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A New Homogeneous Quasi-Continuous Second Order Sliding Mode Control

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Abstract: A new homogeneous quasi-continuous second order sliding mode control algorithm is developed. The Lyapunov function for finite-time stability analysis is constructed. The homogeneity property of the algorithm is analyzed. The chattering attenuation procedure using Lyapunov function of the closed-loop system is discussed. The scheme for control parameters tuning based on linear matrix inequalities is elaborated. The theoretical results are supported by numerical simulations.

1. INTRODUCTION

Introduced 50 years ago sliding mode is the oldest robust control methodology (see, for example, Utkin [1977] and references therein). The main advantage of sliding mode technique is the insensitivity of the closed-loop system to the so-called matched disturbances and uncertainties, see Edwards and Spurgeon [1998], Utkin et al. [2009], Shtessel et al. [2014].

The concept of the second order sliding mode algorithm was introduced in Levant [1993] in order to develop an novel approach to sliding mode control design for systems with relative degree 2 and to reduce of chattering effect (see, Utkin et al. [2009]) of sliding mode systems. The so-called quasi-continuous sliding mode algorithm (see, Levant [2005a]) looks the most perspective in this context, since it is discontinuous only on the surface $\{x :\in \mathbb{R}^n : s(x) = \dot{s}(x) = 0\}$, where $s(x) = 0$ is a conventional sliding surface, $s : \mathbb{R}^n \to \mathbb{R}$. However, in contrast to other second order sliding algorithms the Lyapunov function-based stability analysis of quasi-continuous sliding mode algorithms have never been presented before. The Lyapunov functions for "twisting", "super-twisting" and "nested" second order sliding mode algorithms are designed in (Polyakov and Poznyak [2009], Moreno [2012], Polyakov and Poznyak [2012]).

This paper elaborates the new quasi-continuous second order sliding mode control algorithm. The main advantage of this algorithm is the simplicity of the tuning of control parameters based on the linear matrix inequalities. The stability of the closed-loop system is analyzed using non-smooth Lyapunov function method (see, for example, Polyakov and Fridman [2014]). The finite-time convergence of system trajectories to the sliding set is proven by means of implicit Lyapunov function (ILF) technique (see, Polyakov et al. [2013]. The explicit analytical representation of the Lyapunov function is also derived in order to present explicitly the feedback law. The control algorithm and the Lyapunov function are designed using the framework of homogeneous vector fields (Zubov [1958], Hermes [1986] et al), which is rather popular in the sliding mode control theory(see, for example, Orlov [2005], Levant [2005a], Bernuau et al. [2014]).

2. NOTATION

Through the paper the following notations will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$, where $\mathbb{R}$ is the set of real number;
- $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$, i.e. $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$ for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$;
- $\text{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with the elements λ_i on the main diagonal;
- for a matrix $P \in \mathbb{R}^{n\times n}$, which has the real spectrum, the minimal and maximal eigenvalues are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively;
- if $P \in \mathbb{R}^{n\times n}$ then the inequality $P > 0$ ($P \geq 0$, $P < 0$, $P \leq 0$) means that P is symmetric and positive definite (positive semidefinite, negative definite, negative semidefinite).
- The set of all subsets of a set $M \subseteq \mathbb{R}^n$ is denoted by 2^M.
- $\nabla V(x)$, $x \in \mathbb{R}^n$ is the gradient of the continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$.

3. PROBLEM STATEMENT

The paper deals with the standard problem of the second order sliding mode control design. Let the single input control system be defined as follows:

$$\dot{z}(t) = f(t, z) + g(t, z)u(t), \quad t \in \mathbb{R}_+, \tag{1}$$

where $z \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are smooth functions. Let the C^2 function $s : \mathbb{R}^n \to \mathbb{R}$ be an auxiliary output of the control system defining the sliding manifold. Following the classical methodology of the second order sliding mode design (Levant [2007]), the control have to constrain the system trajectories to evolve onto the augmented sliding surface:

$$\mathcal{S} := \{z \in \mathbb{R}^n : s(z) = \dot{s}(z) = 0\}.$$

In addition, the reaching phase of the surface S is usually required to be realized in a finite-time (Levant [2007], Polyakov and Fridman [2014]).

