Memorias del XVI Congreso Latinoamericano de Control Automático, CLCA 2014
Octubre 14-17, 2014. Cancún, Quintana Roo, México

Frequency Domain Modulating Functions for Continuous-Time Identification of Linear and Nonlinear Systems

Oscar B. Cieza * **Julio C. Tafur** ** **Johann Reger** ***

* *Pontifical Catholic University of Peru, Lima, Peru (e-mail: osbn.cieza@gmail.com).*
** *Pontifical Catholic University of Peru, Lima, Peru (e-mail: jtafur@pucp.edu.pe)*
*** *Control Engineering Group, Technische Universität Ilmenau, Germany, (e-mail: johann.reger@tu-ilmenau.de)*

Abstract: System identification techniques through modulating functions have generated a wide range of methods and applications. In the underlying contribution, we generalize previous modulating function methods and develop a frequency domain modulating function (FDMF) approach that is based on a given frequency spectrum. In particular, we present limitations of integrable and convolvable system representations, both linear and nonlinear, and propose an extension of the tractable system classes, the so-called e-convolvable systems.

Keywords: System identification, modulating functions, Fourier transforms, Hartley transform, continuous-time systems, nonlinear systems.

1. INTRODUCTION

Since its introduction by Shinbrot (1957) the modulation function technique (MFT) for system identification has been developed and established as a signal-based alternative to identification techniques that reside on state space models and estimation. With respect to computational effort these algorithms have reached maturity levels that, when compared with customary Kalman-filter approaches, may allow a similar broad range of application on practical setups. For implementation reasons, many of these algorithms are formulated within the discrete-time framework. However, this may entail various drawbacks as e.g. artificial non-minimum phase zeros, numerical issues, to name but a few, see Rao and Unbehauen (2005) for an overview. For these reasons, in the recent years, continuous-time identification frameworks have reached popularity again, among them, methods that resort to specific modulation functions within the convolution integrals of the identification process.

The MFT is based on the moment function method and has the purpose to transform (modulate) a differential equation, on a finite time interval, into a set of integro-algebraic equations, Pearson and Lee (1985). This transformation is achieved by multiplying the input-output representation of the differential equation with a special function ϕ, called modulating function (MF), which lets exactly integrate, at least in theory, all derivative terms in the equation. The central idea of the method is to carry out integration by parts using particular boundary conditions for the time derivatives $\phi^{(i)}$. This way, derivative terms of the output and input are reduced while at the same time eliminating the need for identifying the initial conditions of the differential equation.

More precisely, the application of the MFT for continuous-time system identification requires a three-stage process. In the first stage, the parametric form of the input-output representation of the differential equation has to be selected. These forms include e.g. linear representations (Byrski and Byrski (2012a); Shen (1993)), integrable (Rao and Unbehauen (2005)), convolvable (Behre (1999); Patra and Unbehauen (1995)), and fractional representations (Janiczek (2010)). The second stage involves the application of a MFT, i.e. for example moment functionals (Fairman (1971)), integral techniques (Rao and Unbehauen (2005)), Fourier and Hartley MF's (Behre (1999); Shen (1993); Pearson and Lee (1985); Rao and Unbehauen (2005)) or Loeb-Cahen MF (Byrski and Byrski (2012b)). Finally, in the third stage, the unknown parameters may be determined with the modulated input-output representation, usually with least squares method (Pearson and Lee (1985)) or weighted least squares methods (Shen (1993)). Also optimal parameter estimation under constraints has been considered, see Byrski and Byrski (2012b).

In our work we considered Fourier and Hartley MF due to its excellent features for expressing nonlinearities in input-output representations. The main contribution of this paper is the unification of Fourier and Hartley MF in a single representation that we termed frequency domain modulating function (FDMF) technique. This adds flexibility for system identification when a frequency spectrum is specified. Additionally, we extend the class of tractable systems to further nonlinearities, i.e. products of functions with arbitrary differentiation order in time.

The paper is organized as follows: After this brief introduction, Section 2 presents the main concepts of the FDMF within a unified framework. In Section 3 the proposed extension of the system class is presented. These theoretical developments are discussed by means of simulations results in Section 4. Finally, we draw our conclusions in Section 5.

2. CONCEPTS OF FREQUENCY DOMAIN MODULATING FUNCTIONS

2.1 FDMF Technique

The modulating function technique (MFT) is based on a finite integral transform of a signal that is associated to a differential equation. Let $\xi : [t_0, t_0 + T] \to \mathbb{R}$ be a time signal associated to a differential equation of maximum order N_d. Then the MF transformation reads

$$\Gamma_{\xi,m} = \int_{t_0}^{t_0+T} \xi(\tau)\phi_m(\tau - t_0)d\tau \tag{1}$$

where ϕ_m is a MF on the domain of integration. When considering the i-th time derivative of signal ξ, i.e. $\xi^{(i)}$ for $i = 0, 1, \ldots, N_d$, in view of partial integration this transformation may serve to eliminate this derivative. To this end, select ϕ_m at least N_d-times continuously differentiable, that is $\phi_m \in \mathcal{C}^{N_d}$ for all $t \in [0, T]$, and let the following homogeneous boundary conditions

$$\phi_m^{(i)}(0) = \phi_m^{(i)}(T) = 0, \quad i = 0, 1, \ldots, N_d - 1 \tag{2}$$

be satisfied conditions, see Pearson and Lee (1985).

