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Stabilization of a class of 2-DOF underactuated mechanical systems with external perturbations

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Abstract: In this paper it is proposed a novel nonlinear controller for underactuated mechanical systems with external perturbations in the unactuated joint. The control objective is to regulate the unactuated variable while the position of the actuated joint remain bounded and its velocity asymptotically decays to zero. The control input is given by terms depending on the positions and velocities of each joint. The closed loop system through simulations show the convergence of the position of the unactuated joint to an equilibrium point which is zero, and it is robust with respect to some uncertainty in the amplitude of the external perturbation of the unactuated joint. Performance issues of the proposed synthesis are illustrated numerically.

Keywords: Mechanical systems, Underactuated systems, Sliding mode control, External perturbations.

1. INTRODUCTION

Control synthesis for underactuated systems is more complex than for fully actuated systems Seto and Baillieul (1994). A few representative papers analyzing some problems about underactuated systems include the study of accessibility Reyhanoglu et al. (1999), stabilization of equilibria through passivity techniques Ortega et al. (2002) and energy shaping Bloch et al. (2000), stabilization and tracking via backstepping control Seto and Baillieul (1994), the use of virtual constraints to produce stable oscillations Shiriaev et al. (2005), path planning Bullo and Lynch (2001), and control of mechanical systems with an unactuated cyclic variable Grizzle et al. (2005), among others.

In the field of underactuated mechanical control several papers have addressed the problem of friction. Linear damping (viscous friction) in both, the unactuated and the actuated joints is considered in Gómez-Estern and Van der Schaft (2004); Woolsey et al. (2001, 2004). Commonly external perturbations on the unactuated joint has been repeatedly left unmatched. A work considering discontinuous friction in the unactuated joint can be found in Martinez and Alvarez (2008) where it is presented a dynamic sliding mode controller to regulate the position of the unactuated joint in 2-DOF underactuated mechanical.

Among other the previous works related to sliding modes are Xu and mit zgner (2008), where a sliding mode control approach is proposed to stabilize a class of underactuated systems which are in cascaded form. In Sankaranarayanan and Mahindrakar (2009) it is presented a sliding mode control algorithm to robustly stabilize a class of underactuated mechanical systems that are not linearly controllable and violate Brockett's necessary condition for smooth asymptotic stabilization of the equilibrium, with parametric uncertainties. Moreover in Martinez et al. (2008) an hybrid control synthesis is proposed for a class of 2-degrees-of-freedom (DOF) underactuated mechanical systems with Coulomb friction in the joints.

More recently in Rascón et al. (2012) it is addressed the position regulation of an underactuated mechanical system with an elastic clearance, using a sliding mode-H_∞ control technique, where the position of the underactuated joint remain bounded around the equilibria due to model uncertainties and disturbances.

For underactuated mechanical systems with a matched disturbance, the problem of compensation, in some cases, can be solved as in Riachy et al. (2006) where a quasi-homogeneous switched controller is proposed.

In this paper, it is proposed a controller for a 2-DOF underactuated mechanical system with external perturbations in the unactuated joint. The control objective is the regulation of the unactuated variable while the position of the actuated joint remain bounded and its velocity converges asymptotically to zero. The only way of influencing the unactuated joint is via the position and the velocity of the actuated one. we propose a sliding mode control that guarantees a zero steady state position error of the underactuated joint. we numerically evaluate the performance of this procedure.

The rest of the paper is outlined as follows: In Section II, we describe the 2-DOF underactuated mechanical system and the control objective. In Section III it is design a controller using the classic technique of sliding modes.

Section IV proves the stability convergence to the sliding surface in finite time. Section V address an application for a mass-spring-damper system using the aforementioned controller design. In Section VI a numerical study is perform using MATLAB®. Finally, Section VII includes a conclusion and future work about the present control approach.