Let us assume that the relative degree of s with respect to control input is equal to 2, then the output dynamics has the form

$$\ddot{s}(t) = d(t, z) + c(t, z)u(t),$$

where $d : \mathbb{R}^{n+1} \to \mathbb{R}$ and $c : \mathbb{R}^{n+1} \to \mathbb{R}$. The functions d and c are traditionally assumed to be unknown but bounded by some numbers as follows:

$$|d(t, z)| \leq d_{max}, \quad 0 < c_1 \leq c(t, z) \leq c_2, \\ \forall t \in \mathbb{R}_+, z \in \mathbb{R}^n. \tag{2}$$

Denote $x_1 = s, x_2 = \dot{s}$ and consider the following extended differential inclusion

$$\dot{x}(t) \in Ax(t) + b\left([c_1, c_2]u(t) + [-d_{\max}, d_{max}]\right), \tag{3}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Evidently, the control law, which will stabilize the origin of the system (3), will provide the required second order sliding mode on the surface $\mathcal{S}$ to the system (1).

Finally, let us assume that only the output s and its derivative $\dot{s}$ can be measured and used for the control purposes.

The problem is to design a feedback control $u = u(s, \dot{s})$ that stabilize the origin of the system (3) *in a finite-time. In addition, the origin of the system* (3) *must be the unique discontinuity point of the control law to be designed (similarly to the so-called quasi-continuous sliding mode algorithm Levant [2005b]).*

4. PRELIMINARIES

4.1 Finite-time stability

Consider the system of the form

$$\dot{x}(t) \in F(x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a nonlinear set-valued vector field.

Assume that F satisfies **the standard conditions**, i.e. $F(x)$ is nonempty, compact and convex for any $x \in \mathbb{R}^n$ and F is upper semi-continuous (see Filippov [1988]).

An absolutely continuous function $x(\cdot, x_0)$ defined on $[0, t^*), t^* \leq +\infty$, is called the solution to the Cauchy problem associated to (4) if $x(0, x_0) = x_0$ and $x(\cdot, x_0)$ satisfies the differential inclusion (4) for almost everywhere on $(0, t_*)$.

Assume that the origin is an equilibrium point of the system (4), i.e. $0 \in F(0)$.

The paper studies only strong global stability properties of the origin of the system (4). We will omit the words *strong and global* in all later definitions and considerations.

Definition 1. (Orlov [2005], Polyakov and Fridman [2014]). The origin of system (4) is said to be **uniformly finite-time stable** if it is asymptotically stable and there exists a **locally bounded** function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that $x(t, x_0) = 0$ for all $t > T(x_0)$, where $x(\cdot, x_0)$ is any solution of the Cauchy problem (4).

The scalar first order sliding mode control system is the classical example of the finite-time stable differential inclusion: $\dot{x}(t) \in -2\overline{\text{sign}}[x(t)] + [-1, 1]$, where $x \in \mathbb{R}$, $t > 0$, $x(0) = x_0$ and $\overline{\text{sign}}[\cdot]$ is the set-valued extension of the sign function based on the standard Fillipov regularization procedure (see, for example, Filippov [1988], Polyakov and Fridman [2014]). Obviously, the settling time function of this system is $T(x_0) = |x_0|$.

Theorem 2. (Roxin [1966], Polyakov and Fridman [2014]). Let a function $V : \mathbb{R}^n \to \mathbb{R}$ be continuous, radially unbounded, continuously differentiable outside the origin and

$$\sup_{y \in F(x)} \nabla^T V(x) y \leq -1 \quad \text{for} \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{5}$$

Then the origin of the system (4) is finite-time stable and

$$T(x_0) \leq V(x_0) \quad \text{for} \quad x_0 \in \mathbb{R}^n, \tag{6}$$

where T is a settling-time function.

The function V satisfying the conditions of Theorem 2 is known as the finite-time Lyapunov function.

Note that the settling-time function may coincide with a finite-time Lyapunov function for the system (4). For instance, the settling-time function $T(x) = |x|$ from the example considered above is evidently the finite-time Lyapunov function for the corresponding first order sliding mode control system. The Lyapunov function to be designed in this paper will have the same property.

Frequently, finite-time stability correlates with the so-called homogeneity property of the system (see, for example, Rosier [1992], Bhat and Bernstein [2000], Orlov [2005], Levant [2005a], Andrieu et al. [2008]).

4.2 Homogeneity

Homogeneity is an intrinsic property of an object which remains consistent with respect to some scaling: level sets (resp. solutions) are preserved for homogeneous functions (resp. vector fields).

Let $\lambda > 0, r_i > 0, i \in \{1, \ldots, n\}$ then one can define:

- the *vector of weights* $r = (r_1, \ldots, r_n)^T$,
- the *dilation matrix*

$$D_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n, \tag{7}$$

note that $D_r(\lambda)x = (\lambda^{r_1}x_1, \ldots, \lambda^{r_i}x_i, \ldots, \lambda^{r_n}x_n)^T$.

