

# Adaptive PD controller to solve the trajectory tracking of a quadrotor unmanned aerial vehicle

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**Abstract:** This paper describes the design of an adaptive output based controller to regulate the path tracking process of a quadrotor robot. This kind of robot represents a relevant system considering that it can be used to complete exploring, personal assistance, home professional medical care and surveillance tasks. The controller used a mixed structure based on adaptive proportional derivative controller to solve the trajectory tracking problem and the super-twisting algorithm as robust observer/differentiator to recover the robot velocity information. A non-smooth Lyapunov function was proposed to prove the ultimate boundedness of the tracking error. The controller was implemented as a class of decentralized structure where each joint was fed with the corresponding reference trajectories. A set of reference trajectories was designed to propose a circular reference trajectory. A set of numerical simulations was implemented to evaluate the controller performance. These simulations showed the effectiveness of the proposed controller to track the reference trajectories.

*Keywords:* Sliding Mode Control, Unmanned Aerial Vehicles, Adaptive Control, Robust Differentiator.

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## 1. INTRODUCTION

The problem of controlling Unmanned Aerial Vehicles (UAVs) has attracted the attention of, both, scientific and technology developers due to the range of applications and the challenge of designing simple to implement but, at the same time, fast, robust and low consumer control schemes due to the natural energetic restrictions of this class of mobile robots.

Many control schemes have been extensively used to solve the path following task as well of regulation, some of them are based on the application of the classical PD and PID control laws. In Tayebi and McGilvray (2006), a quaternion-based model independent PD control was implemented to solve the problem of velocity regulation. Both, proportional-derivative (PD) and proportional-derivative-integral (PID) schemes have shown good results in spite of the narrowed knowledge of the model, and even the presence of disturbances in contrast with purely model based controllers which may present poor results when model variations arise, as reported in Bouabdallah et al. (2004). Also, the usual complexity of model based controllers for nonlinear systems has motivated the use of enhanced model free robust control techniques.

On the other hand, some of the most popular control schemes for uncertain nonlinear systems are based on discontinuous schemes; in particular, sliding mode controllers have shown good robustness for uncertainties in the control systems, as well as the capacity of rejecting matched disturbance inputs Utkin et al. (1999). More

over, the finite-time convergence of these controllers as well as their simplicity have made them suitable for a large variety of applications, from power electronics Utkin (1993), Sira-Ramírez and Silva-Ortigoza (2006), robotic arms Parra-Vega et al. (2003), mechanical systems Bartolini et al. (2003), biological systems Gallardo-Hernández et al. (2013), among others.

These last features have made the combination of sliding mode control, under a PD structure a good alternative for controlling uncertain (and possibly disturbed) nonlinear systems, specially Unmanned Aerial Vehicles, where the necessity of robust tracking schemes, as well as fast convergent algorithms is crucial. Some successful implementations of sliding mode control schemes have been reported in literature (see Zeghlache et al. (2012), Lee et al. (2009), Xu and Ozguner (2006), Ramirez-Rodriguez et al. (2014)) with motivating results.

Even when there are some advantages concerning the application of this technique, two main drawbacks are found within this control scheme: The chattering effect and the high activity in the control action which tends to increment the energetic consumption of the controller Utkin and Poznyak (2013a). Some solutions to increment the energetic efficiency are based on the use of gain scheduling approaches Cheng and Chien (2006), Zeghlache et al. (2012) and adaptive control rules (see Lee et al. (2009), Utkin and Poznyak (2013b), Shtessel et al. (2012) and references therein). The first one is based mainly on

the combination of a sliding mode driven by a neural network structure, or by means of fuzzy models.

