PD controller for second order unstable systems with time-delay

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Abstract: This paper is devoted to the design of a PD controller to overcome the problem of the stabilization and control of linear delayed systems with two unstable poles. Necessary and sufficient conditions for the existence of a PD stabilizing control are determined taking into account frequency domain analysis techniques. Based on this analysis, there were found simple guidelines to compute the complete stability regions for the PD controller parameters.

Keywords: Control systems, Time-delay, PD control.

1. INTRODUCTION

Time-delay systems are frequently encountered in various problems in engineering. These delays are due to different facts, systems with delays arise in engineering, biology, physics, traffic flow modeling, etc. Sipahi et al. (2011). Delays are associated with the transport of material, energy or information. Time delays have been also incorporated into industrial process models by many authors when a high-order system is approximated by low-order systems Skogestad (2003). For instance, in the input-output model approximation for control design purposes.

Time delays are often a source of complex behaviors such as oscillations, bad performance or even instability in dynamical systems, thus considerable attention had been paid on the stability analysis and the control design for time-delay processes, [Niculescu (2001), Richard (2003)]. Hence, the study of systems with delays has been a subject of great interest during the last decades and there are different works devoted to its study, [Loiseau et al. (2009), Chiasson and Loiseau (2007)].

One of the classic approaches to deal with dead time systems is the Smith Predictor Compensator Smith (1957); Palmor (1996). The main limitation of the original SP is the fact that it is restricted to stable plants. To deal with this disadvantage, some modifications to the original SP structure have been proposed (see for instance ?, Rao and Chidambaram (2006)). With a different perspective, (Normey-Rico and Camacho (2008, 2009)), propose a modification to the original Smith’s structure in order to deal with unstable delayed systems. Unfortunately, for the control scheme proposed in Normey-Rico and Camacho (2009), it can be easily proven that in the case of unstable plants, the internal stability is not guaranteed.

For the analysis and control of time-delay systems there are several approaches reported in the literature. A common approach in order of dealing with this kind of systems consists on substitution of the delay operator by means of Taylor or Pad series approximations that leads to non minimum-phase models with rational transfer function representation Munz et al. (2009). When the control law based on these approximations is applied through a digital computer, the discretization of the process with input/output delay yields a rational transfer function in the complex variable z, under some considerations in the selection on the sampling period T. Unfortunately, this approach provide controllers of very high order (see for instance Garca and Albertos (2013)).

Classic controllers P, PI and PID are also studied to delayed processes. For example Silva et al. (2004) design PID controllers for first order unstable systems. An step forward is given by Xiang et al. (2007), where conditions for the stabilization of second order unstable systems with PID controllers are proposed.

Many chemical and biological systems exist whose dynamics present second or higher order behavior. Continuous stirred tank reactors, polymerization reactors and bioreactors are inherently unstable by design; these types of systems can be modeled as open-loop unstable systems plus time delay. The stabilization of linear systems with one unstable pole, n real stable poles and time delay, are tackled in Lee et al. (2010) by static output feedback and PI-PID controllers. In Hernández et al. (2013), necessary and sufficient conditions for the stabilization of linear systems by static output feedback for SISO systems with one unstable pole, a couple of complex conjugate stable poles and time delay, are provided. Using an observer based control scheme, the stabilization of high order systems with real poles, two of them unstable, is dealt in Novella Rodriguez et al. (2014).

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This work proposes a controller in order to stabilize delayed systems with two unstable real poles, necessary and sufficient conditions for the existence of the stabilizing PD controller are given. An frequency domain analysis is used in order to demonstrate such conditions. The controller parameters can be computed taking into account the magnitude and phase equations. The document is organized as follows, section 1 provided a brief introduction. In section 2 we will formulate the problem to be dealt in this work, the class of system studied and some useful preliminaries. In section 3 the main results of the paper will be given. Later, in section 4 an example will be developed in order to illustrate the closed loop performance of the controller. Finally section 5 gives final conclusions.

2. PROBLEM FORMULATION

Let us consider the unstable process given by the transfer function:

\[ G(s) = \frac{\alpha}{(s-a)(s-b)} e^{-\tau s}, \]  

(1)

where, the parameters \(a, b\) and \(\tau\) are positive real constants. The purpose of this work is to provide the conditions for the existence of stabilizing PD controllers represented by:

\[ C(s) = k_p(s + k_d), \]  

(2)

in order to control delayed plants with the form (1), where \(k_p\) is the proportional gain and \(k_d\) is the derivative term. The open loop transfer function can be expressed in the form:

\[ G(s)C(s) = K \frac{N(s)}{sD(s)} e^{-\tau s}, \]  

(3)

where the gain \(K = \alpha K_p, N(s)\) and \(D(s)\) are polynomials of the complex variable \(s\), an unity output feedback yields the closed loop system:

\[ \frac{Y(s)}{R(s)} = \frac{KN(s)e^{-\tau s}}{sD(s) + KN(s)e^{-\tau s}}, \]  

(4)

where the \(Y(s)\) and \(R(s)\) are the output and reference signals respectively.