Definition 3. (Zubov [1958], Levant [2005a]). A vector field $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is said to be r-homogeneous of degree m iff for all $\lambda > 0$ and for all $x \in \mathbb{R}^n$ we have $\lambda^{-m} D_r^{-1}(\lambda) F(D_r(\lambda)x) = F(x)$.

For a given $x \in \mathbb{R}^n$, the set $(D_r(\lambda)x)_{\lambda > 0}$ is a curve on $\mathbb{R}^n$. An object is homogeneous iff the behavior of this object is symmetric along these particular curves.

Theorem 4. (Levant [2005a], Bernuau et al. [2013]). Let a set-valued vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogenous of negative degree and the origin of the system (4) is asymptotically stable, then it is finite-time stable.

Homogeneity automatically provides additional robustness (Input-to-State Stability) properties to the closed-loop control system (see Bernuau et al. [2014]). That is why, this paper designs the homogenous sliding mode control.

4.3 Homogeneous Implicit Lyapunov Function

The theorem given below refines Theorem 2 for implicit definition of Lyapunov function.

Theorem 5. (Polyakov et al. [2013]). Let there exists a continuous function
$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$
$$(V, x) \mapsto Q(V, x)$$
such that

C1) Q is continuously differentiable for all $x \in \mathbb{R}^n \backslash \{0\}$ and $V \in \mathbb{R}_+$;

C2) for any $x \in \mathbb{R}^n \backslash \{0\}$ there exist $V \in \mathbb{R}_+$ such that
$$Q(V, x) = 0;$$

C3) for $\Omega = \left\{(V, x) \in \mathbb{R}^{n+1} : Q(V, x) = 0\right\}$ we have
$$\lim_{\substack{x \to 0 \\ (V,x) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \to 0^+ \\ (V,x) \in \Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\| \to \infty \\ (V,x) \in \Omega}} V = +\infty;$$

C4) the inequality $-\infty < \frac{\partial Q(V,x)}{\partial V} < 0$ holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \backslash \{0\}$;

C5) the inequality $\sup_{y \in F(x)} \frac{\partial Q(V,x)}{\partial x} \leq \frac{\partial Q(V,x)}{\partial V}$ holds for all $(V, x) \in \Omega$.

Then the origin of system (4) is globally finite-time stable with the following settling time estimate $T(x_0) \leq V_0$, where $V_0 \in \mathbb{R}_+$ such that $Q(V_0, x_0) = 0$.

The proof of this theorem is based on combination of Theorem 2 and the classical Implicit Function Theorem Courant and John [2000] providing $\frac{\partial V}{\partial x} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial x}$.

Let us consider the following ILF candidate (Nakamura et al. [2007], Polyakov et al. [2013])
$$Q(V, x) := x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \qquad (8)$$
where $V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $D_r(\cdot)$ is the dilation matrix of the form (7) with $r = (r_1, r_2, ..., r_n)^T \in \mathbb{R}^n_+$, $P \in \mathbb{R}^{(n+1)\times(n+1)}$ is a symmetric positive definite matrix.

The proof of the next corollary can be found in Appendix.

Corollary 6. Let the set-valued vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ be r-homogeneous of degree -1 with the vector of weights $r = (r_1, r_2, ..., r_n)^T \in \mathbb{R}^n_+$. If there exists a matrix $P \in \mathbb{R}^{n\times n}$, which satisfy matrix inequalities
$$\text{diag}(r)P + P\text{diag}(r) > 0, \quad P > 0 \qquad (9)$$
and the inequality
$$\sup_{y \in F(z)} z^T P y + y^T P z \leq -z^T(\text{diag}(r)P + P\text{diag}(r))z, \quad (10)$$
holds for all $z \in \mathbb{R}^n$ such that $z^T P z = 1$, then the system (4) is finite-time stable and the settling time function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ admits the following estimate:
$$T(x_0) \leq V_0 \qquad (11)$$
where $x_0 \in \mathbb{R}^n$ and $V_0 \in \mathbb{R}_+$ is such that $Q(V_0, x_0) = 0$. If the inequality (10) becomes equality then $T(x_0) = V(x_0)$.

This corollary has the simple geometric interpretation: if the homogeneous system (4) admits a Lyapunov function with at least one ellipsoidal level set, then it has a homogeneous ILF defined by the formula (8).

The present paper develops such quasi-continuous homogenous second order sliding mode control algorithm that provides to the closed-loop system a homogeneous ILF with ellipsoidal level sets ("quadratic ILF"). Corollary 6 will be used for the ILF design.