2. STATEMENT OF THE PROBLEM

Consider an underactuated mechanical system represented by

$$\begin{aligned} \ddot{q}_1 &= f_0(q_1,\dot{q}_1,q_2,\dot{q}_2) + g_0(q_1,\dot{q}_1,q_2,\dot{q}_2)u, \\ \ddot{q}_2 &= f_1(q_2,\dot{q}_2) + g_1 q_1 + g_2\dot{q}_1 + w \end{aligned} \tag{1}$$

where $q_1 \in \mathbb{R}$, $q_2 \in \mathbb{R}$, f_0, g_0 are smooth functions, $g_0 \neq 0$, f_1 is a linear function and the control input is given by u. To account for discrepancies in the model, has been introduced an external perturbation satisfying the following upper bounds, $|w| \leq D_0$, $|\dot{w}| \leq D_1$.

The objective is to design a control law u that allows the regulation of the underactuated variable to the equilibrium point which is zero, while the position of the actuated joint remain bounded and its velocity decays asymptotically to zero.

3. CONTROLLER DESIGN

The control u can be designed using the classic technique of sliding modes. The sliding surface s proposed is given by

$$s = \Big\{ (q_1,\dot{q}_1,q_2,\dot{q}_2) \in \mathbb{R}^4 \Big| \; s = \ddot{q}_2 + k_p q_2 + k_d\dot{q}_2 + \dot{q}_1 = 0 \Big\}. \tag{2}$$

The derivative along the trajectories of (1) is given by

$$\begin{aligned} \dot{s} = &\frac{\partial}{\partial q_2} f_1(q_2,\dot{q}_2) + \frac{\partial}{\partial \dot{q}_2} f(q_2,\dot{q}_2) + g_1\dot{q}_1 + k_p\dot{q}_2 + k_d\ddot{q}_2 \\ &+(1+g_2)\ddot{q}_1 + \dot{w} \end{aligned} \tag{3}$$

which has an upper bound given as

$$\begin{aligned} \dot{s} \leq &\frac{\partial}{\partial q_2} f_1(q_2,\dot{q}_2) + \frac{\partial}{\partial \dot{q}_2} f(q_2,\dot{q}_2) + g_1\dot{q}_1 + k_p\dot{q}_2 + k_d\ddot{q}_2 \\ &+(1+g_2)\ddot{q}_1 + D_1. \end{aligned} \tag{4}$$

Due to stability purposes that will be clear later, it is desirable that (4) takes the following form,

$$\dot{s} = -\beta\text{sign}(s). \tag{5}$$

One controller capable to fulfill the dynamics from (4)-(5) is given as follows

$$u = g_0^{-1}\left[-f_0(q_1,\dot{q}_1,q_2,\dot{q}_2) + (1+g_2)^{-1}\tau\right], \tag{6}$$

where τ is given by

$$\begin{aligned} \tau = &-\frac{\partial}{\partial q_2} f_1(q_2,\dot{q}_2) - \frac{\partial}{\partial \dot{q}_2} f_1(q_2,\dot{q}_2) - g_1\dot{q}_1 - k_p\dot{q}_2 \\ &-k_d\left(f_1(q_2,\dot{q}_2) + g_1 q_1 + g_2\dot{q}_1 + \beta\text{sign}(s)\right), \end{aligned} \tag{7}$$

and $\beta \in \mathbb{R}$ is the gain of the discontinuous term, which purpose will be evident in the stability analysis.