Note that the MF ϕ_m may also be represented as a linear combination of the kernel $\psi(w, t)$ of an other integral transform, for some specified w. This idea is extracted from Pearson and Lee (1985) where a sine and cosine transform kernel is employed.

In view of FDMF, we define Fourier and Hartley transforms as follows:

Definition 1. Let $\xi : [t_0, t_0 + T] \to \mathbb{C}$ be an integrable function. Then

$$\widehat{\xi}(kw_0) = \frac{1}{T}\int_{t_0}^{t_0+T} \xi(x)e^{-jw_0kx}dx$$

with $w_0 = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$ is the finite Fourier transform.

Definition 2. Let $\xi : [t_0, t_0 + T] \to \mathbb{R}$ be an integrable function. Then

$$\widehat{\xi}(kw_0) = \frac{1}{T}\int_{t_0}^{t_0+T} \xi(x)\text{cas}(w_0kx)dx$$

with $\text{cas}(a) = \sin(a) + \cos(a)$, $w_0 = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$ is the finite Hartley transform.

Definition 3. Let $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{S}$ be the kernel of a Hartley or Fourier transform for $\mathbb{S} \subseteq \mathbb{C}$ and let $w_i \in \mathbb{R}$ be a scalar. A function $\phi_m : \mathbb{R} \to \mathbb{S}$ is said to be a FDMF if ϕ_m is linear in $\psi(w_i, .)$ and fulfills the boundary conditions (2).

Within the time interval $[t_0, t_0 + T]$, the elements w_i are considered frequency multiples of ω_0, where $\omega_0 = 2\pi/T$. The FDMF according to the previous definition may then be expressed in matrix form as

$$\mathbf{\Phi}(t) := \mathbf{R}\mathbf{\Psi}(\mathbf{w}, t). \tag{3}$$

with the components $\mathbf{\Psi}(\mathbf{w}, t) = [\psi(w_1, t), \ldots, \psi(w_{N_w}, t)]^{\mathrm{T}}$, $\mathbf{w} = [w_1, \ldots, w_{N_w}]^{\mathrm{T}}$, $\mathbf{\Phi} = [\phi_1, \ldots, \phi_{N_w-N_d}]^{\mathrm{T}}$.

Differences within the classes of FDMF may be made up by the chosen kernel ψ and the structure of matrix $\mathbf{R}$. This matrix has two structures developed for Hartley or Fourier MF: One structure is calculated from the null space, the other from a binomial expansion of its basic form. For details of calculating matrix $\mathbf{R}$ see Section 2.4.

2.2 Theorems of FDMF

One of the advantages of MF is that its application may avoid the calculation of derivative terms within some input-output differential equations. This features is established by the following theorems.

Theorem 1. (Integrable functions). Let $\xi : [t_0, t_0+T] \to \mathbb{R}$ be a function of differentiability class $\mathcal{C}^i$. Applying the FDMF transform to $\xi^{(i)}(.)$ yields

$$\mathbf{\Gamma}_{\xi^{(i)}} = \mathbf{R}\mathbf{D}(i)\frac{1}{T}\int_{t_0}^{t_0+T} \xi(\tau)\mathbf{\Psi}(\mathbf{w}\alpha^i, (\tau - t_0))d\tau \tag{4}$$

where $\mathbf{D}(i)$ follows from the time derivative of the kernel vector $\mathbf{\Psi}$ and α depends on the kernel ψ.

Proof. The statement of the theorem is obtained when substituting (3) into (1) followed by replacing Fourier and Hartley transformation kernels. Subsequent integration-by-parts until all the derivatives shift from ξ to ϕ, taking (2) for canceling all the boundary terms, yields the result.

As a consequence, time derivatives of functions may be shifted to derivatives of the vector kernel. Note that the integral transform is the Fourier or Hartley transform for the desired frequencies w_i specified in $\mathbf{w}$.

Obtaining expressions for $\xi^{(i)}$ in terms of function ξ only, is one of the main purposes for the variety of MFT's. However, there are several additional nonlinearities that may be expressed as a multiplication of two functions, i.e. $g\,h^{(i)}$. This signal class is called *convolvable.* Presently, signals of this class can be treated only by Fourier and Hartley MFT's.

In the next theorem, we extend this signal class treatable with FDMF by further differentiations, that is, to handle functions of the form $(g\,h^{(i)})^{(k)}$.