5. DESIGN OF NEW QUASI CONTINUOUS SECOND ORDER SLIDING MODE CONTROL

Originally, the second order sliding mode control was designed based on some geometrical constructions (Levant [2005a], Levant [2007]). Next, the traditional stability analysis based on Lyapunov function approach has been developed Polyakov and Poznyak [2009], Moreno [2012]. This paper introduces "the reverse procedure", which designs the Lyapunov function before the control law. Note that the scheme presented below completely differs from the Control Lyapunov Function approach.

5.1 Design of the Lyapunov function

Let us define the ILF candidate for the system (3) in the form (8), with dilation matrix $D_r(\lambda) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$ for $\lambda > 0$. Such selection of the dilation matrix is motivated by the structure of the system (3). The right-hand side of the open-loop system (3) satisfies Definition 3 with homogeneity degree -1.

For $n = 2$ the equation $Q(V, x) = 0$ can be rewritten as follows
$$V^4 - p_{22} x_2^2 V^2 - 2 p_{12} x_1 x_2 V - p_{11} x_1^2 = 0, \qquad (12)$$
where $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $\{p_{ij}\}$ are elements of the matrix $P > 0$. In order to find the finite-time Lyapunov function, the matrix $P \in \mathbb{R}^{2\times 2}$ should satisfy the conditions of Corollary 6. In this case, there exists a unique smooth (outside the origin) positive definite function V satisfying the equation (12) for all $x \in \mathbb{R}^2$ (see the proof of Corollary 6). For $n = 2$ the function V can be found analytically. We use the procedure from Carpenter [1966] for this purpose (see, Sections 8.2-8.4 for the details)

Proposition 7. If the matrix $P \in \mathbb{R}^{2\times 2}$ satisfies the linear matrix inequalities (9) with $r = (2, 1)^T$, then

1) the smooth (outside the origin) positive definite function $V : Q(V, x) = 0, x \in \mathbb{R}^n \backslash \{0\}$ admits the following explicit representation:
$$V(x) = \begin{cases} \frac{\sqrt{2p_{22}x_2^2 + \frac{4p_{12}x_1x_2}{\sqrt{z_0(x)}} - z_0(x)} + \sqrt{z_0(x)}}{2} & \text{if } p_{12}x_1x_2 > 0, \\ \frac{\sqrt{2p_{22}x_2^2 - \frac{4p_{12}x_1x_2}{\sqrt{z_0(x)}} - z_0(x)} - \sqrt{z_0(x)}}{2} & \text{if } p_{12}x_1x_2 < 0, \\ \sqrt{\frac{p_{22}x_2^2 + \sqrt{p_{22}^2 x_2^4 + 4p_{11}x_1^2}}{2}} & \text{if } \quad p_{12} = 0, \\ \sqrt{|x_1|\sqrt{p_{11}}} & \text{if } \quad x_2 = 0, \\ |x_2|\sqrt{p_{22}} & \text{if } \quad x_1 = 0, \end{cases} \qquad (13)$$
where $z_0(x) = \frac{2p_{22}x_2^2 - C_1(x) - C_2(x)}{3}$,
$$C_i(x) = \sqrt[3]{\frac{(-1)^i \sqrt{\Delta_1^2(x) - 4\Delta_0^3(x)} + \Delta_1(x)}{2}}, \quad i = 1, 2,$$
$\Delta_0(x) = p_{22}^2 x_2^4 - 12 p_{11} x_1^2$, $\Delta_1(x) = 2p_{22}x_2^2(\Delta_0 + 6Dx_1^2)$ and $D = 8p_{11} - 9p_{12}^2/p_{22} > 0$;

2) the function V defined by (13) with $P = X^{-1}$ is homogeneous, positive definite, radially unbounded, continuous in $\mathbb{R}^2$ and continuously differentiable outside the origin.

The proof is given in the Appendix.

5.2 Feedback Design

Based on the ideas of the paper Polyakov et al. [2013] the quasi-continuous homogeneous second order sliding mode control for the system (3) is designed as follows

$$u(x) = kD_r\left(V^{-1}(x)\right)x, \tag{14}$$

where $x \in \mathbb{R}^2$, $D_r(\cdot)$ is the dilation matrix for $r = (2,1)^T$, V is defined by (13) for $P \in \mathbb{R}^{2\times 2}$ satisfying (9) and $k = (k_1, k_2) \in \mathbb{R}^{1\times 2}$ is a constant row vector.

Theorem 8. Let the linear matrix inequalities (LMI)

$$\begin{cases} \tilde{A}X + X\tilde{A}^T + c_i(by + y^T b^T) + d_{\max}(X + bb^T) \le 0, \\ X\mathrm{diag}(r) + \mathrm{diag}(r)X > 0, \quad X > 0, \quad i = 1, 2 \end{cases} \tag{15}$$

be feasible for some $X = X^T \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^{1\times n}$ and $\tilde{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ then

1) the control function (14) with $k = yX^{-1}$ is continuous outside the origin and globally bounded as follows

$$u^2(x) \le \frac{k^T k}{\lambda_{\min}(P)} \quad \text{for all } x \in \mathbb{R}^2;$$

2) the closed-loop system (3), (14) is homogenous and finite-time stable with the settling-time function estimate: $T(x) \le V(x)$.