In this manuscript, a new scheme for controlling quadrotors was designed using an adaptive PD structure in closed-loop with the super-twisting algorithm (STA) working as a robust exact differentiator (RED) Levant (1998). The stability of the equilibrium point for the quadrotor system regulated by this controller was analyzed using the idea given in Moreno and Osorio (2008) and Moreno and Osorio (2012), where a non-smooth Lyapunov function for the STA was introduced. Even when the ideas were taken from Moreno and Osorio (2008) and Moreno and Osorio (2012) this paper analyzed a different contribution when not only the differentiator was considered. Moreover, the STA was implemented in closed-loop with the adaptive PD scheme. This simple modification in this novel PD scheme, has contributed to reduce the well known problems regarding the application of PD such as peaking and overshoots among others.

The application of Lyapunov theory enabled a constructive method to adjust the controller and RED gains. An important remark regarding the capabilities of a PD controller is that this control method can not ensure the convergence of the tracking error to zero under the presence of bounded perturbation and parametric uncertainties. Thus, the controller proposed here, cannot achieve zero tracking error because the STA is acting only in the estimation of the error signal derivative.

## 2. QUADROTOR DESCRIPTION

The following set of ordinary differential equations describes the dynamic behavior of the quadrotor Tayebi and McGilvray (2006):

$$\begin{aligned} \frac{d^2}{dt^2}x &= (\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi)\sin(\theta)) \frac{U_1}{m} \\ \frac{d^2}{dt^2}y &= (-\cos(\psi)\sin(\phi) + \sin(\psi)\cos(\phi)\sin(\theta)) \frac{U_1}{m} \\ \frac{d^2}{dt^2}z &= -g + \cos(\phi)\cos(\theta) \frac{U_1}{m} \\ \frac{d}{dt}\phi &= \frac{I_{YY} - I_{ZZ}}{I_{XX}}\theta\psi - \frac{J_{TP}}{I_{XX}}\theta\Omega + \frac{U_2}{I_{XX}} \\ \frac{d}{dt}\theta &= \frac{I_{ZZ} - I_{XX}}{I_{YY}}\phi\psi - \frac{J_{TP}}{I_{YY}}\phi\Omega + \frac{U_3}{I_{YY}} \\ \frac{d}{dt}\psi &= \frac{I_{XX} - I_{YY}}{I_{ZZ}}\phi\theta + \frac{U_4}{I_{ZZ}} \end{aligned} \quad (1)$$

The first three equations describes the linear acceleration of the vehicle in the direction of  $x$ ,  $y$ , and  $z$  axes, respectively while the last three equations are nominated for the angular accelerations of the vehicle about the same axes, respectively. The parameters  $\phi$ ,  $\theta$  and  $\psi$  represent the Euler angles about the body axes ( $x, y, z$ ), respectively.  $I_{xx}$ ,  $I_{yy}$  and  $I_{zz}$  are the inertial components about the x-axis, y-axis and z-axis, respectively.

The variables  $\frac{d}{dt}x$ ,  $\frac{d}{dt}y$  and  $\frac{d}{dt}z$  are the translational velocity components along the main axes. Finally,  $\rho$  is the air density. The controller actions associated to the tangential velocities presented in the model described above are formally described as follows

$$\begin{aligned} U_1 &= b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ U_2 &= lb(-\Omega_2^2 + \Omega_4^2) \quad U_3 = lb(-\Omega_1^2 + \Omega_3^2) \\ U_4 &= d(-\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2) \end{aligned} \quad (2)$$

The constant  $l, [m]$  is the distance between the center of the quadrotor and the center of a propeller. The derivation of the aerodynamics contributions, thrust  $b, [Ns^2]$  and drag  $d, [Ns^2]$  can be consulted in Bouabdallah et al. (2004). The variables  $U_i$  represents the tangential velocity of each propeller.

The variable  $\Omega_1, [rad/s]$  is the front propeller speed,  $\Omega_2, [rad/s]$  is the right propeller speed,  $\Omega_3, [rad/s]$  is the rear propeller speed and  $\Omega_4, [rad/s]$  is the left propeller speed. Finally, the variable  $\Omega$  refers to the sum of all individual angular velocities.

