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An output feedback scheme for the terminal sliding controller based on Lyapunov functions ⋆

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Abstract: In this paper we provide an output feedback (OF) scheme for an uncertain system with a well-defined relative degree 2 with respect to the measured output. For that, we use two specific second-order sliding mode algorithms to implement the OF. Firstly, we consider the design of a state feedback (SF) control based on a terminal sliding algorithm, which assures finite-time convergence and robustness under bounded perturbations. Secondly, we design a finite-time convergent and robust observer to estimate the state of the plant. The observer is based on the super-twisting (ST) algorithm. For the terminal sliding controller and for the ST based observer, we have proposed smooth strict Lyapunov functions. Finally, the stability analysis of the interconnection of the observer and the controller in the OF scheme is established by Lyapunov arguments, in the same spirit as it is done for the observer-based OF control in conventional nonlinear control. It is shown that the gains of the observer have to be designed taking into account the gains of SF controller in order to ensure the robustness of the OF.

Keywords: Second-order sliding modes, finite-time output feedback stabilization, Lyapunov Methods.

1. INTRODUCTION

Second-order sliding mode controllers (2-sliding controllers) are considered to be effective to keep at zero outputs of relative degree 2, while at the same time, it deals with bounded uncertainties and disturbances. Some 2-sliding controllers with finite time convergence has been successfully implemented for solution of real problems. For example, the suboptimal controller Bartolini et al. (1998), the terminal sliding mode controller Man et al. (1994), the twisting controller, Levant (1993), and some others algorithms given in Levant (2007).

However, 2-sliding controllers require the measurement of the output σ and its derivative $\dot{\sigma}$, but in practice, it is common that only σ is available. The robust finite-time differentiator is used to obtain the information of $\dot{\sigma}$. Such a differentiator is based on the well-known super-twisting (ST) algorithm, Levant (1993).

In general, for nonlinear systems, a separated design of stable observers and controllers does no ensure the stability of their combination (no separation principle). Generally, the interconnection of a 2-sliding controller and a finite-time observer, is argued to be stable due to the finite-time convergence of the individual algorithms Levant (2005). Due to the finite-convergence of the observer, the controller is turning on after certain time at which the observer has already converged. However, this arguments are not addressed from the point of view of the classical observer-based output feedback (OF) control.

The study of the problem of finite-time OF stabilization has been successful for continuous control laws. Particulary, for the double integrator system some important results has been established in Hong et al. (2001), Orlov et al. (2011), Bernuau et al. (2012). In Orlov et al. (2011), the stability of the observer-based OF control is concluded due to the finite-time convergence of the controller and the observer as it usually is done in Levant (2005). On the other hand, Hong et al. (2001) and Bernuau et al. (2012) show that the controller and the observer can work at the same time providing finite-time stabilization of the OF. In Hong et al. (2001), Orlov et al. (2011), the perturbed case is also analyzed giving more general results for the OF than those reported in Bernuau et al. (2012).

Motivated by the classical observer-based OF control, an robust OF scheme for the Twisting controller is proposed in Moreno (2012). This work shows that under a correct design of the gains of the controller and observer, it is unnecessary turning on the controller until the observer converges because they can work at the same time. This output feedback scheme provides finite-time stabilization.

The purpose of this paper is to provide an OF scheme using two specific 2-sliding controllers and a Lyapunov-based approach. The idea is to establish by Lyapunov arguments an observer-based OF control, in the same spirit as it is done conventional in non-linear control literature. With this in mind, we firstly design an state feedback (SF) control based on a 2-sliding controller. It assures finite-time convergence and robustness under bounded perturbations. These properties are characterized by using a smooth strict LF. Also, the LF allows to design the controller gains and to estimate the convergence time. After this, we design a finite-time convergent and robust observer to estimate the state of the plant. The observer is based on the ST algorithm. For the ST based observer, we also propose a smooth LF. It differs from the previous one proposed by Moreno (2011) which is not Lipschitz.

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We show that the interconnection of the 2-sliding controller and the observer does not satisfy the separation principle, since the observer designs have to be designed taking into account the controller gains. In contrast with the arguments used for the establishment of an OF scheme in sliding mode control Bartolini et al. (2000),Levant (2007), we propose a Lyapunov-based approach.

2. PRELIMINARIES

Let us introduce some important concepts. Consider the system

$$\dot{x} = f(x),\ x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the associated vector field. Assume that the origin is an asymptotically stable (AS) equilibrium point, i.e.,$f(0) = 0$. Solutions of (1) are understood in the sense of Filippov (1988). Define an open ball centered at the origin with radius $r > 0$ by $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$.

Definition 1. Bhat and Bernstein (2005). The origin of system (1) is finite-time stable (FTS) if it is AS and for every $x_0 \in B_r \setminus \{0\}$, any solution $x(t,x_0)$ of (1) reaches $x(t,x_0) = 0$ at some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

The notion is global if it satisfies Definition 1 with $B_r = \mathbb{R}^n$.

Our approach is based on homogeneous of homogeneous systems. Then, we recall some important concepts related to this.

Definition 2. Let $\Lambda^r_\varepsilon x := (\varepsilon^{r_1}x_1,...,\varepsilon^{r_n}x_n) = diag(\varepsilon^{r_i})x$ be the dilation operator $\forall \varepsilon > 0$ and $\forall x \in \mathbb{R}^n$, where $r_i > 0$, $i = 1,...,n$, are weights of the coordinates.