Proof.

1) The equation $Q(V, z) = 0$ implies

$$x^T D_r(V^{-1})PD_r(V^{-1})x = 1 \Rightarrow \|D_r(V^{-1})x\|^2 \le \frac{1}{\lambda_{\min}(P)}.$$

and

$$\|u(x)\| \le \|k\| \cdot \|D_r(V^{-1})x\| \le \frac{\|k\|}{\sqrt{\lambda_{\min}(P)}}.$$

2) Let us denote the right-hand side of the closed-loop system (3), (14) by

$$F(x) = Ax + b\left([c_1, c_2]u(x) + [-d_{\max}, d_{\max}]\right).$$

Taking into account homogeneity of the Lyapunov function function we easily derive $u(D_r(\lambda)x) = u(x)$ and $F(D_r(\lambda)x) = \lambda^{-1}D_r(\lambda)F(x)$, $\forall \lambda \in \mathbb{R}_+$.

Let $z \in \mathbb{R}^n$ be such that $z^T Pz = 1$. The equality $Q(V, z) = 0$ implies $V(z) = 1$, $u(z) = kz$ and

$$\begin{aligned} &\sup_{y\in F(z)} y^T Pz + z^T Py = z^T(A^T P + PA)z + \\ &\sup_{\alpha\in[c_1,c_2]} \alpha(z^T Pbkz + z^T k^T b^T Pz) + \\ &\sup_{\beta\in[-d_{\max},d_{\max}]} \beta(z^T Pb + b^T Pz) = \\ &\sup_{\lambda\in[0,1]} \lambda z^T(A^T P + PA + c_1(Pbk + k^T b^T P))z + \\ &(1-\lambda)z^T(A^T P + PA + c_2(Pbk + k^T b^T P))z + \\ &\sup_{\beta\in[-d_{\max},d_{\max}]} \beta(z^T Pb + b^T Pz) \end{aligned}$$

The LMI (15) implies for $i = 1, 2$

$$\begin{aligned} &A^T P + PA + c_i(Pbk + k^T b^T P) \le \\ &-\mathrm{diag}(r)P - P\mathrm{diag}(r) - d_{\max}(P + Pbb^T P) < 0. \end{aligned}$$

Since $-z^T Pz - zTPbb^T Pz \pm (z^T Pb + b^T Pz) \le 0$ for $z^T Pz = 1$ then

$$\sup_{y\in F(z)} y^T Pz + z^T Py \le -z^T(\mathrm{diag}(r)P + P\mathrm{diag}(r))z.$$

Applying Corollary (6) we finish the proof. ■

Note that if $c = c_1 = c_2$ and $d_{\max} = 0$ then replacing the first inequality in (15) with $\tilde{A}X + X\tilde{A} + c(by + y^T b^T) = 0$ and repeating the proof of Theorem 8 it can be shown that $T(x) = V(x)$, i.e. *the Lyapunov function* (13) *of the closed-loop system* (1), (14) *coincides with the settling-time function in the disturbance-free case.*

Remark 9. The condition $\|u(x)\| \le u_0$, $u_0 \in \mathbb{R}_+$ can be represented in the form of LMI

$$\begin{pmatrix} X & y^T \\ y & u_0^2 \end{pmatrix} \ge 0. \tag{16}$$

Indeed, the inequality $u^2 = x^T D_r(V^{-1})k^T kD_r(V^{-1})x \le u_0^2 = u_0^2 x^T D_r(V^{-1})PD_r(V^{-1})x$ holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ such that $Q(V, x) = 0$, then $k^T k \le u_0^2 P$.

5.3 Remark on chattering reduction

Practical importance of the high order sliding mode control concept is usually motivated by essential reduction of the chattering phenomenon. In practice, any high gain control (even linear) may be subjected by chattering. The Lyapunov function method provides the additional tool for chatting attenuation. Following classical ideas of the chattering reduction (see Utkin et al. [2009]) we linearize our control close to sliding surface, i.e.

$$u(x) = \begin{cases} kD(V_{\min}^{-1})x & \text{for } V(x) \le V_{\min} \\ kD\left(V^{-1}(x)\right)x & \text{for } V(x) \ge V_{\min} \end{cases} \tag{17}$$

where $V_{\min} > 0$ is a parameter. Note that the level set of the Lyapunov function defined by inequality $V(x) \le V_{\min}$ is an ellipsoid in $\mathbb{R}^n$, which is positively invariant for the closed-loop system. So, in order to reduce the chattering we replace the quasi-continuous second order sliding mode control law with linear one inside the ellipsoid $\{x \in \mathbb{R}^n : V(x) \le V_{\min}\}$. The size of the ellipsoid is defined by $V_{\min}$.

