k_1 = -\frac{\mathrm{Re}\{G^{-1}(j\omega)\}}{D_1 A^\gamma}, \quad k_2 = -\frac{\mathrm{Im}\{G^{-1}(j\omega)\}}{D_2\omega^\alpha A^{\alpha-1}}; \tag{12}$$

where $\gamma = 2(\alpha-1)/(2-\alpha)$. Note that (12) allows computation of the controller gains k_1 and k_2 that will enforce a periodic solution at the output x_1 of system (1) for a determined amplitude A, frequency ω and a given α.

In certain applications, depending on the order of the system and controller complexity, (11) cannot be solved analytically; therefore, graphical methods such as Nyquist plots can be used to determine the existence of periodic solutions (Slotine and Li, 1991). In Fig. 1 the Nyquist plot of $G(j\omega)$ and $N^{-1}(A,\omega)$ is shown for different frequencies and amplitudes, where the numbers denote the frequency at that point. It can be seen that G and N^{-1} interception points are infinite; the solution to the harmonic balance equation is the point where amplitude and frequency of G and N^{-1} match.

4.2 Poincaré-Bendixson's Theorem

The Poincaré-Bendixson's theorem states that if a trajectory of the system enters a region in the plane and never leaves it, i.e. it is a positive invariant set, then either there is a critical point or a closed trajectory inside the region (Khalil, 1996). To show that there exists a region M in the plane, which is positive invariant for system (1)-(2) and it does not contain a critical point, let us make use of a positive definite function from Orlov et al. (2011) to define a region

$$M_1 = \{\, (x_1,x_2) \mid r < V(x_1,x_2)\,\},$$

where

$$V = \eta_1|x_1|^{\rho_1} + \frac{1}{2}x_2^2 + \frac{\gamma}{2}x_1^2. \tag{13}$$

The derivative of V along the trajectories of the system is

$$\begin{aligned}\dot V = {}& (\rho\eta|x_1|^{\rho-1}|x_2| - k_1|x_1|^{\frac{\alpha}{2-\alpha}}|x_2|)\,\mathrm{sgn}(x_1x_2)\\ &+(\gamma-h_1)x_1x_2 - h_2x_2^2 - k_2|x_2|^{\alpha+1}.\end{aligned}$$

Let $\gamma = h_1$, $\rho\eta = k_1$, $\rho = \frac{2}{2-\alpha}$ and define k_2 as in (12), it follows that:

$$\dot V = -h_2|x_2|^2 + \frac{h_2}{D_2(\omega A)^{\alpha-1}}|x_2|^{\alpha+1}. \tag{14}$$

From (14), it is clear that the $\dot V$ changes sign when the trajectories are close to the origin. Let $\dot V = 0$ to find where (14) changes sign. Solving for $|x_2|$ from (14) we arrive at the following cases:

$$\dot V \begin{cases} <0, & |x_2| > \omega A D_2{}^{\frac{1}{\alpha-1}},\\ >0, & |x_2| < \omega A D_2{}^{\frac{1}{\alpha-1}},\\ =0, & |x_2| = \omega A D_2{}^{\frac{1}{\alpha-1}}.\end{cases} \tag{15}$$

The meaning of these cases is as follows: inside the stripe

$$-\omega A D_2^{\frac{1}{\alpha-1}} < x_2 < \omega A D_2^{\frac{1}{\alpha-1}}, \tag{16}$$

the trajectories of the system along the closed curve

$$V = \frac{1}{2}(\omega A)^2 D_2^{\frac{2}{\alpha-1}} \tag{17}$$

point outwards, as can be seen in Fig. 2. Outside the stripe (16), the trajectories of the system point inwards the closed curve (17). Hence, we can conclude that once the system trajectories enter set M_1, they cannot leave it towards the origin. Note that the radius r of the set M_1 can be chosen as the right hand side of (17) or smaller (see Fig. 2).

Now, let us show that the trajectories do not escape to infinity. In a region of the plane away from the origin, such that $|x_1|, |x_2| \gg 0$ the linear terms of the system (1)-(2) dominate the nonlinear terms of the controller. The system may be approximated as

$$\begin{aligned}\dot x_1 &= x_2,\\ \dot x_2 &\approx -h_1x_1 - h_2x_2.\end{aligned} \tag{18}$$

Let us define the following region to bound the trajectories that may leave to infinity:

$$M_2 = \{\, (x,y) \mid W \le R\,\},$$

where

$$W = \frac{1}{2}(x_1^2 + x_2^2). \tag{19}$$

The constant $R \in \mathbb{R}^+$ is chosen large enough such that assumption (18) is valid. The derivative of W along the trajectories of the system (18) is readily available as