Theorem 2. (Convolvable functions). Let functions $\xi, \nu : [t_0, t_0 + T] \to \mathbb{R}$, where $\xi \in \mathcal{C}^{i+k}$ and $\nu \in \mathcal{C}^k$. Then the FDMF transform of $(\nu\,\xi^{(i)})^{(k)}$ has the form

$$\mathbf{\Gamma}_{(\nu\xi^{(i)})^{(k)}} = \mathbf{R}\mathbf{D}(k)\sum_{s=-\infty}^{\infty} \widehat{\nu}_s(-1)^i\frac{1}{T} \times \int_{t_0}^{t_0+T} \xi(\tau)\Big(\psi_{sw_0}^{-1}(t)\mathbf{\Psi}(\mathbf{w}\alpha^k, (\tau - t_0))\Big)^{(i)}d\tau \tag{5}$$

where $\mathbf{D}(i)$ and α are as in Theorem 1, $\widehat{\nu}_s$ is the Fourier or Hartley transform coefficient of ν, and ψ^{-1} is the inverse transform kernel.

Proof. Based on Theorem 1 we may solve for the first k derivatives of the pair $\nu\,\xi^{(i)}$. Consequently, $\nu(t)$ is replaced by a series of the selected kernels. Then use integration by parts until all the derivatives shift from ξ to $\psi_{sw_0}^{-1}\mathbf{\Psi}(\mathbf{w}\alpha^k, .)$ while employing (2) so as to let vanish all boundary terms.

Theorem 2 may easily be applied to classical convolvable signals by setting $k = 0$. However, the complexity is to solve the integral transform and the summation. The next section is dedicated to the determination of $\mathbf{D}(i)$ and α, and the reduction of (5) by means of selecting the kernels.

2.3 Kernels of FDMF

Complex Fourier modulating function For $\mathbf{w} \in \mathbb{R}^{N_w}$ the i-th time derivative of ϕ_m can be expressed as

$$\begin{aligned}\mathbf{\Phi}^{(i)}(t) &= \mathbf{R}\mathbf{\Psi}^{(i)}(\mathbf{w},t)\\ &= \mathbf{R}(-1)^i \underbrace{j^i \begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_{N_w}\end{pmatrix}^i}_{\mathbf{D}(i)} \underbrace{\begin{pmatrix} e^{-jw_1 t} \\ \vdots \\ e^{-jw_{N_w} t}\end{pmatrix}}_{\mathbf{\Psi}(\mathbf{w},t)}\end{aligned} \tag{6}$$

where ψ is the complex Fourier kernel transform, $j = \sqrt{-1}$ and w_i is multiple of $\omega_0 = \frac{2\pi}{T}$ in the case of finite time interval $[t_0, t_0+T]$. For reducing the integral expression of Theorem 2 we substitute (6) into (5) and obtain

$$\mathbf{\Gamma}_{(\nu\xi^{(i)})^{(k)}} = \mathbf{R}\mathbf{D}(k)(\widehat{\nu} * \mathbf{D}(i)\widehat{\xi})(\mathbf{w}), \tag{7}$$

where $\widehat{(.)}$ represents the Fourier transform of a function and $(f * g)(\bar{w}) = \int_{t_0}^{t_0+T} f(w)g(\bar{w}-w)dw$ is the finite convolution of two functions f and g.

Hartley modulating function For $\mathbf{w} \in \mathbb{R}^{N_w}$ the i-th time derivative of ϕ_m can be written as

$$\begin{aligned}\mathbf{\Phi}^{(i)}(t) &= \mathbf{R}\mathbf{\Psi}^{(i)}(\mathbf{w},t)\\ &= \mathbf{R}(-1)^i \underbrace{cas'\left(\frac{i\pi}{2}\right) \begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_{N_d}\end{pmatrix}^i}_{\mathbf{D}(i)}\\ &\quad\times \underbrace{\begin{pmatrix} cas(w_1 t(-1)^i) \\ \vdots \\ cas(w_{N_w} t(-1)^i)\end{pmatrix}}_{\mathbf{\Psi}(\mathbf{w}(-1)^i,t)}\end{aligned} \tag{8}$$

where ψ is the Hartley kernel transform, $\text{cas}(a) = \cos(a) + \sin(a)$, $\text{cas}'(a) = \cos(a) - \sin(a)$, and w_i is multiple of $\omega_0 = \frac{2\pi}{T}$ in the case of finite time interval $[t_0, t_0+T]$. For reducing the integral expression of Theorem 2 we substitute (8) into (5) and obtain

$$\mathbf{\Gamma}_{(\nu\xi^{(i)})^{(k)}} = \mathbf{R}\mathbf{D}(k)(\widehat{\nu} \otimes \mathbf{D}(i)\widehat{\xi})(\mathbf{w}(-1)^i), \tag{9}$$

where $\widehat{(.)}$ represents the Hartley transform of a function and $f(w) \otimes g(w) = 0.5[f(w) * g(w) - f(-w) * g(-w) + f(w) * g(-w) + f(-w) * g(w)]$ is the Hartley convolution of two functions f and g.