4. STABILITY ANALYSIS

Now, we ensure the existence of sliding modes by verifying $s\dot{s} < 0$ and taking $\dot{s}$ from (4). To this end, note that the discontinuous friction amplitude is bounded by an upper bound M then

$$\begin{aligned} s\dot{s} &= s\left(k_d w - k_d\beta\text{sign}(s) + \dot{w}\right) \\ &\leq -k_d\beta|s| + (k_d D_0 + D_1)\, s \\ &\leq -k_d\left(\beta - \left(D_0 + \frac{D_1}{k_d}\right)\right)|s|. \end{aligned} \tag{8}$$

Can be concluded the existence of sliding modes on the sliding surface s while the condition $\beta > D_0 + D_1/k_d$ be satisfied. This gives a guide to tune the parameter β of the controller (6)-(7). In fact, we can demonstrate that the trajectories reach the surface $s = 0$, in finite time, using the quadratic function

$$V(s) = s^2, \tag{9}$$

whose time derivative throughout the solutions of (1), satisfied

$$\begin{aligned} \dot{V}(s(t)) &= -2k_d\left(\beta - (D_0 + D_1/k_d)\right)|s| \\ &= -2k_d\left(\beta - (D_0 + D_1/k_d)\right)\sqrt{V(s(t))}, \end{aligned} \tag{10}$$

given an initial condition $V(t_0) = V_0$ and while satisfying $\beta > D_0 + D_1/k_d$, it can be guarantee the existence of a time t_f such for all $t > t_f$, the function $V(t)$ goes to zero. This time can be calculated directly from (10), where

$$V(t) = 0, \;\; \text{for} \;\;\; t \geq t_0 + \frac{\sqrt{V_0}}{k_d(\beta - (D_0 + D_1/k_d))} = t_f. \tag{11}$$

thus, $V(t)$ converges to zero in finite time and consequently a motion occurs along the set $s = 0$ in the system (1) from time t_f. When the trajectories of (1) are on sliding mode this is when $s = \dot{s} = 0$, the remain dynamics of the system are given as follows

$$\begin{aligned} \dot{q}_1 &= \dot{q}_1 \\ \ddot{q}_1 &= (1+g_2)^{-1}\Big(-\frac{\partial}{\partial q_2} f_1(q_2,\dot{q}_2) - \frac{\partial}{\partial \dot{q}_2} f_1(q_2,\dot{q}_2) \\ &\quad -(g_1 - k_d)\dot{q}_1 - (k_p - k_d^2)\dot{q}_2 \\ &\quad +k_p k_d q_2 - \dot{w}\Big) \\ \dot{q}_2 &= \dot{q}_2 \\ \ddot{q}_2 &= -\dot{q}_1 - k_p q_2 - k_d\dot{q}_2 \end{aligned} \tag{12}$$

the stability proof of system (12) is under development and will be published elsewhere.

5. APPLICATION

5.1 Example 1

Let us consider the system shown in Figure 1, described by

$$\begin{aligned} m_1\ddot{q}_1 + k_a q_1 + k_b(q_1 - q_2) + f_b(\dot{q}_1 - \dot{q}_2) = u \\ m_2\ddot{q}_2 + k_b(q_2 - q_1) + f_b(\dot{q}_2 - \dot{q}_1) = w \end{aligned} \tag{13}$$

where q_i are the mass positions, $\dot{q}_i$ the speeds, m_i the masses, for $i = 1, 2$; k_a and k_b are the spring stiffness, f_b stands for viscous friction of the damper, and the control input is given by u. To account for discrepancies in the model, a external perturbation is considered, with

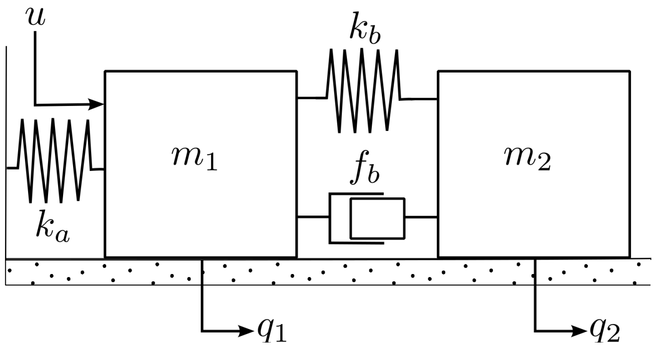

Fig. 1. A mass-spring-damper system

an amplitude level denoted as $\sup_t |w(t)| \leq D_0$, $D_0 > 0$, and $\sup_t |\dot{w}(t)| \leq D_1$, $D_1 > 0$ which are assumed known a priori.