## 3. PROBLEM FORMULATION

The problem considered in this paper is to design a feedback controller to force the asymptotic convergence of the trajectory tracking error between the states of (1) and the stable reference model given by  $\zeta^*$ . Formally, the previous statement can be reformulated as: to design an output feedback controller such that  $\lim_{t \rightarrow \infty} \|e(t)\| = 0$  where  $e = \zeta - \zeta^*$ . The vector  $\zeta$  is given by  $\zeta^T = [z \ \phi \ \theta \ \psi]$ .

The variables  $x$  and  $y$  are indirectly controlled by the application of an indirect like strategy Bouabdallah et al. (2004). The reference trajectory  $\zeta^*$  satisfies

$$\frac{d^2}{dt^2}\zeta^*(t) = h\left(\frac{d}{dt}\zeta^*(t), \zeta^*(t)\right) \quad (3)$$

with the reference trajectory  $\zeta^* \in \mathbb{R}^8$ . In the previous set of second order differential equations, the initial conditions  $\zeta^*(0)$  and  $\frac{d}{dt}\zeta^*(0)$  are given vectors. The function  $h(\zeta^*, \frac{d}{dt}\zeta^*)$  is a Lipschitz function. The system (3) can be transformed using the change of variables  $\zeta_a^* = \zeta^*$  and  $\zeta_b^* = \frac{d}{dt}\zeta^*$ .

The reference system (3) has a stable equilibrium point and by the converse Lyapunov theorem, one can ensure that the system in the new coordinates  $[\zeta^*]^T = [(\zeta_a^*)^T \ (\zeta_b^*)^T]$  satisfies

$$\|\zeta^*(t)\|^2 \leq \zeta_+^*, \quad \zeta_+^* \in \mathbb{R}^+, \quad \forall t \geq 0 \quad (4)$$

Due to the previous inequality is valid and considering that both functions  $f$  and  $h$  are continuous, then the following assumption can be straightforwardly verified:

*Assumption 1.* There is a positive constant  $h^+$  such that the following inequality is valid:

$$\|f(\zeta^*) - h(\zeta^*)\| \leq h^+ \quad (5)$$

$\forall \zeta^* \in \mathbb{R}^n$  solution of (3).

The problem of trajectory tracking between (1) and (3) can be solved by some well developed control schemes. The most common manner for solving the controller design is the so-called PD scheme. The classical PD and PID controllers are the most successfully control techniques implemented in real applications Skoczowski et al. (2005), Yu et al. (2005). Several approaches have been designed for improving the robustness of classical PD and PID control

schemes Boubertakh et al. (2010), Matausek and Ribic (2009).

All the previous control designs are structures that need the derivative of the error signal. This condition is hardly fulfilled in real quadrotors due to the necessity of setting up many sensors and wasting large amounts of energy. The main problem has been to calculate the error derivative signal. The most useful methods to approximate the numerical derivative are based on linear filters and observers Yu and Ortiz (2005). The main approach for designing a linear differentiator is to approximate a transfer function within a finite frequency band. Additionally, low-pass filters are also used to damp noises Levant (1998). Another important technique is related to the use of Luenberger state estimators. Unfortunately, the exact description of the model associated with the signal is demanded and the observer (differentiator) parameters can not be tuned easily for reducing sensitivity under measurement noises or perturbations Dridi et al. (2010). For uncertain, perturbed and even unknown systems, sliding modes based differentiators are a suitable option for overcoming the main drawbacks described for linear differentiators. Indeed, without noises, the exact information on the derivative can be obtained by averaging high frequency switching signals.

Despite the natural benefits offered by these observers, the estimation of velocity in the flying of a quadrotor requires the application of the so-called finite time RED. The most remarkable of these estimators is the STA which posses many relevant properties that can be useful to solve the trajectory tracking problem presented in this study Yu and Ortiz (2005) Dridi et al. (2010), Mboup et al. (2007), Levant (1998).