6. NUMERICAL SIMULATIONS

For $c_1 = 1, c_2 = 1.2$, $d_{\max} = 0.2$ and $u_0 = 2$ we solve the LMI (15):

$$P = \begin{pmatrix} 23.7436 & 10.0502 \\ 10.0502 & 4.7858 \end{pmatrix} \text{ and } k = (-5.9663, -3.6786).$$

We design the sliding mode control in the form (17) with $V_{\min} = 0.05$. Figure 1 presents the simulation results for the system

$$\dot{x}(t) = Ax(t) + b\left((1 + 0.2\sin x_1)u(t) + 0.2\sin x_2\right).$$

The simulation has be done using explicit Euler method with the step size 0.01.

7. CONCLUSION

The homogeneous quasi-continuous second order sliding mode algorithm is developed. The LMI-based algorithm for parameter tuning is elaborated. The robust finite-time stability analysis based on Lyapunov function method is provided. The chattering reduction scheme based on Lyapunov function method is presented and demonstrated on numerical simulations.

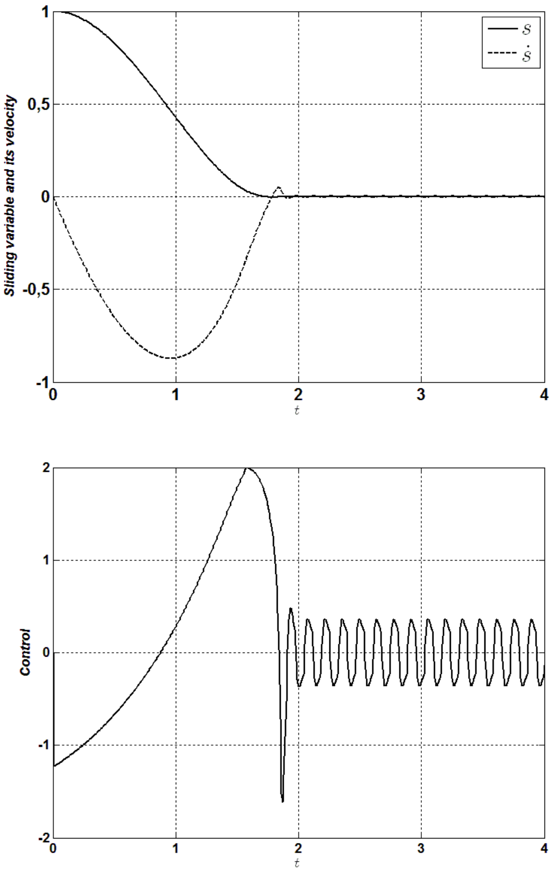

Fig. 1. The simulation results

8. APPENDIX

8.1 Proof of Corollary 6

I. The function $Q(V,x)$ defined by (8) satisfies the conditions *C1*)-*C3*) of Theorem 5. Indeed, it is continuously differentiable for all $V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n$.

Since $P > 0$ then the following chain of inequalities

$$\frac{\lambda_{\min}(P)\|x\|^2}{\max\limits_i\{V^{2r_i}\}} \leq Q(V,x)+1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{\min\limits_i\{V^{2r_i}\}}$$

implies that for any $z \in \mathbb{R}^n\backslash\{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+ : Q(V^-,x) < 0 < Q(V^+,x)$. Due to continuity of Q the condition *C2)* also fulfilled.

If $Q(V,x)=0$ then the same chain of inequalities gives

$$\frac{\min\limits_i\{V^{2r_i}\}}{\lambda_{\max}(P)} \leq \|x\|^2 \leq \frac{\max\limits_i\{V^{2r_i}\}}{\lambda_{\min}(P)}.$$

Therefore, the condition *C3*) of Theorem 5 holds.

Since

$$\frac{\partial Q}{\partial V} = -\frac{x^T D_r(V^{-1})(\text{diag}(r)P+P\text{diag}(r))D_r(V^{-1})x}{V},$$

then

$$\text{diag}(r)P + P\text{diag}(r) > 0$$

implies $\frac{\partial Q}{\partial V} < 0$ for $\forall V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n\backslash\{0\}$. So, the condition *C4*) of Theorem 5 also holds.