For a zero force input $u = 0$ and zero perturbation $w = 0$, the system (13) has the following equilibrium point $\bar{q_1} = 0$ and $\bar{q_2} = 0$.

The sliding surface s proposed is given by

$$s = \ddot{q}_2 + k_p q_2 + k_d \dot{q}_2 + \dot{q}_1. \tag{14}$$

according to (6), u is designed as

$$u = k_a q_1 + k_b(q_1 - q_2) + f_b(\dot{q}_1 - \dot{q}_2) + \frac{m_1 m_2}{f_b + m_2}\tau, \tag{15}$$

where τ is given by

$$\tau = \frac{k_b}{m_2}(\dot{q}_2 - \dot{q}_1) - k_p \dot{q}_2 + \left(k_d - \frac{f_b}{m_2}\right)\left(\frac{k_b}{m_2}(q_2 - q_2) + \frac{f_b}{m_2}(\dot{q}_2 - \dot{q}_1) - \frac{\beta}{m_2}\operatorname{sign}(s)\right). \tag{16}$$

The derivative along the trajectories of (13) goes by

$$\dot{s} \leq -\frac{k_b}{m_2}(\dot{q}_2 - \dot{q}_1) + k_p \dot{q}_2 + \left(k_d - \frac{f_b}{m_2}\right)\left(-\frac{k_b}{m_2}(q_2 - q_1) - \frac{f_b}{m_2}(\dot{q}_2 - \dot{q}_1) + \frac{w(t)}{m_2}\right) + \left(\frac{f_b}{m_2} + 1\right)\ddot{q}_1 + \frac{\dot{w}(t)}{m_2}. \tag{17}$$

We ensure the existence of sliding modes by verifying $s\dot{s} < 0$

$$s\dot{s} \leq -\left(\frac{k_d}{m_2} - \frac{f_b}{m_2^2}\right)\left(\beta - D_0 - \left(\frac{m_2}{k_d m_2 - f_b}\right) D_1\right)|s|. \tag{18}$$

can be concluded the existence of sliding modes on the surface while the condition $\beta > D_0 + \left(\frac{m_2}{k_d m_2 - f_b}\right) D_1$ be satisfied. Moreover, we can demonstrate that the trajectories reach the surface $s = 0$, in finite time, using the quadratic function

$$V(s) = s^2,$$

and compute its time derivative along the solutions of (13),

$$\dot{V}(s(t)) \leq -2\left(\frac{k_d m_2 - f_b}{m_2^2}\right) \times \left(\beta - D_0 - \left(\frac{m_2}{k_d m_2 - f_b}\right) D_1\right)\sqrt{V(s(t))}. \tag{19}$$

From (19) it follows that $V(t) = 0$ for

$$t \geq t_0 + \frac{\sqrt{V(t_0)}}{\left(\frac{k_d}{m_2} - \frac{f_b}{m_2^2}\right)\left(\beta - D_0 - \left(\frac{m_2}{k_d m_2 - f_b}\right) D_1\right)} = t_f. \tag{20}$$

Hence, $V(t)$ converges to zero in finite time and, in consequence, a motion along the manifold $s = 0$ occurs in the discontinuous system (13). Thus, in the following development, we assume that system (13) is in sliding mode; therefore, $s = \dot{s} = 0$ for $t \geq t_f$. We have that the dynamics of system (13), once in sliding mode, are reduced to

$$\begin{aligned} \dot{q}_1 &= \dot{q}_1 \\ \ddot{q}_1 &= \frac{m_2}{f_b + m_2}\left(\Delta_1 \dot{q}_1 + \Delta_2 q_2 + \Delta_3 \dot{q}_2 - \frac{\dot{w}}{m_2}\right) \\ \dot{q}_2 &= \dot{q}_2 \\ \ddot{q}_2 &= -\dot{q}_1 - k_p q_2 - k_d \dot{q}_2 \end{aligned} \tag{21}$$

where

$$\begin{aligned} \Delta_1 &= -\frac{k_b}{m_2} - \frac{f_b}{m_2} + k_d \\ \Delta_2 &= -\frac{k_p f_b}{m_2} + k_p k_d \\ \Delta_3 &= -k_p + k_d^2 - \frac{f_b k_d}{m_2} + \frac{k_b}{m_2}. \end{aligned} \tag{22}$$