2.1 Preliminaries

We proceed to state some preliminary results which are useful in order to formulate the main results stated in this work. First, the well known Nuyquist criterion is cited as follows:

**Nyquist Stability Criterion** Given the open-loop transfer function \(G(s)C(s)\) in (3) with \(P\) unstable poles, the closed-loop system (4) is stable if and only if the Nyquist plot of \(G(s)C(s)\) encircles the critical point, \((-1, 0)\), \(P\) times in anticlockwise direction.

Additionally, another preliminary result is presented, which will be used for the stability analysis in the development of the main result of this work.

Lemma 1. Xiang et al. (2007) Given the open-loop transfer function \(G(s)\) defined in (3), a necessary condition for the closed-loop stability is that the polynomial \(H(s)\) defined as:

\[ H(s) = \frac{D(s)e^{-\tau s}}{s^{m+1} + s^m \cdots s + 1}, \]  

(5)

has all its zeros in the open left half plane, where \(m\) is the degree of \(N(s)\) and \(d\) indicates the derivative. The proof of the previous Lemma can be developed taking into account the conditions on the stabilization of time delay systems provided in Kharitonov et al. (2005).

3. MAIN RESULT

This section is devoted to the analysis for the stabilization of a second order system with two unstable poles plus time-delay (1). The first result can now be stated.

![Fig. 1. Closed loop controller configuration](image)

**Theorem 2.** The time-delay system given by (1) can be stabilized with a PD controller characterized by equation (2), (connected in the configuration shown in Figure 1), if and only if:

\[ \tau < \frac{1}{a} + \frac{1}{b} - \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}, \]  

(6)

**Proof 1. Sufficiency**

We proceed to study the case of a PD controller given by \(k_p(s + k_d)\). Let us assume that the condition (6) holds, considering the open loop system frequency response given by:

\[ C_{PD}(j\omega)G_s(j\omega) = K (j\omega + k_d) \]  

(7)

with \(K = \alpha k_p\), then we can obtain the magnitude equation:

\[ M_{PD}(\omega) = K \sqrt{\frac{k_d^2 + \omega^2}{a^2 + \omega^2}}, \]  

(8)

and the phase expression can be written as:

\[ \Phi_{PD}(\omega) = \left(-\pi + \arctan \left(\frac{\omega}{a}\right)\right) + \left(-\pi + \arctan \left(\frac{\omega}{b}\right)\right) - \cdots \]  

\[ -\tau \omega + \arctan \left(\frac{\omega}{k_d}\right) \]  

(9)

From the Nyquist stability criterion, it can be seen that for the system (7), the number of unstable roots \(P = 2\), then the Nyquist plot should encircle twice the critical point \((-1, 0)\) in counterclockwise. In order to obtain a double
Fig. 2. Nyquist Plot: PD stabilization encirclement of the point (−1, j0) in the Nyquist trajectory (Fig. 2), the magnitude equation (8) should satisfy $M_{PD}(\omega_1) < M_{PD}(\omega_2)$ to obtain a correct direction of the trajectory, where $\omega_1$ and $\omega_2$ are the frequencies where the Nyquist plot cross the negative real axis. Taking into account the following condition on the derivative of the magnitude equation:

$$\frac{dM_{PD}(\omega)}{d\omega} > 0,$$

(10)

where:

$$\frac{dM_{PD}(\omega)}{d\omega} = k_\omega \left( -\omega^4 - 2\omega^2 k_a^2 - k_a^2 a^2 - k_a^2 b^2 + a^2 k_b^2 \right) \sqrt{\frac{(k_a^2 + \omega^2)(a^2 + \omega^2)^2}{(a^2 + \omega^2)(b^2 + \omega^2)^2}}.$$

(11)

From (10) it is possible to conclude that a term $k_d > -ab/\sqrt{a^2 + b^2}$ assures that the magnitude of the system will increase for frequencies $\omega \to 0$, obtaining a correct direction on the Nyquist path.

On the other hand, considering the Bode plot of the system, an additional requirement to obtain the desired trajectory is that the phase equation decreases when $\omega \to 0$. Then, considering that the derivative of the phase equation is given by:

$$\frac{d\Phi_{PD}(\omega)}{d\omega} = -\tau \omega + \frac{a}{a^2 + \omega^2} + \frac{b}{b^2 + \omega^2} + \frac{k_d}{k_a^2 + \omega^2},$$

(12)

taking into account that the condition (6) holds, it is possible to conclude that if the phase equation satisfies:

$$\frac{d\Phi_{PD}(\omega)}{d\omega} \bigg|_{\omega \to 0} = -\tau + \frac{1}{a} + \frac{1}{b} + \frac{1}{k_d} < 0,$$

(13)

then, the trajectories of the Nyquist plot when $\omega \to 0$, will advance in the correct direction.