- **i)** A function $V : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $m \in \mathbb{R}$ with respect to (w.r.t.) $\Delta^r_\varepsilon x$, if $V(\Delta^r_\varepsilon x) = \varepsilon^m V(x)$ holds, Bacciotti and Rosier (2005).
- **ii)** A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $l \in \mathbb{R}$ w.r.t. $\Delta^r_\varepsilon x$, if $f(\Delta^r_\varepsilon x) = \varepsilon^l \Delta^r_\varepsilon f(x)$ holds, Bacciotti and Rosier (2005).
- **iii)** A vector-set field $F(x) \subset \mathbb{R}^n$, is homogeneous of the degree $l \in \mathbb{R}$ w.r.t. $\Delta^r_\varepsilon x$, if $F(\Delta^r_\varepsilon x) = \varepsilon^l \Delta^r_\varepsilon F(x)$ holds,Levant (2005).
- **iv)** System (1) is homogeneous of degree $l \in \mathbb{R}$ iff the vector field (or the vector-set field) f is so.

Stability of homogeneous systems (homogeneous differential inclusions (DI's)) can be studied by means of homogeneous Lyapunov functions (HLF's), Bacciotti and Rosier (2005),Bhat and Bernstein (2005), Nakamura et al. (2002). For a homogeneous continuous vector field f of degree l with locally AS equilibrium point, a C^p HLF of degree m exists if $m > p \cdot \max_i\{r_i\}$ for any $p \in \mathbb{N}$, Bacciotti and Rosier (2005). For a homogeneous Filippov DI, we have the following result.

Theorem 3. Nakamura et al. (2002). Assume that the origin of a homogeneous Filippov DI, $\dot{x} \in F(x)$, is uniformly globally AS. Then, there exists a C^∞ homogeneous strong LF.

Along this paper the operator $\lceil z \rfloor^m := |z|^m \operatorname{sign}(z)$, $z \in \mathbb{R}$, $m \geq 0$, is used.

3. PROBLEM STATEMENT

Consider a single-input single-output dynamical system given by

$$\dot{x} = f(x,t) + g(x,t)u,\ \sigma = h(t,x), \tag{2}$$

where $x \in \mathbb{R}^n$ defines the state, $u \in \mathbb{R}$ is the control input and σ is the measured output. The functions f and g are smooth but unknown. We assume that the system has well defined relative degree 2 w.r.t. σ. Under these conditions and defining the state variables $z = \sigma$ and $z_2 = \dot{\sigma}$, system (2) can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= a(x,t) + b(x,t)u, \end{aligned} \tag{3}$$

where $a(\cdot), b(\cdot)$ are some unknown smooth scalar functions which are assumed to be globally uniformly bounded, i.e.

$$|a(x,t)| \leq C,\ K_m \leq b(x,t) \leq K_M,\ \forall x \in \mathbb{R}^2,\ \forall t \geq 0, \tag{4}$$

for some known positive constants C, K_m, K_M. This condition shows that $a(\cdot)$ and $b(\cdot)$ do not need to be exactly known. Since functions $a(\cdot)$ and $b(\cdot)$ are uncertain, a continuous SF controller could not stabilize system (3).

The control problem consists in using a discontinuous feedback control to drive the output $\sigma(x,t)$ to vanish in finite time and keep $\sigma = 0$, even in spite of disturbances, and using the only the measured output σ. To solve the problem, we use the controller

$$u(z_2,z_1) = -k_2 \operatorname{sign}\left(\lceil z_2 \rfloor^2 + k_1^2 z_1\right), \tag{5}$$

where the gains k_2 and k_1 are appropriately designed. Controller (5) is similar to the controller with prescribed convergence law $u = -k_2 \operatorname{sign}(z_2 + k_1 \lceil z_1 \rfloor^{\frac{1}{2}})$, Levant (2007), because they are discontinuous on the same curve $z_2 = -k_1 \lceil z_1 \rfloor^{\frac{1}{2}}$. Solutions of system (2) with the controller (5) are understood in the Filippov's sense, Filippov (1988).

The unmeasurable state $z_2 = \dot{\sigma}$ is determined by means of an observer which only uses the measured states $z_1 = \sigma$. Due to the uncertainties in (3), a continuous observer can not provide the exact estimation of $\dot{\sigma}$ in finite time. Therefore, we use the discontinuous ST observer

$$\begin{aligned} \dot{\hat{z}}_1 &= -l_1 \lceil \hat{z}_1 - z_1 \rfloor^{1/2} + \hat{z}_2, \\ \dot{\hat{z}}_2 &= -l_2 \operatorname{sign}(\hat{z}_1 - z_1) + K_m u, \end{aligned} \tag{6}$$

The main goal of this paper is to design an OF finite-time stabilizing controller by considering the OF control law

$$u(\hat{z}_2,z_1) = -k_2 \operatorname{sign}\left(\lceil \hat{z}_2 \rfloor^2 + k_1^2 z_1\right), \tag{7}$$

where $\widehat{z}_2$ is provided by the observer (6). We will proceed in three steps: (i) we show that the origin of (3) FTS with the SF controller (5); after this, (ii) we design the ST observer; and finally (iii) we show that interconnection (3)-(7)-(6) is robust and FTS.

4. SMOOTH LF FOR THE DISCONTINUOUS CONTROLLER

When both states (z_2, z_1) are available, the uncertain system (3) in feedback with the discontinuous controller (5) leads to

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= a(x,t) - b(x,t)k_2 \operatorname{sign}\left(\lceil z_2 \rfloor^2 + k_1^2 z_1\right). \end{aligned} \tag{8}$$

Note that system (8) is homogeneous of degree -1 with weights $(2,1)$. Under hypothesis (4), we obtain the following result.