Performance issues and robustness properties of the proposed controller are additionally tested in numerical experiments. In the simulations, performed with MATLAB®, the mechanical system (13) is studied with the parameters shown in Table 1.

Table 1. Parameteres of the mass-spring-damper system and controller.

Notation	Description	Value	Units
m_1	Mass 1	1	kg
m_2	Mass 2	1	kg
k_a	Spring stiffness	1	N/m
$w(t)$	External perturbation	$0.1\sin(t)$	N
k_b	Spring stiffness	1	N/m
f_b	Damper coefficient	2	kg/s
β	controller parameter	0.3	N
k_p	controller parameter	17	kg/s^2
k_d	controller parameter	4	kg/s

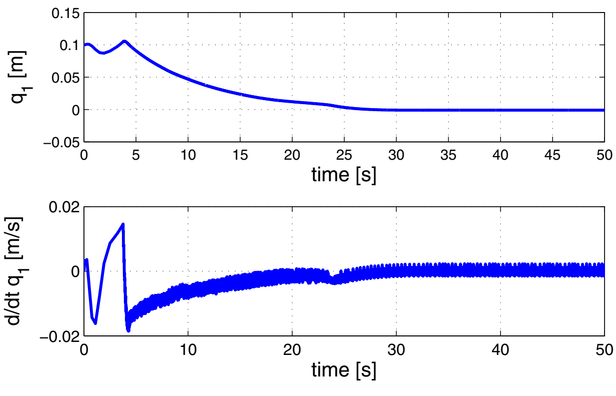

Fig. 2. Mass position q_1 and velocity $\dot{q}_1$.

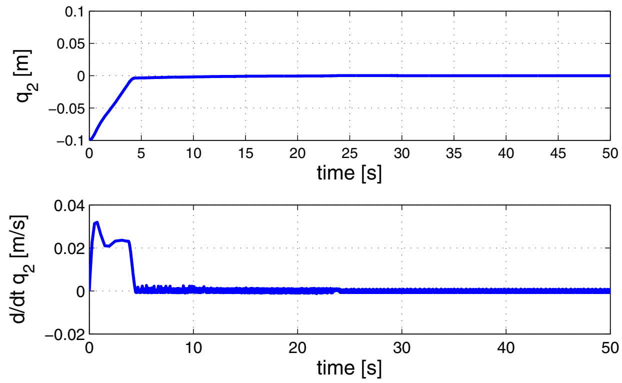

Fig. 3. Mass position q_2 and velocity $\dot{q}_2$.

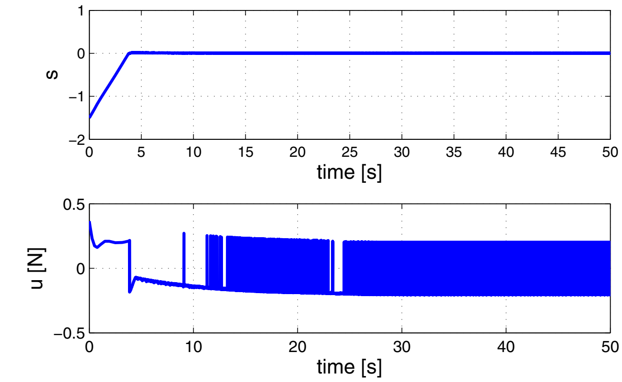

Fig. 4. Sliding variable s and signal control u.