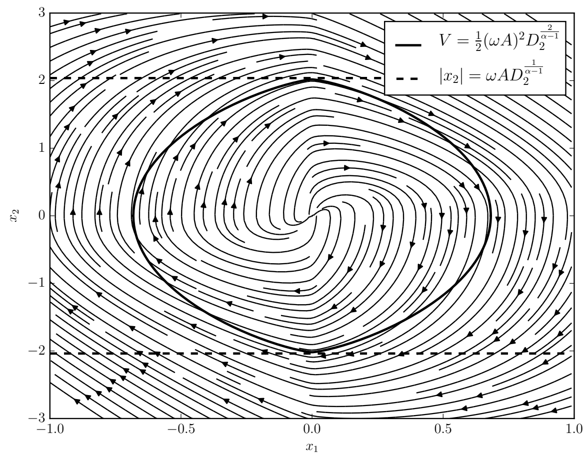

Fig. 2. Closed curve (17).

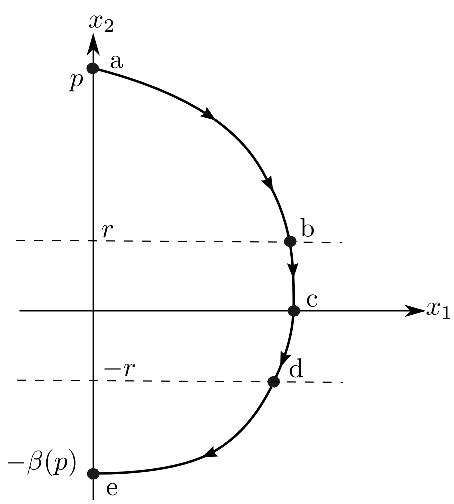

Fig. 3. Trajectories.

$$\dot{W} = -h_2 x_2^2. \tag{20}$$

Thus, the trajectories of the system cannot escape to infinity through the set M_2. Note that (20) is negative except on the x_1 axis, but from (1) it can be seen that the trajectories cannot be confined to the x_1 axis. From these results, we can define an annular region

$$M = M_1 \cap M_2 \tag{21}$$

that is closed and bounded from below by (17) and from above by (19). For this reason, the vector field points inwards the set M from below and above, thus, it can be concluded that it is a positively invariant set. Furthermore, the system (1)-(2) has no critical points inside M. By Poincaré-Bendixson's theorem, the existence of a periodic solution inside M is proved.

5. STABILITY OF PERIODIC SOLUTIONS

To show that the limit cycle generated by the closed loop system is stable, we will make use of two methods. The first one, is based on the analysis of the Taylor series expansion of a perturbation of the harmonic balance equation around its solutions. This only guarantees local stability due to the assumption that harmonic balance still holds in a region close to the solutions of the equations. The second method is based on Lyapunov stability theory. It is a stronger method as it allows to guarantee global stability of the limit cycles.

5.1 Local Stability

To derive the conditions for stability of the limit cycle, Loeb's criterion for stability (Loeb, 1956) is used. Suppose that equation (11) predicts a limit cycle of amplitude A_0 and frequency ω_0. A small perturbation in the limit cycle amplitude and frequency is allowed introducing the change of variables: $A_0 \to A_0 + \Delta A$, $\omega_0 \to \omega_0 + \Delta\omega + j\Delta\sigma$ while harmonic balance still holds:

$$N(A_0 + \Delta A, \omega_0 + \Delta\omega + j\Delta\sigma) + G^{-1}(\omega_0 + \Delta\omega + j\Delta\sigma) = 0.$$

The reasoning to determine the stability of the limit cycles is to find the conditions such that the relation $\Delta\sigma/\Delta A > 0$ holds. The physical meaning is that, if the perturbation injects energy into the system, the damping takes a positive value and the energy injected by the perturbation is dissipated. The converse is also true: if the perturbation decreases the energy of the system, a negative damping injects energy back into the system to bring the limit cycle to its nominal values (A_0, ω_0).

Given that amplitude and frequency perturbations are small, the Taylor series expansion of the perturbed harmonic balance equation can be taken around the equilibrium point (A_0, ω_0) while neglecting high order derivatives. For simplicity, the arguments are omitted:

$$\frac{\partial N}{\partial A}\Delta A + \frac{\partial N}{\partial \omega}(\Delta\omega + j\Delta\sigma) + \frac{\mathrm{d}\,G^{-1}}{\mathrm{d}\,\omega}(\Delta\omega + j\Delta\sigma) = 0$$