Theorem 4. Suppose that condition (4) holds and choose the controller (5) for system (3). If the gains are chosen such that the inequality $k_2 K_m > 2k_1^2 + C$ holds with $k_1 > 0$, then the origin of the closed-loop system (8) is robust and globally FTS.

According to Levant (2007), with the gain conditions stated by the previous Theorem, any trajectory inevitably hits the curve $\lceil x_2 \rfloor^2 + k_1^2 x_1 = 0$ and slides on it until the origin is reached in finite-time. In few words, a sliding mode of first order is established.

The proof of this result is derived from the following Proposition.

Proposition 5. The continuously differentiable function

$$V = \frac{1}{3}|z_2|^3 + k_1^2 z_1 z_2 + \frac{4}{3}k_1^3 |z_1|^{\frac{3}{2}}, \tag{9}$$

is a robust LF for the feedback system and the time derivative $\dot{V}$ along the trajectories of (8) satisfies

$$\dot{V} \leq -a_{22}|\sigma_2| - k_1^4|z_1|, \sigma_2 = \lceil z_2 \rfloor^2 + k_1^2 z_1, \tag{10}$$

where $a_{22} = k_2 K_m - C - 2k_1^2$. Moreover, the time derivative of the LF satisfies

$$\dot{V} \leq -\kappa_c V^{\frac{2}{3}}, \tag{11}$$

where $\kappa_c > 0$ is a scalar depending on the selection of k_1, k_2 and C. Furthermore, any state trajectory reaches the origin, from any initial state $z_0 \in \mathbb{R}^2$, in a finite-time smaller than

$$T(z_0) \leq \frac{3}{\kappa_c} V^{\frac{1}{3}}(z(0)). \tag{12}$$

Proof. First, positive definiteness of the LF is proved. From Lemma 14 (see Appendix), it follows that $-(\frac{2}{3}\gamma_l^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}\gamma_l^{-3}|z_2|^3) \leq |z_1||z_2| \leq \frac{2}{3}\gamma_u^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}\gamma_u^{-3}|z_2|^3$. Then, (9) can be lower and upper bounded as

$$a_1|z_1|^{\frac{3}{2}} + a_2|z_2|^3 \leq V \leq b_1|z_1|^{\frac{3}{2}} + b_2|z_2|^3,$$

where $a_1 = \frac{2}{3}k_2^2(2k_1 - \gamma_l^{\frac{3}{2}})$, $a_2 = \frac{1}{3}(1 - k_1^2\gamma_l^{-3})$, $b_1 = \frac{2}{3}k_2^2(2k_1 + \gamma_u^{\frac{3}{2}})$, $b_2 = \frac{1}{3}(1 + k_1^2\gamma_u^{-3})$. The LF is positive definite if $a_1 > 0$ and $a_2 > 0$, i.e., if $\frac{2}{3}k_2^2(2k_1 - \gamma_l^{\frac{3}{2}}) > 0$ and $\frac{1}{3}(1 - k_1^2\gamma_l^{-3}) > 0$, or equivalently, iff $(2k_1)^2 > \gamma_l^3 > k_1^2$. It is not difficult to see that γ_l always exists for any $k_1 > 0$. Then, the LF is positive definite and decreasing. Now, define $s_2 = z_2 + k_1\lceil z_1 \rfloor^{\frac{1}{2}}$. It is worth to note that s_2 and σ_2 are equals to $z_2 = -k_1\lceil z_1 \rfloor^{\frac{1}{2}}$ when $s_2 = \sigma_2 = 0$. Hence, taking the time derivative of V along the trajectories of (8), we have

$$\dot{V} = \sigma_2(a(x,t) + b(x,t)u) + k_1^2 s_2 \dot{z}_1 + k_1^3\lceil z_1 \rfloor^{\frac{1}{2}}\dot{z}_1.$$

Under the hypothesis (4) and the control input (5), we get

$$\dot{V} \leq -(k_2 K_m - C)|\sigma_2| - k_1^4|z_1| + k_1^2|s_2|^2,$$

which is continuous everywhere. From Lemma 15 (see Appendix), we deduce that $|s_2|^2 \leq 2|\sigma_2|$, from this we arrive at (10). The time derivative $\dot{V}$ is always negative definite if $a_{22} > 0$ and $k_1 > 0$, i.e., if the gains are chosen as in Theorem 4. Note that the functions $V(z)$ and $\dot{V}(z)$ are homogeneous continuous functions of degrees $m_V = 3$ and $m_{\dot{V}} = 2$ w.r.t. weights $r_i = \{2, 1\}$. Respectively, it follows from Lemma 16 that (11) holds, $\forall z \in \mathbb{R}^2$, where $\kappa_c = \min_{\{x:V(z)=1\}}\{-\dot{V}(z)\}$. Note that $\kappa_c > 0$ since $-\dot{V}(x)$ is positive definite. The right-hand side of the inequality (11) characterizes the finite-time stability. Finally, from the comparison principle, Khalil (2002), for $V_0 = V(x_0)$, the solution $V(t)$ satisfies $V(t) \leq \min(V_0^{\frac{1}{3}} - \frac{1}{3}\kappa_c t)^{\frac{3}{1}}$. It leads to find (19).

The following result establishes the trajectories will converge to zero in finite time even if the perturbation is initially larger than C but its bound becomes eventually smaller than C.