We conclude that the equation $Q(V,x) = 0$ implicitly defines a proper positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that for any $x \in \mathbb{R}^n$ we have $Q(V(x),x) = 0$.

II. Let us calculate the time derivative of the function V along the trajectories of the system (4):

$$\dot{V}(x) \leq \sup_{y\in F(x)} -\left[\frac{\partial Q}{\partial V}\right]^{-1}\frac{\partial Q}{\partial x}y =$$

$$\sup_{y\in F(x)} -\left[\frac{\partial Q}{\partial V}\right]^{-1}(z^TPD_r(V^{-1})y + y^T(t,x_0)D_r(V^{-1})Pz),$$

where $x \in \mathbb{R}^n\backslash\{0\}$ and $z = D_r(V^{-1})x$. The set-valued vector field F is homogeneous, i.e. $\lambda^{-m}D_r^{-1}(\lambda)F(D_r(\lambda)x) = F(x)$, $\forall\lambda \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n$, where $m \in \mathbb{R}$ is the degree of homogeneity. Hence, $V^{-1}D_r^{-1}(V^{-1})F(z) = F(x)$ for $\lambda = V^{-1}$ and

$$\dot{V}(x) \leq \sup_{y\in F(z)} -\left[\frac{\partial Q}{\partial V}\right]^{-1}V^{-1}(z^TPy + y^TPz) =$$

$$\frac{\sup_{y\in F(z)} z^TPy+y^TPz}{z^T(\text{diag}(r)P+P\text{diag}(r))z}.$$

Since $Q(V,x) = 0$ then $z^TPz = 1$ and the inequality (10) implies $\dot{V}(x) \leq -1$ for $x \neq 0$.

8.2 Cubic equation

Consider the cubic equation

$$z^3 + pz^2 + qz + r = 0, \quad p,q,r \in \mathbb{R}.$$

Introduce the following numbers

$$C_1 = \sqrt[3]{\frac{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \quad C_2 = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \tag{18}$$

where

$$\Delta_0 = p^2 - 3q, \quad \Delta_1 = 2p^3 - 9pq + 27r.$$

If $\Delta_1^2 - 4\Delta_0^3 \geq 0$ then

$$z_0 = -\frac{p + C_1 + C_2}{3}$$

is the real root of the cubic equation.

8.3 Quartic equation

Let us consider the quartic equation

$$V^4 + aV^2 + bV + c = 0, \quad a,b,c \in \mathbb{R}$$

and the adjoint cubic equation

$$z^3 + 2az^2 + (a^2 - 4c)z - b^2 = 0.$$

If $z_0 \in \mathbb{R}$ is a root of the cubic equation the roots of the quartic one are the following

$$V_1 = \frac{-\sqrt{z_0}+\sqrt{-z_0-2a+2b/\sqrt{z_0}}}{2}, V_2 = \frac{-\sqrt{z_0}-\sqrt{-z_0-2a+2b/\sqrt{z_0}}}{2},$$

$$V_3 = \frac{\sqrt{z_0}+\sqrt{-z_0-2a-2b/\sqrt{z_0}}}{2}, V_4 = \frac{\sqrt{z_0}-\sqrt{-z_0-2a-2b/\sqrt{z_0}}}{2},$$

$$z_0 = -\frac{2a + C_1 + C_2}{3}$$

where C_1 and C_2 are defined by (18) with

$$\Delta_0 = a^2 + 12c, \quad \Delta_1 = -2a^3 + 72ac - 27b^2.$$

8.4 Proof of Proposition 7

I. Taking into account the condition

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} > 0$$

we derive $p_{11} > 0$, $p_{22} > 0$ and $8p_{11}p_{22} - 9p_{12}^2 > 0$.

II. Consider the cubic equation

$$z^3 - 2p_{22}x_2^2z^2 + (p_{22}^2x_2^4 + 4p_{11}x_1^2)z - 4p_{12}^2x_1^2x_2^2 = 0,$$

and find its solution using Cordano formulas (see, for example, Section 8.3). Evidently, this equation does not have real negative roots. Let us find a real non-negative root of this equation using the Cordano formulas (see above). First, let us calculate $\Delta_0 = p_{22}^2x_2^4 - 12p_{11}x_1^2$, $\Delta_1 = 2p_{22}x_2^2(\Delta_0 + 6Dx_1^2)$, where $D = 8p_{11} - 9p_{12}^2/p_{22} > 0$.