The nature of PID controllers forces controller energy that may be unsuitable for regulating the path tracking problem of quadrotor systems. Therefore, the adaptive structure may bring an option to save energy from the quadrotor and increase its flying period.

## 4. ADAPTIVE PD CONTROLLER

### 4.1 Class of Nonlinear Systems

The nonlinear system used to describe the dynamical behavior of a quadrotor system can be represented by the following second order nonlinear differential equation

$$\begin{aligned} \frac{d^2}{dt^2}\zeta(t) &= f\left(\frac{d}{dt}\zeta(t), \zeta(t)\right) + g(\zeta(t))u(t) + \\ &\eta\left(\frac{d}{dt}\zeta(t), \zeta(t), t\right) \quad (6) \\ \zeta(0) &= \zeta_0 \text{ and } \frac{d}{dt}\zeta(0) = \zeta_{d0} \text{ given} \\ &\zeta_0, \zeta_{d0} \in \mathbb{R}^n \end{aligned}$$

Here  $\zeta \in \mathbb{R}^n$  and  $\frac{d}{dt}\zeta \in \mathbb{R}^n$ ,  $\zeta(0)$  and  $\frac{d}{dt}\zeta(0)$  are the initial conditions for the differential equation. The drift term  $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$  and the input associated term  $g: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ . The nonlinear function  $\eta: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^n$  represents some uncertainties affecting the nonlinear system satisfying

$$\|\eta\|^2 \leq \eta_0 + \eta_1 \left\| \frac{d}{dt}\zeta \right\|^2, \quad \eta_0, \eta_1 \in \mathbb{R}_+ \quad (7)$$

The signal  $\zeta$  is the available output information. The control action is represented by  $u \in \mathbb{R}^n$ . The class of systems considered in (1) is a rough generalization of many mechanic, electromechanical, electric, thermodynamic and hydrodynamic systems. The system presented above in (1) with the selection of  $\zeta_a = \zeta$  and  $\zeta_b = \frac{d}{dt}\zeta$  can be represented as (by the state variable technique)

$$\begin{aligned} \frac{d}{dt}\zeta_a(t) &= \zeta_b(t) \\ \frac{d}{dt}\zeta_b(t) &= f(\zeta(t)) + g(\zeta_a(t))u(t) + \eta(\zeta(t), t) \quad (8) \\ y(t) &= \zeta_a(t) \end{aligned}$$

where  $\zeta = [\zeta_a^T \zeta_b^T]^T \in \mathbb{R}^{2n}$ .

Throughout the paper, the following assumptions are assumed to be fulfilled

A1. The nonlinear function  $f(\cdot)$  is unknown but satisfies the Lipschitz condition

$$\|f(\zeta) - f(\zeta')\| \leq L_1 \|\zeta - \zeta'\|, \quad \forall \zeta, \zeta' \in \mathbb{R}^{2n}, \quad L_1 \in \mathbb{R}_+ \quad (9)$$

A2. The nonlinear system (8) is controllable, and therefore the function  $g(\zeta)$  is known and satisfies

$$0 < g^- \leq \|g(x_\zeta)\| \leq g^+ < \infty, \quad \forall \zeta_a \in \mathbb{R}^n \quad (10)$$

By this assumption, the matrix  $g(\zeta_a)$  is invertible  $\forall t \geq 0$ .

### 4.2 Design of the Adaptive PD controller

In general, a PD controller is designed using the following structure

$$u(t) = -k_1 e(t) - k_2 \frac{d}{dt}e(t) \quad (11)$$

where  $k_1, k_2 \in \mathbb{R}^n$  are the controller gains that must be adjusted and  $e \in \mathbb{R}^n$  is the output error given by  $e(t) := \zeta(t) - \zeta^*(t)$ . However, this controller is hardly to be implemented considering that  $\frac{d}{dt}e(t)$  and  $e(t)$  are rarely measured simultaneously without an important resources investment. Therefore, in classical literature, one can find two important solutions: to construct an observer or using a first order filter to approximate the error derivative. The first one requires the system structure (that is in this paper is assumed to be unknown) and in the second case, the derivative approximation is usual poor, specially if the output information is contaminated with noises. One additional option is considering a class of RED that can provide a suitable and accurate approximation of the error derivative. STA has demonstrated to be one of the best RED many times Levant (1993).