Theorem 6. Consider the closed-loop system (8) with bounded perturbation satisfying

$$|a(t,x)| \leq C + \omega(t) \leq C + \Omega,$$

instead of (4), where $\omega(t) > 0$, $\forall t > 0$, is an integrable function for every finite $t > 0$. Assume that the gains are selected as in the Theorem 4. Suppose that $\omega(t)$ vanishes asymptotically and it is, e.g. absolutely integrable, then $x(t) \to 0$ in finite time.

Proof. It follows from the LF (9). For this case, the time derivative satisfies

$$\dot{V} \leq -\kappa_c V^{\frac{2}{3}} + \omega(t)(|\sigma_2| + |z_1|) - \omega(t)|z_1|.$$

Note that the functions $|\sigma_2| + |z_1|$ is a homogeneous function of degree $m_{\dot{V}} = 2$ w.r.t. weights $r_i = \{2, 1\}$. Respectively, it follows from Lemma 16 that $(|\sigma_2| + |z_1|) \leq \lambda_{\max} V^{\frac{2}{3}}(z)$ holds, $\forall z \in \mathbb{R}^2$, where $\lambda_{\max} = \max_{\{x:V(z)=1\}}\{|\sigma_2| + |z_1|\}$. Therefore,

$$\dot{V} \leq -(\kappa_c - \lambda_{\max}\omega(t))V^{\frac{2}{3}}.$$

Integrating the equation $\dot{v} = -(\kappa_c - \lambda_{\max}\omega(t))v^{\frac{2}{3}}$, we obtain

$$\begin{aligned} v(t) &= (v_0^{1/3} - \frac{\kappa_c}{3}t + \frac{\lambda_{\max}}{3}\int_0^t \omega(\tau)d\tau)^{\frac{1}{3}} \\ &\leq (v_0^{1/3} - \frac{1}{3}(\kappa_c - \lambda_{\max}\Omega)t)^{\frac{1}{3}} \end{aligned}$$

Note that $v(t)$ remains bounded for every time and for every bounded perturbation if $\kappa_c \geq \lambda_{\max}\Omega$. Moreover, if the perturbation vanishes asymptotically and $\int_0^t \omega(\tau)d\tau$ is finite, then $v(t) \to 0$ as $t \to \infty$.

5. SMOOTH LF FOR THE ST OBSERVER

The stability properties of the ST algorithm has been analyzed by means of a like-quadratic and nonsmooth LF in Moreno (2011). In this section, we propose a smooth LF to show that the ST observer (6) ensures that the estimation error converge exactly, robustly and in finite time to the plant's values. *Exact* means that finite-time convergence is achieved despite of bounded perturbations.

Define the estimation errors as $e_1 = \hat{z}_1 - z_1$ and $e_2 = \hat{z}_2 - z_2$. Using (3) and (6), the dynamics of the observation error is calculated as

$$\begin{aligned} \dot{e}_1 &= -l_1\lceil e_1 \rfloor^{1/2} + e_2, \\ \dot{e}_2 &= -l_2 \operatorname{sign}(e_1) + \rho_2(t,x,u), \end{aligned} \tag{13}$$

where the perturbation term

$$\rho_2(t,x,u) = -a(x,t) - (b(x,t) - K_m)u. \tag{14}$$

This function satisfies $|\rho_2(t,x,u)| \leq \rho$, where the bound $\rho = C + (K_M - K_m)k_2$. Note that $\rho_2(t,x,u)$ is a nonvanishing perturbation.

Remark 7. In the case when $b(x,t) = K_m$, being K_m a known positive constant, $\rho_2(x,t) = -a(x,t)$ and $|\rho_2(x,t)| \leq C$. Therefore, the bound of the perturbation does not depend on the gain k_2 of the controller.

Note that the error dynamic (13) is homogeneous of degree -1 with weights $(2,1)$. By properly selecting the parameters of the ST observer, the zero solution of the error dynamic (13) is FTS.

Theorem 8. Suppose that condition (4) holds and chose the observer (6) for system (3). If the gains are chosen such that the inequality

$$l_1^2 > 2\frac{(l_2+\rho)|(3l_2+\rho)^2 - 4(l_2+\rho)^2|}{|l_2-\rho|^2} - \frac{4(l_2+\rho)^2}{|l_2-\rho|} \tag{15}$$

holds with $l_2 > \rho$, then the origin $e = 0$ of (13) is robust and globally FTS.

The proof of this Theorem follows from the following Proposition.

Proposition 9. The continuously differentiable function

$$V = \frac{2}{3l_1^2}|e_2|^3 - e_1e_2 + \frac{2}{3}l_1|e_1|^{\frac{3}{2}}, l_1 > 0, \quad (16)$$

is a robust global LF for system (13) and the time derivative $\dot{V}$ along the trajectories of (13) satisfies

$$\dot{V} \leq -l_1^2|s_2|^2 + 2(l_2+\rho)|s_{2d}| - (l_2-\rho)|e_1|, \quad (17)$$

where $s_2 = \lceil e_1 \rfloor^{1/2} - \frac{e_2}{l_1}$, $s_{2d} = e_1 - \frac{\lceil e_2 \rfloor^2}{l_1^2}$ and l_1 as in (15). Moreover, the time derivative of the LF satisfies

$$\dot{V} \leq -\kappa_o V^{\frac{2}{3}}, \quad (18)$$

where $\kappa_o > 0$ is a scalar depending on the selection of l_1, l_2 and ρ. Furthermore, any state trajectory reaches the origin, from any initial state $e_0 \in \mathbb{R}^2$, in a finite-time smaller than

$$T(e_0) \leq \frac{3}{\kappa_o}V^{\frac{1}{3}}(e(0)). \quad (19)$$

Remark 10. In comparison with Moreno (2011), the gains provided by the smooth LF's are relatively conservative.