Note that trigonometric Fourier modulating functions (TFMF), developed in Pearson and Lee (1985), may also be derived from the FDMF approach.

2.4 Determination of the linear combination matrix

Null space approach This approach was introduced by Pearson and Lee (1985). Here, matrix $\mathbf{R}$ is obtained by substituting the FDMF (3) into the boundary conditions (2), subsequent reordering in $[\mathbf{\Phi}^{(0)}(T), \ldots, \mathbf{\Phi}^{(N_d-1)}(T)]^{\mathrm{T}} = \mathbf{M}\mathbf{R}^{\mathrm{T}}$ and calculating

$$\mathbf{R} = \text{null}(\mathbf{M})^{\mathrm{T}}, \tag{10}$$

where $\mathbf{M}^{\mathrm{T}} = [\mathbf{\Psi}(\mathbf{w},T), \mathbf{\Psi}^{(1)}(\mathbf{w},T), \ldots, \mathbf{\Psi}^{(N_d-1)}(\mathbf{w},T)]$ and $\mathbf{R}$ is of dimension $(N_w - N_d) \times N_w$ with N_w the number of frequencies used in the MF. N_d is the maximum derivative order of the differential equation and null(.) represents the null space of a matrix. Among others, two approaches have been prosped to find the null space of a matrix: using an orthonormal[1] basis or a rational basis. The former is computed from the singular value decomposition whereas the latter from the reduced row echelon form, see Strang (1993).

The matrix $\mathbf{R}$ calculated from the orthonormal basis has no specific structure whereas the 'rational' basis leads to the following structure:

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & \ldots & r_{1,N_d} & 1 & 0 \\ \vdots & \vdots & \vdots & & \ddots \\ r_{N_w-N_d,1} & \ldots & r_{N_w-N_d,N_d} & 0 & 1 \end{pmatrix}. \tag{11}$$

Binomial matrix approach Here the matrix $\mathbf{R}$ takes its name from the binomial expansion $(a+b)^k = \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i$. This approach was employed in Rao and Unbehauen (2005); Pearson (1992) for both Fourier and Hartley MF.

In this approach, matrix $\mathbf{R}$ has the following structure:

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & \ldots & r_{1,N_d} & & \ldots & 0 \\ & r_{2,1} & \ldots & r_{2,N_d} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \ldots & & r_{N_w-N_d,1} & \ldots & r_{N_w-N_d,N_d} \end{pmatrix} \tag{12}$$

with rows $\mathbf{r}_l$ calculated by taking the null space of the vector $\mathbf{\Psi}([w_l, \ldots, w_{l+N_d}]^{\mathrm{T}}, t)$ in the boundary conditions (2), similar to the approach leading to (10).

An example is the complex Fourier modulating function (CFMF) developed in Pearson (1992). It uses a vector $\mathbf{w}$ that is formed by $\omega_0 k$, for $\omega_0 = 2\pi/T$ where k are consecutive integers. Then the MF $\phi_m(t)$ is given by

$$\phi_m(t) = e^{-jm\omega_0 t}(e^{-j\omega_0 t} - 1)^{N_d} \tag{13}$$

where $j = \sqrt{-1}$ and N_d is the maximum derivative order of the differential equation. Using the binomial expansion the CFMF (13) may be rewritten as

$$\phi_m(t) = \sum_{k=0}^{N_d}(-1)^{N_d-k}\binom{N_d}{k}e^{-j\omega_0(m+k)t}. \tag{14}$$

Therefore, $r_{i,k} = (-1)^{N_d-k}\binom{N_d}{k}$.

Note that the two approaches explained here are not unique, there are more solutions.

3. E-CONVOLVABLE SYSTEMS

For many system identification tasks it is desirable to have a simple, but accurate mathematical representation which incorporates most of the common nonlinearities. In this context, the main limitations of integrable and convolvable systems shall be explained in view of the nonlinearities expressed by multiplications of functions $f^{(i)}(y,u)$, $y^{(k)}$ and $u^{(p)}$.

[1] Two orthogonal vectors are orthonormal if they have Euclidian norm equal to 1.

3.1 Limitation of integrable systems

Integrable systems may be represented as

$$\sum_{i=0}^{n_1}\sum_{p=0}^{n_2} a_{i,p}\, f_{i,p}^{(i)}(u(t),y(t)) = 0 \tag{15}$$

for a set of known functions $f_{i,p}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, at least of class $\mathcal{C}^i$, with $a_{i,p}$ constant coefficients.