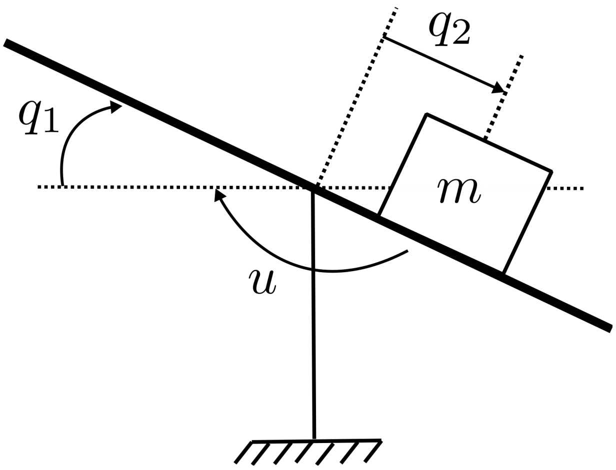

Fig. 5. A mass and beam system

5.2 Example 2

Consider now the mass and beam system shown in Figure (5). This system is modeled by the equations

$$\begin{aligned} (J + mq_2^2)\ddot{q}_1 + 2mq_2\dot{q}_1\dot{q}_2 - mgq_2cos(q_1) &= u \\ \ddot{q}_2 - q_2\dot{q}_1^2 - g\sin(q_1) + f_{v2}\dot{q}_2 &= w \end{aligned} \tag{23}$$

where q_1 is the beam (angular) position, q_2 the mass (linear) position, $\dot{q}_1$ and $\dot{q}_2$ the respective velocities, m the mass, f_{v2} is the viscous friction level, w is the external perturbation which is considered bounded, J is the moment of inertia of the beam, g the gravitational acceleration and u the control input.

It is desired to obtain a control law u so that the mass m will be regulated around the value $[q_1, \dot{q}_1, q_2, \dot{q}_2]^T = 0^T \in \mathbb{R}^4$

In order to apply the previous result, we must simplify this model, assuming the beam angle and beam velocity are small. Under this condition, system (23) can be simplified to

$$\begin{aligned} (J + mq_2^2)\ddot{q}_1 + 2mq_2\dot{q}_1\dot{q}_2 - mgq_2\cos(q_1) &= u, \\ \ddot{q}_2 - gq_1 + f_v\dot{q}_2 &= w. \end{aligned} \tag{24}$$

Let us set the control u as

$$u = 2mq_2\dot{q}_1\dot{q}_2 - mgq_2\cos(q_1) + (J + mq_2^2)\tau \tag{25}$$

where τ is given by

$$\tau = -g\dot{q}_1 - k_p\dot{q}_2 - (k_d - f_{v2})(gq_1 - f_{v2}\dot{q}_2 + \beta\text{sign}(s)). \tag{26}$$

The sliding surface stands by

$$s = \ddot{q}_2 + k_pq_2 + k_d\dot{q}_2 + \dot{q}_1. \tag{27}$$

The derivative along the trajectories of (24) is given by

$$\begin{aligned} \dot{s} \;=\; & \dot{w} + g\dot{q}_1 + k_p\dot{q}_2 - (f_v - k_d)(w + gq_1 - f_v\dot{q}_2) \\ & + \left(\frac{1}{J + mq_2^2}\right)(u - 2mq_2\dot{q}_1\dot{q}_2 + mgq_2\cos(q_1)) \end{aligned} \tag{28}$$

which in closed-loop can be reduced to

$$\dot{s} \;=\; -(k_d - f_v)\left(\beta\text{sign}(s) - w - \frac{\dot{w}}{k_d - f_v}\right) \tag{29}$$

now, we ensure the existence of sliding modes by verifying $s\dot{s} < 0$ and taking $\dot{s}$ from (27). To this end, note that the external perturbation and its derivative are bounded by upper bounds D_0 and D_1, respectively

$$\begin{aligned} s\dot{s} &\leq -s\,(k_d - f_v)\left(\beta\text{sign}(s) - D_0 - \frac{D_1}{k_d - f_v}\right) \\ &\leq -(k_d - f_v)\left(\beta - D_0 - \frac{D_1}{k_d - f_v}\right)|s|. \end{aligned} \tag{30}$$