Proof. First, positive definiteness of the LF is proved. From Lemma 14, it follows that $-(\frac{2}{3}\gamma_l^{\frac{3}{2}}|e_1|^{\frac{3}{2}} + \frac{1}{3}\gamma_l^{-3}|e_2|^3) \leq |e_1||e_2| \leq \frac{2}{3}\gamma_u^{\frac{3}{2}}|e_1|^{\frac{3}{2}} + \frac{1}{3}\gamma_u^{-3}|e_2|^3$. Then, (16) can be lower and upper bounded as

$$a_1|e_1|^{\frac{3}{2}} + a_2|e_2|^3 \leq V \leq b_1|e_1|^{\frac{3}{2}} + b_2|e_2|^3,$$

where $a_1 = \frac{2}{3}(l_1 - \gamma_l^{\frac{3}{2}})$, $a_2 = \frac{1}{3}(\frac{2}{l_1^2} - \gamma_l^{-3})$, $b_1 = \frac{2}{3}(l_1 + \gamma_l^{\frac{3}{2}})$, $b_2 = \frac{1}{3}(\frac{2}{l_1^2} + \gamma_l^{-3})$. The LF is positive definite if $a_1 > 0$ and $a_2 > 0$, i.e., if $\frac{2}{3}(l_1 - \gamma_l^{\frac{3}{2}}) > 0$ and $\frac{1}{3}(\frac{2}{l_1^2} - \gamma_l^{-3}) > 0$, or equivalently, iff $l_1^2 > \gamma_l^3 > \frac{l_1^2}{2}$. It is not difficult to see that γ_l always exists for any $l_1 > 0$. Then, the LF is positive definite and decreasing. Taking the time derivative of V along the trajectories of (13), we have

$$\dot{V} = (l_1\lceil e_1 \rfloor^{1/2} - e_2)\dot{e}_1 - s_{2d}\dot{e}_2 + \frac{1}{l_1^2}\lceil e_2 \rfloor^2 \dot{e}_2.$$

Under the hypothesis (4) and the control input (5), we arrive at (17), which is continuous everywhere. It is worth to note that s_2 and s_{2d} are equals to $e_2 = l_1\lceil z_1 \rfloor^{\frac{1}{2}}$ when $s_2 = s_{2d} = 0$. In this case, Lemma 15 is useless to determine under what gain conditions the time derivative is negative definite, since $|s_2|^2 \leq 2|s_{2d}|$, but Lemma 17 can help us. To attain this, let us rewrite (17) as

$$-\dot{V} \geq l_1^2|s_2|^2 - 2(l_2+\rho)|s_{2d}| + (l_2-\rho)|e_1| > 0,$$

the last two terms of this inequality can be written as

$$l_1^2|s_2|^2 > 2(l_2+\rho)|s_{2d}| - (l_2-\rho)|e_1|,$$

Then from Lemma 17, fixing $\alpha = 2(l_2+\rho)$ and $\beta = l_2 - \rho$, we conclude that $\dot{V} < 0$ when the gains are chosen as in (15). Note that the functions $V(x)$ and $\dot{V}(x)$ are homogeneous continuous functions of degrees $m_V = 3$ and $m_{\dot{V}} = 2$ w.r.t. weights $r_i = \{2,1\}$. Again, it follows from Lemma 16 that (18) holds $\forall x \in \mathbb{R}^2$, where $\kappa_o = \min_{\{x:V(x)=1\}}\{-\dot{V}(x)\}$. Note that $\kappa_o > 0$ since $-\dot{V}(x)$ is positive definite. Finally, from the comparison principle Khalil (2002), for $V_0 = V(x_0)$, the solution $V(t)$ satisfies $V(t) \leq \min(V_0^{\frac{1}{3}} - \frac{1}{3}\kappa_o t)^{\frac{3}{1}}$. It leads to find (19).

6. CLOSED-LOOP SYSTEM STABILITY

Using the coordinates (z_1, z_2, e_1, e_2), the dynamic of closed-loop system is described by

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= a(x,t) - b(x,t)k_2\,\mathrm{sign}(\sigma_2) + \chi(t,x,e_2,z_2),\\ \dot{e}_1 &= -l_1\lceil e_1 \rfloor^{1/2} + e_2,\\ \dot{e}_2 &= -l_2\,\mathrm{sign}(e_1) + \rho_2(t,x,u),\end{aligned} \quad (20)$$

where $\rho_2(t,x,u)$ is given in (14) and the term

$$\chi(t,x,e_2,z_2) = b(x,t)k_2[\mathrm{sign}(\sigma_2) - \mathrm{sign}(\lceil z_2+e_2 \rfloor^2 + k_1^2\lceil z_1 \rfloor)],$$

is associated with the effect of the observation error in the control loop, and when $e_2 = 0$, it vanishes $\chi(t,x,0,z_2) = 0$. As we can see,

$$|\chi(t,x,e_2,z_2)| \leq 2K_Mk_2.$$

Note that the closed-loop system (20) is homogeneous of degree -1 w.r.t. to the dilation

$$\Delta_\varepsilon^r = (\varepsilon^2 z_2, \varepsilon z_1, \varepsilon^2 e_2, \varepsilon e_1). \quad (21)$$