The main drawback of these systems lies in the number of repeated coefficients that depend on the order of the system. To clarify this, let us use the composite function $f\circ y:\mathbb{R}\to\mathbb{R}$ and denote the partial derivatives wrt. t and y as per $(.)^{(i_t,i_y)}$. In light of this, the i-th time derivative of the composite function $f^{(i)}(y(.))$ may be expressed as

$$\begin{aligned} f^{(1)} &= f^{(0,1)}y^{(1)} \\ f^{(2)} &= f^{(0,2)}\left(y^{(1)}\right)^2 + f^{(0,1)}y^{(2)} \\ f^{(3)} &= f^{(0,3)}\left(y^{(1)}\right)^3 + 3f^{(0,2)}y^{(1)}y^{(2)} + f^{(0,1)}y^{(3)} \\ \vdots &= \vdots \end{aligned} \tag{16}$$

From these steps of calculation it is clear that $f^{(2)}$ in terms of $f^{(0,2)}\left(y^{(1)}\right)^2$ and $f^{(0,1)}y^{(2)}$ cannot be calculated based on Theorem 1 alone. Thus, these terms always occur together with the same coefficient for the second and further derivative orders of f.

3.2 Limitation of convolvable systems

Convolvable systems may be represented as

$$\sum_{i=0}^{n_1}\sum_{p=0}^{n_2}\sum_{q=0}^{n_3} a_{i,p,q}\, g_q(u(t),y(t))\, f_{i,p}^{(i)}(u(t),y(t)) = 0 \tag{17}$$

for a set of known functions $f_{i,p}, g_q:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, the former of at least of class $\mathcal{C}^i$, with $a_{i,p,q}$ constant coefficients.

These systems, however, may handle the stated situation until the second order invoking $g\,f^{(i)}$ functions, only. However, for third and higher order a similar problem arises because it is not possible to represent the expression $y^{(1)}y^{(2)}$ with the disposable expressions.

3.3 Introduction of e-convolvable systems

In light of the above-presented limitations in Fourier and Hartley MF, we propose the class of *e-convolvable systems*. These are treatable with FDMF and show a representation in terms of form $g^{(k)}f^{(i)}$. Hence, for a set of known functions $f_{i,p}, g_{q,k}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, the former at least of class $\mathcal{C}^{i+k}$ and the latter of class $\mathcal{C}^k$, an e-convolvable system has the representation

$$\sum_{i,p,q,k=0}^{n_1,n_2,n_3,n_4} a_{i,p,q,k}\, f_{i,p}^{(i)}(u(t),y(t))\, g_{q,k}^{(k)}(u(t),y(t)) = 0. \tag{18}$$

The sum n_1+n_4 is the maximum derivative order and $a_{i,p,q,k}$ are constant coefficients. Such a representation is not directly feasible by applying Theorems 1 or 2. To prepare the ground for this it is necessary to express the expressions $g^{(k)}f^{(i)}$ in terms of $(gf^{(i)})^{(k)}$, $g^{(k)}$, and $f^{(i)}$. The following lemma can be proved:

Lemma 1. Let functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ such that $f\in\mathcal{C}^{i+k}$ and $g\in\mathcal{C}^k$. Then the multiplication of $g^{(k)}f^{(i)}$ can be rewritten as

$$g^{(k)}f^{(i)} = \sum_{s=0}^{k}(-1)^s\binom{k}{s}\left(gf^{(i+s)}\right)^{(k-s)}. \tag{19}$$

By means of Lemma 1 it is now possible to apply the FDMF method also to the class of systems represented by (18). Indeed, using the equations from Theorem 2 and Lemma 1 in (18) finally results in

$$\sum_{i,p,q,k=0}^{n_1,n_2,n_3,n_4} a_{i,p,q,k}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\mathbf{\Gamma}_{\left(g_{q,k}\,f_{i,p}^{(i+s)}\right)^{(k-s)}} = 0. \tag{20}$$

4. SIMULATION RESULTS

The theoretical developments of the previous sections have been tested with 5 classes of FDMF (illustrated in Table 1) and 2 examples. A linear system serves to verify the generalization and correct identification of the 5 obtained classes. A nonlinear system serves to analyze the e-convolvable extension.

Table 1. FDMF Methods

FDMF variations	Calculation of matrix $\mathbf{R}$	Transform kernel
NSFC	Null space	Complex Fourier
NSFT	Null space	Trigonometric Fourier
NSH	Null space	Hartley
BF	Binomial	Complex Fourier
BH	Binomial	Hartley

These classes are combinations of Fourier and Hartley transforms, and different selections of matrix $\mathbf{R}$. Among these, the types NSFT, BF and BH for especial cases of the vector $\mathbf{w}$ and matrix $\mathbf{R}$ are the methods used by Pearson and Lee (1985), Pearson (1992), Shen (1993), Rao and Unbehauen (2005) for linear, integrable, and convolvable systems.