It is clear that this equation can be expressed in terms of its real and imaginary parts; and for it to hold true, both parts must vanish:

$$\frac{\partial N_\mathrm{R}}{\partial A}\Delta A + \frac{\partial (G_\mathrm{R}^{-1} + N_\mathrm{I})}{\partial \omega}\Delta\omega - \frac{\partial (G_\mathrm{I}^{-1} + N_\mathrm{R})}{\partial \omega}\Delta\sigma = 0, \tag{22}$$

$$\frac{\partial N_\mathrm{I}}{\partial A}\Delta A + \frac{\partial (G_\mathrm{I}^{-1} + N_\mathrm{I})}{\partial \omega}\Delta\omega + \frac{\partial (G_\mathrm{R}^{-1} + N_\mathrm{R})}{\partial \omega}\Delta\sigma = 0; \tag{23}$$

where the subindices $(\cdot)_\mathrm{R}$, $(\cdot)_\mathrm{I}$ represent the real and imaginary parts, respectively. Solving for $\Delta\omega$ from (22) and substituting in (23), an expression for stability such that $\Delta\sigma/\Delta A > 0$ can be determined as follows:

$$\frac{\partial N_\mathrm{R}}{\partial A}\left(\frac{\partial N_\mathrm{I}}{\partial \omega} + \frac{\mathrm{d}\,G_\mathrm{I}^{-1}}{\mathrm{d}\,\omega}\right) > \frac{\partial N_\mathrm{I}}{\partial A}\left(\frac{\partial N_\mathrm{R}}{\partial \omega} + \frac{\mathrm{d}\,G_\mathrm{R}^{-1}}{\mathrm{d}\,\omega}\right). \tag{24}$$

From (6), the derivative of the transfer function's inverse is readily available as

$$\frac{\mathrm{d}\,G^{-1}}{\mathrm{d}\,\omega} = -2\omega + jh_2;$$

and from (9) the partial derivatives are defined as follows

$$\frac{\partial N}{\partial \omega} = jk_2 D_2 \alpha (\omega A)^{\alpha-1},$$

$$\frac{\partial N}{\partial A} = \gamma k_1 D_1 A^{\gamma-1} + j(\alpha-1)k_2 D_2 \omega^\alpha A^{\alpha-2}.$$

Using (24), (12); after some algebraic manipulations, the stability condition

$$(1-\alpha) > 0 \tag{25}$$

is defined. One can conclude that the limit cycle is always asymptotically stable, at least locally, as α always lies in the half-open interval $[0, 1)$. Thus, condition (25) always holds. For $\alpha = 1$, (25) is not met. This is due to the fact that control (2) coincides with a PD controller for such α. It is known from stability theory that a linear system cannot exhibit limit cycles.

5.2 Global Stability

Let us address the problem of global stability of the system (1)-(2) in the sense of Lyapunov (Vidyasagar, 2002). The function (13) can be used as a Lyapunov function. Indeed, taking its derivative along the trajectories of the system yields (14). It is stated that inside the stripe (16) the vector field points away from the closed curve (17). To show that trajectories do not escape to infinity through the stripe, let us define as p the point where trajectories hit the positive x_2 semi-axis, and as $-\beta(p)$ the point in the negative x_2 semi-axis as shown in Fig. 3. When p is smaller than $r > 0$ the curve *abcde* lies completely inside the stripe and

$$V(-\beta(p)) - V(p) > 0.$$

When p is greater than the radius r, it is clear from (14) that

$$V(b) - V(a) < 0, \quad V(d) - V(b) > 0, \quad V(e) - V(d) < 0;$$

Indeed, as p gets larger, the negative part of the curve becomes more dominant; it follows that

$$\lim_{p\to\infty} V(-\beta(p)) - V(p) = -\infty. \tag{26}$$

Hence, the trajectories cannot escape to infinity because, as the point p gets larger, the trajectories are "pulled harder" towards the origin. Moreover, the trajectories cannot stay completely inside the strip (16), where (14) is always positive, as it is not an invariant set. Thus, global stability in the sense of Lyapunov is concluded.

Furthermore, stability analysis allows us to show the uniqueness of the periodic solution. Due to the oddness of (1)-(2), its trajectories are symmetric with respect to the x_2 axis, and for a limit cycle to exist the following condition must hold

$$V(p) = V(-\beta(p)), \tag{27}$$

for some $p > 0$. By continuity of the Lyapunov's function derivative (14), there can exist only one p for which condition (27) is true; hence, the limit cycle is unique. A similar proof for the Van der Pol oscillator can be found in Khalil (1996).