Therefore, there exists a radially unbounded positive definite C^1 function $V(z,e)$ for (20), Hong et al. (2001). The construction of a such LF is not straightforward and other arguments need to be applied. For example, combining the FL's of Proposition 5 and Proposition 9, we define the following candidate LF

$$V_{LC} = \alpha V_1(z) + \beta V_2(e),$$

for the closed-loop system (20), where $\alpha, \beta > 0$ are suitable constants. Clearly, V_{LC} is positive definite, radially unbounded and a C^1 homogeneous function of degree $m_{V_{LC}} = 3$ w.r.t. the dilation (21). Taking the time derivative of V_{LC} along of all the trajectories of (20), we have

$$\begin{aligned}\dot{V}_{LC} &= \alpha\dot{V}_1(z) + \beta\dot{V}_2(e),\\ \dot{V}_1(z) &\leq -a_{22}|\sigma_2| - k_1^4|z_1| + |\chi||\sigma_2|, \sigma_2 = \lceil z_2 \rfloor^2 + k_1^2 z_1,\\ \dot{V}_2(e) &\leq -l_1^2|s_2|^2 + 2(l_2-\rho)|s_{2d}| - (l_2-\rho)|e_1|,\end{aligned}$$

and it is not easy to establish that the interconnection works.

Theorem 11. For system (3), consider that

(1) the functions $a(\cdot)$ and $b(\cdot)$ satisfy condition (4);
(2) the gains k_1 and k_2 are chosen according to Theorem 4;
(3) the gains l_1 and l_2 are chosen according to Theorem 8.

Then, the origin of closed-loop system (20) is robustly and GFTS.

Proof. From Theorem 8, we know that the estimation error converges to zero in finite time, i.e. there exists a $T(e_0) > 0$ such that $\forall t > T(e_0)$, it follows that $e_1(t) = e_2(t) = 0$. Note that the trajectories of system cannot escape to infinity in finite time. Moreover, the perturbation signal χ is also bounded, and since e_2 converges to zero in finite time, then it follows that $\chi(t,x,0,z_2) = 0, \forall t > T(e_0)$. Since $\int_0^t \omega(\tau)d\tau$ is finite, Theorem 6 ensures that $z(t)$ converges to the zero in finite time.

Remark 12. Since the origin of (13) is GTFS with settling time $T(e_0)$, we have $u(\hat{z}_2, z_1) = u(z_1, z_1)$, $t \geq T(e_0)$.

The gains l_1 and l_2 of the ST observer depend on the controller's gains k_1 and k_2 through the term $b(x,t)$. So they cannot be designed completely independently from each other. Then, Theorem 11 almost provides a separation principle.

7. ACADEMIC EXAMPLE

As an illustration, we design an observer-based OF for a second-order mechanical system, whose dynamic is described by

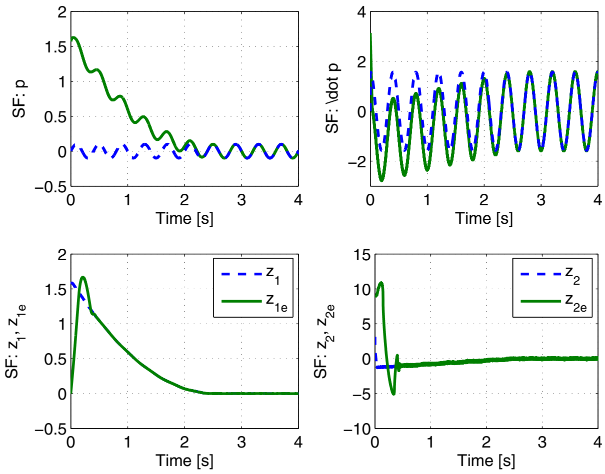

Fig. 1. Closed-loop behaviour for the state variables and their estimation under SF.

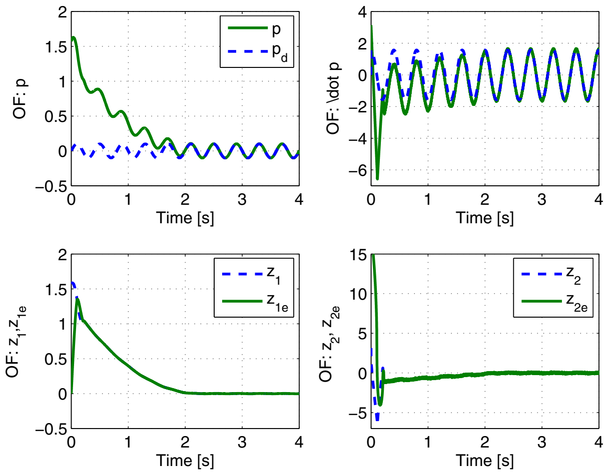

Fig. 2. Closed-loop behaviour for the state variables and their estimation under OF.