The simulations for the identification using FDMF are processed in the following way:

(1) The dynamic system is rearranged into an input-output differential equation that shows linear, integrable, convolvable or e-convolvable system structure.
(2) The frequency vector $\mathbf{w}$ is selected.
(3) The system is simulated for inputs $u=\sin(\frac{\pi}{T}F_m t^2)$ obtaining outputs y. The function LSIM() of Matlab is used to simulate the linear differential equation, and Euler method the nonlinear non-stiff systems.
(4) The values of u and y used to identify the system are corrupted with white Gaussian noise generated by the Matlab function AWGN() for a 50% (high noise influence) of noise-signal ratio (NSR) defined as

$$\mathrm{NSR}_x = \frac{P_{\eta_x}}{P_x}\cdot 100\%,$$

with P_{η_x} average power of the additive noise on the system signal and P_x average power of the noiseless system signal x.
(5) Calculate integral transforms and $\mathbf{\Gamma}$.

(6) Reorder the differential equation as per

$$\boldsymbol{\Gamma}_{y^{(n)}} + \mathbf{V}\boldsymbol{\Theta} = \epsilon.$$

There, $\boldsymbol{\Gamma}_{y^{(n)}}$ is the modulated function of $y^{(n)}$, $\mathbf{V}$ is the matrix with all the remaining modulated functions, $\boldsymbol{\Theta}$ are the coefficients of the modulated functions, and ϵ is the modulated (transformed) error.

(7) Compute coefficient estimates $\bar{\boldsymbol{\Theta}}$ with least squares.

(8) Repeat steps 4–7 for the required number of experiments N_{exp} for a Monte Carlo simulation (set to 200).

(9) The identification performance is assessed using the mean square bias (MSB) and mean variance (MV) of the coefficients. The MSB is related with the accuracy and the MV with the precision of the identification. They read

$$\text{MSB}_\Theta = \frac{1}{N_\Theta}\sum_{k=1}^{N_\Theta}(\underbrace{\theta_k - \text{mean}(\bar{\theta}_k)}_{\text{bias}})^2$$

$$\text{MV}_\Theta = \frac{1}{N_\Theta}\sum_{k=1}^{N_\Theta}\text{var}(\bar{\theta}_k)$$

Therein, θ_k is the true k-th coefficient of the differential equation, $\bar{\theta}_k$ is an estimated coefficient, N_Θ is the number of coefficients. The mean and variance operations are with respect to N_{exp}.

Example 1. (Linear system). A second order linear non-minimum phase system with a large influence of the non-minimum phase zero is considered. The transfer function reads

$$\frac{Y(s)}{U(s)} = \frac{2s - 0.2}{s^2 + 3.8s + 2.4}.$$

The maximum input frequency F_m is set to 0.7 Hz based on the system's bandwidth. The integration time is $T = 10$ sec., the sampling frequency $F_s = 100$ Hz and the vector $\mathbf{w}$ is fixed to be successive and symmetric with maximum frequency $w_{N_w} \in [0.3, 0.9]$Hz.

Figure 1 shows the evaluation of the MSB and MV of the proposed classes. It can be seen that all classes of the generalized MFT are able to identify the system. Besides, it may be taken notice that the identification performance changes according to the selected vector $\mathbf{w}$ even for the most simple case (successive and symmetric). It may be further observed that BF and BH as NSFC and NSH have the same behavior wrt. the same vector $\mathbf{w}$.

Example 2. (e-convolvable system). For assessing the e-convolvable approach from Section 3 to handle systems represented by summations of two multiplied functions, a random differential equation was selected that may be treated as e-convolvable or and convolvable, for a fair comparison. It is given by

$$y^{(2)} + 0.5(y^{(1)})^2 + 1.2y^{(1)} + 0.9y = 1.5u.$$

Note that the identification of convolvable systems uses

$$(y^{(1)})^2 = \frac{1}{2}(y^2)^{(2)} - yy^{(2)}$$

and for the e-convolvable Lemma 1:

$$(y^{(1)})^2 = y^{(1)}y^{(1)} = (yy^{(1)})^{(1)} - yy^{(2)}.$$

We chose the maximum input frequency $F_m = 0.6$ Hz, $T = 10$ sec., and $F_s = 100$ Hz. The convolution frequencies are equal to $\mathbf{w}$, and $\mathbf{w}$ is successive and symmetric with maximum frequency $w_{N_w} \in [0.3, 0.9]$ Hz. Figure 2 shows on the top the evaluation of MSB_Θ for e-convolvable systems and on the bottom for convolvable.