6. SIMULATION

To validate the proposed approach, consider the following second order linear system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - x_2 + u, \\ y &= x_1; \end{aligned} \tag{28}$$

with control law as in (2). The open loop system transfer function is given by

$$G(s) = \frac{1}{s^2 + s + 1}. \tag{29}$$

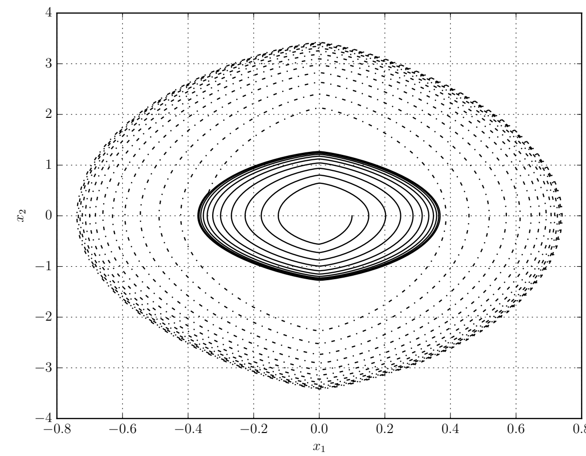

Fig. 4. Phase plot of the system from the simulation.

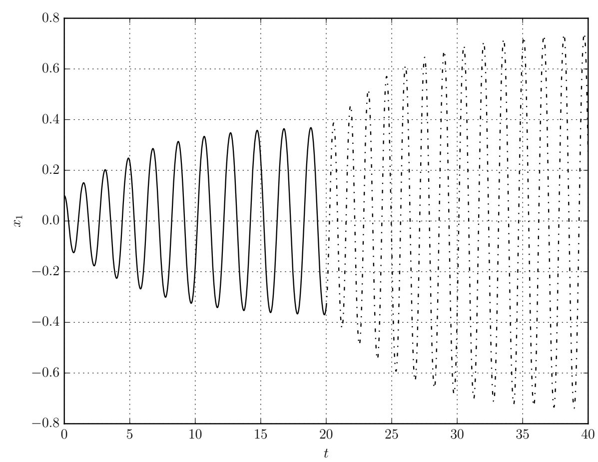

Fig. 5. Time response of the system from the simulation.

From equation (12), the controller gains can be found such that the closed loop system (28)-(2) exhibits an asymptotically stable limit cycle at its output x_1; with arbitrary values for amplitude and frequency. To demonstrate the capability of the closed loop system to jump between orbits, the gains are modified at $t \approx 20$s; such that the amplitude and frequency for the first orbit are $A = 0.4$ and $\omega = 3$rad/s, respectively; and $A = 0.8$ and $\omega = 4$rad/s for the second orbit. Thus, the controller gains are $k_1 = 3.745$, $k_2 = -0.984$ and later changed to $k_1 = 11.147$, $k_2 = -1.607$ and $\alpha = 0.5$ for both cases.

The phase plot of the system is shown in Fig. 4. It can be seen that the amplitudes of the oscillations are closed to the desired value. The point in which the controller gains are modified is represented by the line style modification. The initial conditions for the simulations are $x_0 = [0.1, 0.0]^{\mathrm{T}}$. The time response of the system is shown in Fig. 5; in a similar way, the time instant in which the controller gains are modified is shown by line style modification. From both plots it can be seen that the amplitude is closed to the desired values, as well as the frequency.

7. DISCUSSION AND FURTHER WORK

For limit cycles to exist in system (1)-(2), the gain k_2 must be negative, while k_1 is not sign definite as it can be seen from (12). But to use (2) as a stabilizing feedback, the controller gains can be designed as given by Santiesteban et al. (2013); thus, the controller gains can never be negative! One can conclude that system (1)-(2) cannot be affected by chattering in the absence of parasitic dynamics; however, in real life applications this is not the case. The method presented here might be useful for estimation of chattering effect in the closed loop system if a model for parasitic dynamics is available. Such model might be obtained through experimentation.

It should be mentioned that, in this work, (2) is not used as a stabilizing feedback, but as a way to self-excite oscillations of the system under study. The system does not reach the limit cycle in finite time, nor high frequency switching would be observed on the limit cycle given that the system cannot be in sliding mode anywhere, as it is known that a sliding mode only exists at the origin of system (1)-(2) (Orlov et al., 2011). Finally, note that the main result (12) is valid for linear higher order systems as the dynamics are in terms of the transfer function; however, Poincar-Bendixson's theorem and Lyapunov's stability analysis is not valid anymore.

8. CONCLUSION

Existence conditions of limit cycles at the output of a linear system in closed loop with a continuous modification of the twisting controller were derived. A mathematical expression allows for controller design to induce a stable limit cycle of desired amplitude and frequency. The derived mathematical relation is a generalization of that for the twisting controller shown in (Aguilar et al., 2009b). It was shown, also, that the system always has at least one limit cycle if the gains are chosen as in (12). It is shown that the limit cycle is globally asymptotically stable. The results were validated through computer simulations.

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