From (13), it can be determined the lower bound for the $k_d$ gain, this is $k_d > ab/\tau ab - a - b$. Assuming that the condition (6) holds, the trajectory of the Nyquist plot will start decreasing from $-\pi$, then it should change its direction, in order to have a double encirclement there should exist a frequency $\omega^*$ such that the phase is greater than $-\pi$, this is $\Phi_{PD}(\omega^*) > -\pi$. Finally, the direction of the Nyquist trajectory changes its direction to continue decreasing. Then, considering (13), there are two intersections with the negative real axis, namely $\Phi_{PD}(\omega_{k1}) = -\pi$, with $k = 1, 2$ for positive frequencies $\omega_{k1}$, considering that there is an increasing magnitude, we can assume that $M_{PD}(\omega_{k1}) < M_{PD}(\omega_{k2})$, obtaining a couple of anticlockwise rounds of the Nyquist path.

**Remark 1.** As it was described before, the derivative gain can be selected in the following range:

$$k_d^{min} < k_d < k_d^{max},$$

(14)

where the lower bound for the controller gain $k_d$ can be found from (13), considering $d\Phi_{PD}/d\omega|_{\omega=0} < 0$ and solving for $k_d$, which yields:

$$k_d^{min} = \frac{ab}{\tau ab - a - b}.$$

On the other hand, the upper bound for the derivative term can be stated as follows:

$$k_d^{max} = \{ max(k_d) : max\Phi_{PD}(\omega) > -\pi, M_{PD}(\omega_{k1}) < M_{PD}(\omega_{k2}) \},$$

Therefore, selecting a gain $k_d$ such that $k_d^{min} < k_d < k_d^{max}$, the Nyquist plot has two anticlockwise rounds, in order to have a closed loop stable system, the anticlockwise encirclement must round the critical point (−1, j0), from the magnitude equation (8), the proportional gain $k_p$ should satisfy:

$$\sqrt{\frac{(a^2 + \omega_{k1}^2)(b^2 + \omega_{k1}^2)}{a^2 k_d^2 + \omega_{k1}^2}} < k_p < \sqrt{\frac{(a^2 + \omega_{k2}^2)(b^2 + \omega_{k2}^2)}{a^2 k_d^2 + \omega_{k2}^2}},$$

(15)

where $\omega_{k1}$ and $\omega_{k2}$ are the frequencies where the Nyquist plot cross the negative real axis.

**Necessity.** Taking into account the Lemma 1, the stability of the closed-loop system with a PD controller system requires that the polynomial:

$$H_{PD}(s) = \tau^2 s^2 - \tau^2 a s - \tau^2 b + 4 \tau s + 4 \tau^2 a - 2 \tau a - 2 \tau b + 2,$$

has all its zeros in the open left half plane, since $\tau$ is a positive constant, a necessary condition for the stabilizability is that the independent term $\tau^2 a - 2 \tau a - 2 \tau b + 2 > 0$, namely:

$$\tau < \frac{1}{a} + \frac{1}{b} - \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}.$$


4. EXAMPLE

Let us consider a time-delay second order system given by the transfer function:
\[ G(s) = \frac{5e^{-\tau s}}{(s - 2)(s - 0.5)} , \]
from Theorem 2 we can conclude that the delayed system can be stabilized by means a PD controller when the time delay is on \(0 < \tau < 0.4384\). It is possible to compute the complete family of PD controllers. Figure 3 shows the set of controller gains that assure closed loop stability with respect of the time delay size.

Fig. 3. All the stabilizing PD controller gains

It can be noticed that the possible region of stabilizing controller gains is reduced as the time-delay increases. In order to analyze the closed-loop performance of the PD controller, let us consider a time-delay \(\tau = 0.2\), then, we choose a PD controller given by:

\[ C(s) = 0.7(s + 0.05) , \]
the closed loop behavior of the controlled plant is presented in Figure 4. Figure 4 shows the closed loop performance of the delayed system. Continuous line shows the performance of the controller with a nominal time delay. Dashed line illustrate the behavior of the controller with an uncertain time delay (a time delay \(\tau = 0.25\)). It can be seen that the controller yields a stable performance and is robust with respect to parameter uncertainties. Figure 5 shows the control signal for both cases.

Fig. 4. Closed loop performance

Fig. 5. Control signal for the closed loop system

4.1 A large time delay

In this section, we will analyze the delayed system studied in this example, but now with a larger time delay, with respect to the parameters of the system, in this case we assume a delay \(\tau = 0.4\). The controller selected for this system is:

\[ C(s) = 0.44(s - 0.45) , \]
the closed-loop performance is stable as is shown in the Figure 6, it can be noticed from Figure 7 that the control effort is greater with respect to the system with a small delay.

Fig. 6. Closed loop performance for a large time-delay

Fig. 7. Control signal for a large time-delay

5. CONCLUSION

Systems with time delay and unstable dynamics represents a challenge for their analysis and the design of stabilizing controllers. This paper proposed a PD controller to
stabilize a particular class of delayed systems, process with two unstable real poles. A frequency domain analysis was developed in order to provide necessary and sufficient conditions for the existence of a stabilizing PD controller. Based on this analysis is possible to compute the controller parameters that assures closed-loop stability. An example is provided in order to illustrate the performance and numerical simulations are developed to show the stable behavior of the closed loop system.

REFERENCES