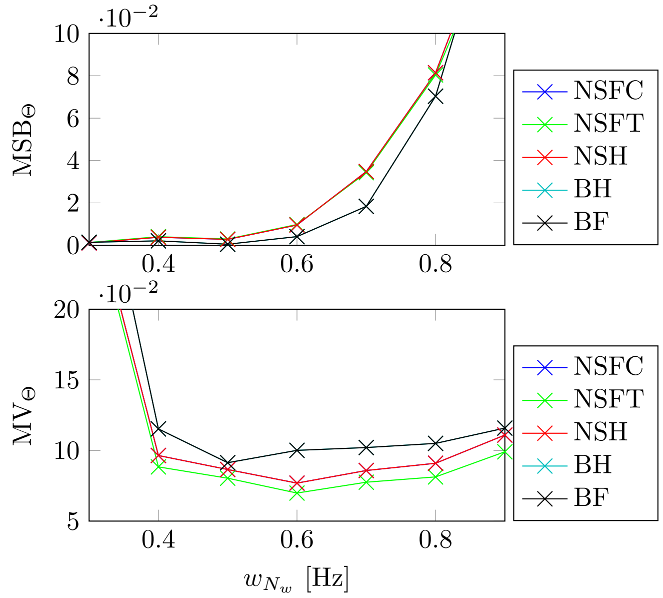

Fig. 1. MSB_Θ and MV_Θ comparison for the linear system.

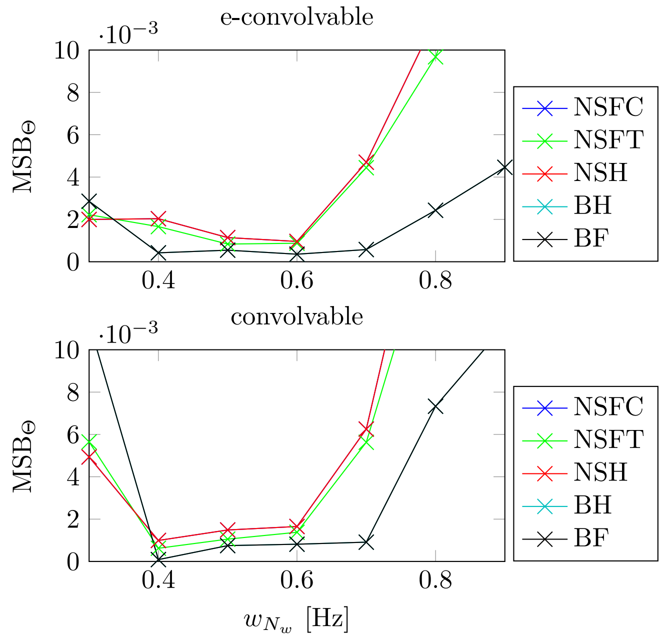

Fig. 2. MSB_Θ of e-convolvable and convolvable identification.

In addition to this nonlinear systems we compared the term $(y^{(1)})^2$ in the context of modulated signals, i.e., for the ideal case $\boldsymbol{\Gamma}_{(y^{(1)})^2}$ (noiseless signal $y^{(1)}$), the convolvable $\frac{1}{2}\boldsymbol{\Gamma}_{(y^2)^{(2)}} - \boldsymbol{\Gamma}_{yy^{(2)}}$ (y data corrupted with noise) and the e-convolvable $\boldsymbol{\Gamma}_{(yy^{(1)})^{(1)}} - \boldsymbol{\Gamma}_{yy^{(2)}}$ (y data corrupted with noise) for $w_{N_w} = 0.6$ Hz. Figure 3 shows the modulated signal $(y^{(1)})^2$ for the NSFC method. We concerned only this class since others show similar performance for convolvable and e-convolvable methods.

From Figure 2 we observe, in general, that the e-convolvable identification yields more accurate coefficients

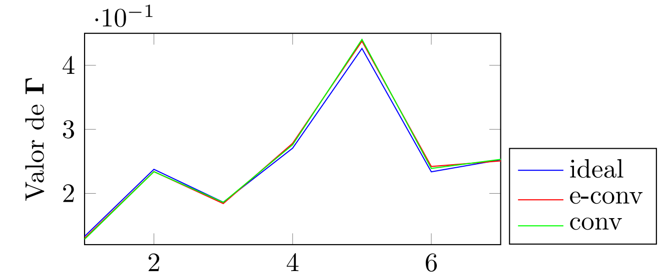

Fig. 3. Comparison of $\mathbf{\Gamma}_{(y^{(1)})^2}$ for the NSFC method.

(low MSB_Θ) than convolvable. However, if focusing on $w_{N_w} \in [0.4, 0.6]$ Hz both have similar performance. This is confirmed by analysis of Figure 3 where the vectors $\mathbf{\Gamma}_{(y^{(1)})^2}$ obtained with the convolvable and e-convolvable approaches are almost the same as the ideal.

Finally we apply Theorem 2 and Lemma 1 on four more complex modulated signals, i.e. $\mathbf{\Gamma}_{(y^{(1)}u^{(1)})}$, $\mathbf{\Gamma}_{(y^{(1)}u^{(2)})}$, $\mathbf{\Gamma}_{(y^{(2)}u^{(1)})}$ and $\mathbf{\Gamma}_{(y^{(2)}u^{(2)})}$, using the same data ($y$ and u) and parameters (T, NSR, w_{N_w}, aso.) from Example 2. The result are depicted in Figure 4 where we only show the NSFC class due to similar performances. From there we draw that the e-convolvable approach yields similar results as the ideal case in all the modulated signals. Besides, we can appreciate in our example that for bigger derivative order of y, the performance decrease.