Can be concluded the existence of sliding modes on the sliding surface while the conditions $k_d > f_v$ and $\beta > D_0 + D_1/(k_d - f_v)$ remain satisfied. This gives a guide to tune the parameter β of the controller (25). In fact, we can demonstrate that the trajectories reach the surface $s = 0$, in finite time, using the quadratic function

$$V(s) = s^2,$$

and compute its time derivative along the solutions of (24),

$$\dot{V}(s(t)) \leq -2(k_d - f_v)\left(\beta - D_0 - \frac{D_1}{k_d - f_v}\right)\sqrt{V(s(t))}. \tag{31}$$

From (31) it follows that

$$V(t) = 0 \qquad \text{for}$$

$$t \quad \geq \quad t_0 + \frac{\sqrt{V(t_0)}}{(k_d - f_v)\left(\beta - D_0 - \frac{D_1}{k_d - f_v}\right)} = t_f. \tag{32}$$

Hence, $V(t)$ converges to zero in finite time and, in consequence, a motion along the manifold $s = 0$ occurs in the discontinuous system (24). Notice that the reaching time can be reduced by increasing the value of parameter β. Thus, in the following development, we assume that system (24) is in sliding mode; therefore, $s = \dot{s} = 0$ for $t \geq t_f$.

Now let us show that, while the system remains in $s = 0$, from (25) and (27), we have that the dynamics of system (24), once in sliding mode, are reduced to

$$\begin{aligned} \dot{q}_1 &= \dot{q}_1 \\ \ddot{q}_1 &= -\dot{w} - g\dot{q}_1 - k_p\dot{q}_2 - (k_d - f_v)(w + gq_1 - f_v\dot{q}_2) \\ \dot{q}_2 &= \dot{q}_2 \\ \ddot{q}_2 &= -\dot{q}_1 - k_p q_2 - k_d\dot{q}_2. \end{aligned} \tag{33}$$

In the simulations, performed with MATLAB®, the mechanical system (24) is studied with the parameters shown in Table 2.

Table 2. Parameteres of the mass and beam system and controller.

Notation	**Description**	**Value**	**Units**
J	Inertia	1	kg.m
m	Mass	1	kg
g	Gravity	9.82	m/s^2
f_v	Viscous friction	1	kg/s
$w(t)$	External perturbation	$0.1\sin(t)$	N.m
β	controller parameter	0.3	N.m
k_p	controller parameter	17	N
k_d	controller parameter	4	N.s

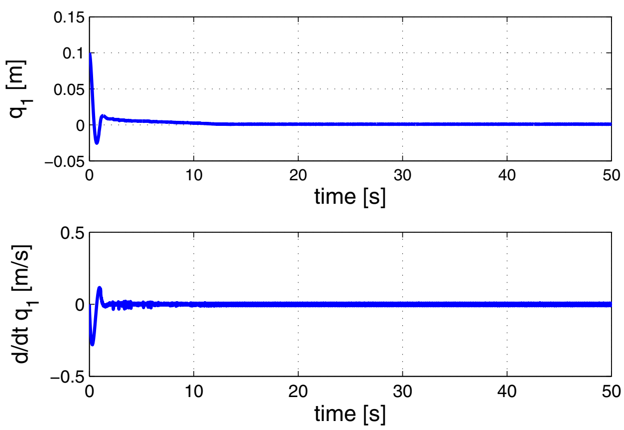

Fig. 6. Mass position q_1 and velocity $\dot{q}_1$.