$$(1+\cos^2(p))\ddot{p}+a\,\mathrm{sign}(\dot{p})+g\sin(p)=u.$$

where p is the measured variable of position and $\dot{p}$ is the velocity variable, a is the coefficient of Coulomb friction and g is the constant of gravity acceleration. For simulation purposes: $a=0.5$, $g=9.8$, $p(0)=[\pi\ 0]^T$ and sampling time $\tau=0.001$ with Euler's integration method. The control target is to ensure finite-time tracking of the reference trajectory $p_d = 0.1\cos(5\pi t)$. Define $z_1=p-p_d$ and $z_2=\dot{p}-\dot{p}_d$. Therefore, system can be rewritten as

$$\dot{z}_1=z_2,\ \dot{z}_2=\frac{1}{1+\cos^2(p)}u+f(p),$$

where $f(p)=(-a\,\mathrm{sign}(\dot{p})-g\sin(p)-\ddot{p}_d)/(1+\cos^2(p))$, represents the unknown forces acting on the system. It is quite clear that $|f(p)|\le C$, with $C=34.98$, and $0.5=K_m\le|\frac{1}{1+\cos^2(p)}|\le K_M=1$. Basically, we compare the

(1) SF control law $u=k_2\,\mathrm{sign}\left(\lceil z_2\rfloor^2+k_1^2 z_1\right)$, since $K_m=0.5$, then with $k_1=1$, we chose $k_2=74$.
(2) observer-based OF control law $u=k_2\,\mathrm{sign}\left(\lceil \hat{z}_2\rfloor^2+k_1^2 z_1\right)$, with same gains as in the SF. The observer is given by

$$\dot{\hat{z}}_1=-l_1\lceil\hat{z}_1-z_1\rfloor^{1/2}+\hat{z}_2,$$
$$\dot{\hat{z}}_2=-l_2\,\mathrm{sign}(\hat{z}_1-z_1)+0.5u.$$

By Theorem 8, we select the gains $l_1=53.1$ and $l_2=110$.
(3) the controller $u=0$, $0\le t<0.23$, and controller $u=k_2\,\mathrm{sign}\left(\lceil \hat{z}_2\rfloor^2+k_1^2 z_1\right)$, $\forall t\ge 0.23$, that is, once the ST observed has converged.

Remark 13. Since the variable p is measured, the term $\frac{1}{1+\cos^2(p)}$ is known. This fact can be used to design the gains of the observer independently of the gains of the controller.

The simulation results for the SF controller are shown in Fig.1. The control target is achieved in finite-time. The observer provides the exact estimation of the measured variable z_1 and its time derivative z_2 after a finite-time. The simulation results for the OF controller are shown in Fig.2. The observer-based controller provides similar dynamical behavior as the SF. Actually, the performance of the OF controller is very similar to the one with the SF. On the other hand, using the approach proposed in Levant (2005), the tracking has a delayed response, see Fig.3.

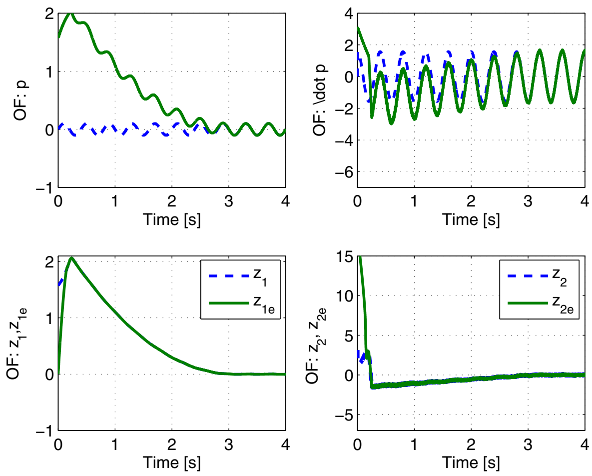

Fig. 3. Closed-loop behaviour for the state variables and their estimation for the point (3).

8. CONCLUSION

We provide a Lyapunov-based approach to design an OF using the terminal sliding controller for uncertain systems with a well-defined relative degree 2. The controller/observer scheme drives the output and its derivative to zero in finite time despite of unknown bounded perturbations. The OF scheme requires the output and its derivative. Since, only the output is measured, its time derivative is estimated by means of the ST observer. If the observer is appropriately designed, the resulting OF controller will reach the target robustly and in finite time. It was shown that observer gains has to be designed after the controller gains. However, the separation principle can take place if the system is not totally unknown.

The stability of the interconnection of the controller and the observer in the OF scheme has been established by Lyapunov arguments. We have proposed a smooth LF for the ST observer which provides a set of gain relatively conservative in comparison with its non smooth LF.

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Appendix A. TECHNICAL LEMMAS

Lemma 14. Moreno (2011). For any real numbers $a>0$, $b>0$, $c>0$, $p>1$, $q>1$, with $\frac{1}{p}+\frac{1}{q}=1$, the inequality $ab\le c^pa^p/p+c^{-q}b^q/q$ holds.

Lemma 15. For $x_1,x_2\in\mathbb{R}$, and p,q nonzero real numbers, such that $0<p\le q$, the inequality

$$0\le|\lceil x_2\rfloor^p+\lceil x_1\rfloor^p|^{1/p}\le 2^{\frac{1}{p}-\frac{1}{q}}|\lceil x_2\rfloor^q+\lceil x_1\rfloor^q|^{1/q} \tag{A.1}$$

holds. Furthermore, equality holds if and only if either $p=q$ or $x_1=x_2$.