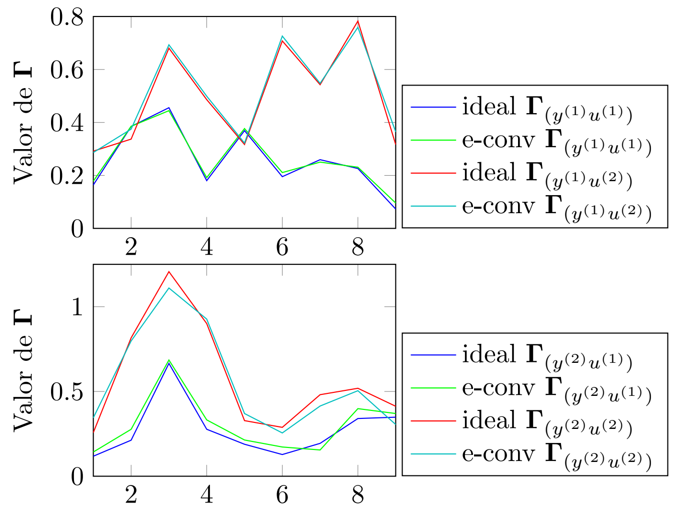

Fig. 4. Comparison of $\mathbf{\Gamma}_{(y^{(1)}u^{(1)})}$, $\mathbf{\Gamma}_{(y^{(1)}u^{(2)})}$, $\mathbf{\Gamma}_{(y^{(2)}u^{(1)})}$ and $\mathbf{\Gamma}_{(y^{(2)}u^{(2)})}$ for the NSFC method.

5. CONCLUSION

The FDMF, considered as a unified framework, is a powerful method for the identification of continuous-time models. We have given evidence to its capabilities, building 5 MF classes for its assessment. All have been able to successfully identify (low MSB) a linear and nonlinear system under considerable noise influence. Besides, due to the general representation of the FDMF some flexibility is given by selection of the frequency vector, kernels, and linear combination matrix. This variants grant possibilities for further increasing the identification convergence rate.

The proposed extension, called 'e-convolvable', has shown promising simulation results for the identification of nonlinear systems, expressed by a multiplication of two arbitrary time-dependent functions. Furthermore, even on the presence of high noise influence, the modulation of the signals produced very accurate results for low differentiation orders.

Future research on the FDMF will have a focus on real-world e-convolvable system models, e.g. centrifugal and coriolis forces, performance optimization, real-time conditions for identification, design of experiments; all including tentatively more nonlinearities.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my supervisors, Johann Reger and Julio Tafur, who gave me the opportunity to successfully complete my Double-Master degree under their guidance.

REFERENCES

Behre, D. (1999). *Parameter Identification of Nonlinear Continuous-time Systems Using the Hartley Modulating Functions Method*. Dissertation, Ruhr-University Bochum, Bochum, Germany.

Byrski, W. and Byrski, J. (2012a). The Improvement of Modulating Function Method for Fast Identification of Parameters Faults in Linear Continuous Systems. In *Proceedings of AsiaMIC 2012*.

Byrski, W. and Byrski, J. (2012b). The Role of parameter Contraints in EE and OE Methods for optimal identification of continuous LTI Models. In *Int. J. Appl. Math. Comput. Sci.*, volume 22, 379–388.

Fairman, F.W. (1971). Parameter identification for a class of multivariable non-linear processes. In *International Journal of Systems Science*, 291–296.

Janiczek, T. (2010). Generalization of the modulating functions method into the fractional differential equations. *Bulletin of the Polish Academy of Science*.

Patra, A. and Unbehauen, H. (1995). Identification of a class of nonlinear continuous-time systems using Hartley modulating functions. *International Journal of Control*, 1431–1451.

Pearson, A.E. (1992). Explicit Parameter Identification for a Class of Nonlinear Input/Output Differential Operator Models. In *Proceedings of the 31st Conference on Desicion and Control*.

Pearson, A.E. and Lee, F.C. (1985). On the Identification of Polynomial Input-Output Differential Systems. In *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, volume AC-30, 778–782.

Rao, G. and Unbehauen, H. (2005). Identification of continuous-time systems. In *IEE Proc.-Control Theory Appl.*, volume 153, No. 2.

Shen, Y. (1993). *System Identification and Model Reduction Using Modulating Functions Techniques*. Dissertation, Brown University, Rhode Island, United States.

Shinbrot, M. (1957). On the analysis of linear and nonlinear system. In *Trans. ASME*, volume 79, 547–552.

Strang, G. (1993). *Introduction to Linear Algebra*. Wellesley-Cambridge Press.