6. CONCLUSION AND FUTURE WORK

6.1 Conclusions

Underactuated mechanical systems with unmatched perturbations, have a problem when they compensate the input to reach the equilibrium position. Taking this into account, a sliding mode controller is proposed for a 2-DOF underactuated mechanical system with external perturbations in the unactuated joint. The control objective is to regulate the unactuated variable to a reference position while the position of the actuated joint remains bounded and its velocity asymptotically decays to zero. The proposed controller through simulations guarantees the convergence of the unactuated variable to the equilibrium

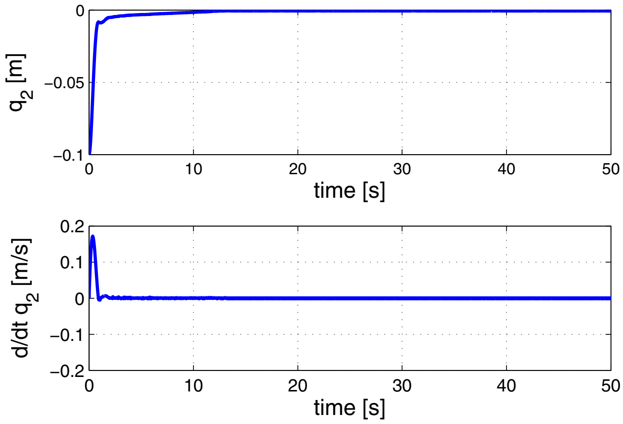

Fig. 7. Mass position q_2 and velocity $\dot{q}_2$.

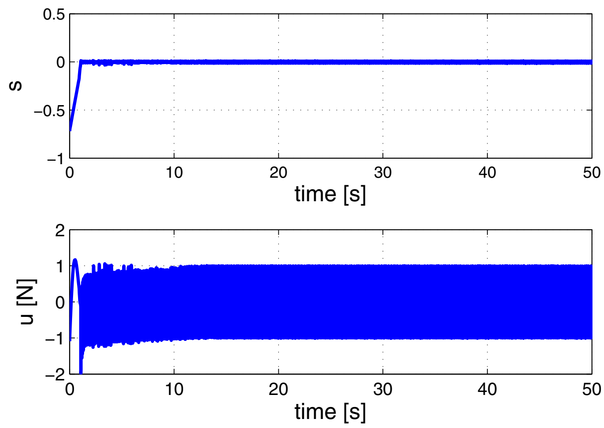

Fig. 8. Sliding variable s and signal control u.

point. It is shown that the closed-loop system is robust with respect to some uncertainty in the amplitude of the external perturbation, being necessary just to know their upper bound. The control is obtained using a sliding mode technique. Some advantages of using this technique are insensibility against disturbances and the choice of surface $s = 0$, this allows us to choose a priori the closed-loop dynamics. However, some disadvantages are the chattering phenomenon and that the trajectories are not robust against uncertainties during the reaching phase. Numerical simulations validate the proposed theoretical approach. The use of MatLab to perform the simulations saves the need to acquire an additional software to simulate, and opens the possibility for other teams to advance or modify the present work using resources available in their institution.

6.2 Future Work

Design a stability proof to guarantee asymptotic stability when the trajectories are on the sliding surface, besides of extend the present approach to underactuaded mechanical systems with discontinuous friction. Design a methodology to validated experimentally the simulations done in MatLab, in which an observer is needed for velocity feedback and to estimate the upper bound of the external perturbation.

The so called phenomena of chattering can be reduced using some afore proposed techniques as low-pass filtering the control signal Tseng and Chen (2010), compensate the uncertainties through the usage of a disturbance estimator Yang et al. (2013), design a second order sliding mode controller (SOSM) Bartolini et al. (1998), among others techniques Cheng et al. (2011), Su et al. (2010), Lee et al. (2004). Also the reaching time to the sliding surface can be reduced using an exponential reaching law Fallaha et al. (2011). Some of these techniques will be approach in future works

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