Proof. The cases $x_1=0$, $x_2=0$ or $x_1=x_2=0$ are straightforward. Therefore, (A.1) is proved for any $x_1\neq0$ and $x_2\neq0$. Consider a change of variables $z_1=\lceil x_1\rfloor^p$ and $z_2=\lceil x_2\rfloor^p$. Therefore, inequality (A.1) is equivalent to

$$|s+1|^r\le 2^{r-1}|1+\lceil s\rfloor^r|,\ r=q/p\ge1. \tag{A.2}$$

where $s=z_1/z_2$ and valid for all $z_2\neq0$ (or equivalently, $x_2\neq0$). Define the function $f(s)=|s+1|^r/|1+\lceil s\rfloor^r|$, we will find the maximum of this function. Note that $\lim_{s\to0}f(s)=\lim_{s\to\infty}f(s)=1$. By simple calculus, there exists a maximum or minimum if $|s|^{r-1}=1$, i.e., at points $s=\pm1$. Since $f(s)$ is not defined at $s=-1$, the use the L'Hopital's rule yields $\lim_{s\to-1}f(s)=0$. For $s=1$, we have $f(s)=2^{r-1}$. Therefore,

$$0\le|s+1|^r/|1+\lceil s\rfloor^r|\le2^{r-1},\ r\ge1.$$

It proves (A.2). When $x_1=x_2$, (A.2) reduces to $2^r=2^r$. Finally, for $p=q$, it reduces to $|s+1|\le|1+\lceil s\rfloor|$, from it follows that $|s+1|=|1+s|$, since $\lceil s\rfloor=s$.

Lemma 16. Bhat and Bernstein (2005). Suppose V_1 and V_2 are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degree $l_1>0$ and $l_2>0$, respectively, and V_1 is positive definite. Then, for every $x\in\mathbb{R}^n$,$[\min_{\{z:V_1(z)=1\}}V_2(z)][V_1(x)]^{\frac{l_2}{l_1}}\le V_2(x)\le[\max_{\{z:V_1(z)=1\}}V_2(z)][V_1(x)]^{\frac{l_2}{l_1}}$.

Lemma 17. For $x_1,x_2\in\mathbb{R}$, and α,β,γ,p,q nonzero real numbers, such that $0<p\le q$, the inequality

$$\alpha|\lceil x_2\rfloor^q+\lceil x_1\rfloor^q|-\beta|x_1|^q\le\gamma|\lceil x_2\rfloor^p+\lceil x_1\rfloor^p|^{\frac{q}{p}} \tag{A.3}$$

holds, where

$$\gamma=\begin{cases}\alpha & \text{if } r=\dfrac{q}{p}=1,\\ \dfrac{\alpha|-1+(1+(\beta/\alpha))^{\frac{r}{r-1}}|-\beta}{|-1+(1+(\beta/\alpha))^{\frac{1}{r-1}}|^r} & \text{if } r=\dfrac{q}{p}>1.\end{cases}$$

Proof. The cases $x_1=x_2$ and $x_1=x_2=0$ are straightforward. For the cases

- $x_1=0$, we have $\alpha|x_2|^q\le\gamma|x_2|^q$, then, $\alpha\le\gamma$.
- $x_2=0$, we have $(\alpha-\beta)|x_1|^q\le\gamma|x_1|^q$, then, $\alpha-\beta\le\gamma$. Since $\gamma>0$, we only pay attention to $\alpha>\beta$. Note that if $\alpha>\beta$, then, $\alpha-\beta<\alpha$. Therefore, we chose $\alpha\le\gamma$.

Then, (A.3) is proved for any $x_1\neq0$ and $x_2\neq0$. Consider a change of variables $z_1=\lceil x_1\rfloor^p$ and $z_2=\lceil x_2\rfloor^p$. Therefore, inequality (A.3) is equivalent to

$$\alpha|1+\lceil s\rfloor^r|-\beta|s|^r\le\gamma|s+1|^r,\ r=q/p\ge1. \tag{A.4}$$

where $s=z_1/z_2$ and valid for all $z_2\neq0$ (or equivalently, $x_2\neq0$). Define the function $f(s)=(\alpha|1+\lceil s\rfloor^r|-\beta|s|^r)/|s+1|^r$, we will find the maximum of this function. Note that $\lim_{s\to0}f(s)=\alpha$ (i.e. when $x_1=0$) and $\lim_{s\to\infty}f(s)=\alpha-\beta$ (i.e. when $x_2=0$). Since $f(s)$ is not defined at $s=-1$, the use the L'Hopital's rule yields $\lim_{s\to-1}f(s)=-\infty$. By simple calculus, may be there exists a maximum at points where

$$\alpha|s+1|\lceil s\rfloor^r-\alpha|\lceil s\rfloor^r+1|s=\beta|s|^r$$

is satisfied. These points are

(1) $s=-(\frac{\alpha}{\alpha+\beta})^{\frac{1}{r-1}}$ and $s=\pm(\frac{\alpha}{\alpha-\beta})^{\frac{1}{r-1}}$, $\forall r>1$, if $\alpha>\beta$.
(2) $s=-(\frac{\alpha}{\alpha+\beta})^{\frac{1}{r-1}}$, $\forall r>1$, if $\alpha\le\beta$.
(3) $s=0$, if $r=1$.

A maximum exists at points $s_M=-(\frac{\alpha}{\alpha+\beta})^{\frac{1}{r-1}}$ if $\forall r>1$, and $s=0$ if $r=1$. Therefore, we have $f(s_M)=\gamma$. Moreover,

$$-\infty\le(\alpha|1+\lceil s\rfloor^r|-\beta|s|^r)(|s+1|^r)\le\gamma,\ r\ge1.$$

It proves (A.4). When $x_1=0$ and $\alpha=\gamma$, (A.4) reduces to $\gamma=\gamma$. When $x_2=0$ and $\alpha=2\gamma$, $\beta=\gamma$, (A.4) reduces to $\gamma=\gamma$.