On the on hand, if $\Delta_0 \le 0$ then $\Delta_1^2 - 4\Delta_0^3 \ge 0$. On the other hand, since

$$\begin{aligned}\Delta_1^2 - 4\Delta_0^3 &= 4p_{22}^2x_2^4(\Delta_0 + 6Dx_1^2)^2 - 4\left(p_{22}^2x_2^4 - 12p_{11}x_1^2\right)\Delta_0^2 \\ &= 48p_{11}x_1^2(3D^2x_2^4 + Dx_2^4\Delta_0 + \Delta_0^2).\end{aligned}$$

then $\Delta_1^2 - 4\Delta_0^3 \ge 0$ for $\Delta_0 > 0$.

Therefor the real non-negative root of the cubic equation is the following:

$$z_0(x_1, x_2) = \frac{2p_{22}x_2^2 - C_1 - C_2}{3},$$

where C_1 and C_2 are represented by (18).

III. Let us study the case $z_0(x_1, x_2) = 0$ or, equivalently, $2p_{22}x_2^2 = C_1 + C_2$. The last equality can be rewritten as follows:

$$\Delta_1 + 6\Delta_0 p_{22}x_2^2 = 8p_{22}^3x_2^6.$$

Hence, $z_0(x_1, x_2) = 0$ is equivalent to $x_{11}x_{22} = 0$.

IV. Finally taking into account the formulas for the roots of the quartic equation we finish the proof of the formula (13).

V. Since the inequality (9) holds for $P = X^{-1}$, then the Lyapunov function candidate V is well-defined by the equation $Q(V, x) = 0$ (see, the proof of Corollary 6). Since $Q(V, D_r(\lambda)x) = Q(\lambda V, x)$ and $0 = Q(V(D_r(\lambda)x), D_r(\lambda)x)$ then $\lambda V(x) = V(D_r(\lambda)x), \forall \lambda \in \mathbb{R}_+$, i.e. V is homogeneous function.

REFERENCES

V. Andrieu, L. Praly, and A. Astolfi. Homogeneous approximation, recursive obsertver and output feedback. *SIAM Journal of Control and Optimization*, 47(4):1814–1850, 2008.

E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov. On homogeneity and its application in sliding mode control. *Journal of The Franklin Institute*, 351(4):1866–1901, 2014.

E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti. On extension of homogeneity notion for differential inclusions. In *Proceeding of the European Control Conference*, pages 2204–2209, 2013.

S.P. Bhat and D.S. Bernstein. Finite-time stability of continuous autonomous systems. *SIAM Journal of Control and Optimization*, 38(3):751–766, 2000.

W. Carpenter. On the solution of the real quartic. *Mathematics Magazine*, 39(1):28–30, 1966.

R. Courant and F. John. *Introduction to calculus and analysis (Vol. II/1)*. New York: Springer, 2000.

C. Edwards and S. Spurgeon. *Sliding Mode Control: Theory And Applications*. CRC Press., 1998.

A.F. Filippov. *Differential equations with discontinuous right-hand sides*. Kluwer, Dordrecht, 1988.

H. Hermes. Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24:731–736, 1986.

A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6): 1247–1263, 1993.

A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41:823–830, 2005a.

A. Levant. Quasi-continuous high-order sliding mode controllers. *IEEE Transactions on Automatic Control*, 50(11):1812–1816, 2005b.

A. Levant. Construction principles of 2-sliding mode design. *Automatica*, 43(4):576–586, 2007.

J Moreno. Strict lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4):1035 – 1040, 2012.

H. Nakamura, N. Nakamura, and H. Nishitani. Stabilization of homogeneous systems using implicit control lyhapunov functions. In *7th IFAC Symposium on Nonlinear Control Systems, Pretoria, Shouth Africa*, pages 561–566, 2007.

Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4):1253–1271, 2005.

A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time stabilization using implicit lyapunov function technique. In *9th Symposium on Nonlinear Control Systems,*, pages 140–145, 2013.

A. Polyakov and L. Fridman. Stability notions and lyapunov functions for sliding mode control systems. *Journal of The Franklin Institute*, 351(4):1831–1865, 2014.

A. Polyakov and A. Poznyak. Lyapunov function design for finite-time convergence analysis: "twisting" controller for second-order sliding mode realization. *Automatica*, 45:444–448, 2009.

A. Polyakov and A. Poznyak. Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems. *IMA Journal of Mathematical Control and Information*, 29(4):529–550, 2012.

L. Rosier. Homogenous lyapunov function for homogenous coninuous vector field. *System & Control Letters*, 19: 467–473, 1992.

E. Roxin. On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15(3):273–283, 1966.

Y. Shtessel, C. Edwards, L. Fridman, and A. Levant. *Sliding Mode Control and Observation*. Birkhauser, 2014.

V.I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2):212 – 222, 1977.

V.I. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electro-Mechanical Systems*. CRC Press., 2009.

V.I. Zubov. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica. (in Russian)*, 1:80–88, 1958.