### 4.3 Super-Twisting Algorithm

In counterpart of some others second order sliding modes algorithms, the STA can be used with systems having relative degree one with respect to the chosen output. The STA has been used as a controller, a state estimator and as a RED Moreno and Osorio (2012). The STA application as a RED is described as follows. If  $w_1(t) = r(t)$  where  $r(t) \in \mathbb{R}$  is the signal to be differentiated,  $w_2(t) = \frac{d}{dt}r(t)$  represents its derivative and under the assumption of  $\left| \frac{d^2}{dt^2}r(t) \right| \leq r^+$ , the following auxiliary equations are gotten

$$\frac{d}{dt}w_1(t) = w_2(t) \quad \frac{d}{dt}w_2(t) = \frac{d^2}{dt^2}r(t) \quad (12)$$

The previous differential equations are a state representation of the signal  $r(t)$ . The STA to obtain the derivative of  $r(t)$  looks like

$$\begin{aligned} \frac{d}{dt}\bar{w}_1(t) &= \bar{w}_2(t) - \lambda_1 |\bar{w}_1(t)|^{1/2} \text{sign}(\bar{w}_1(t)) \\ \frac{d}{dt}\bar{w}_2(t) &= -\lambda_2 \text{sign}(\bar{w}_1(t)) \\ \tilde{w}_1 &= \bar{w}_1 - w_1; \quad d(t) = \frac{d}{dt}\bar{w}_1(t) \end{aligned} \quad (13)$$

where  $\lambda_1, \lambda_2 > 0$  are the STA gains. Here  $d(t)$  is the output of the differentiator Levant (1998). In this equation,

$$\text{sign}(\nu) = \begin{cases} 1 & \text{if } \nu > 0 \\ \in [-1, 1] & \text{if } \nu = 0 \\ -1 & \text{if } \nu < 0 \end{cases} \quad (14)$$

To apply the STA as differentiator, let us represent the uncertain system (8) as the composition of the following  $n$  second order systems

$$\begin{aligned} \frac{d}{dt}\zeta_{a,i}(t) &= \zeta_{b,i}(t) \\ \frac{d}{dt}\zeta_{b,i}(t) &= f_i(\zeta(t)) + g_i(\zeta_a(t))u_i(t) + \xi_i(\zeta(t), t) \end{aligned} \quad (15)$$

where  $\zeta_{a,i}$  and  $\zeta_{b,i}$  are the  $i$ -th and  $(n+i)$ -th states of (8), respectively. The nonlinear functions  $f_i(\cdot)$  and  $g_i(\cdot)$  are the functions associated to the states  $\zeta_{a,i}$  and  $\zeta_{b,i}$ . Similarly,  $\xi_i(\cdot, \cdot)$  is the corresponding uncertainty to the same subsystem.

Before the STA can be applied, the dynamics of the tracking error also must be treated as in the case of (15). Then, the definition for the individual elements of the vector  $e$  is given by

$$\frac{d}{dt}e_i = \frac{d}{dt}\zeta_{a,i} - \frac{d}{dt}\zeta_{a,i}^* \quad (16)$$

must be obtained using (1)

$$\begin{aligned} \frac{d}{dt}e_i(t) &= e_{i+n}(t) \\ \frac{d}{dt}e_{i+n}(t) &= f_i(\zeta(t)) + g_i(\zeta_a(t))u_i(t) - h_i(\zeta^*(t)) + \xi_i(\zeta(t), t) \end{aligned} \quad (17)$$

The function  $h_i(\zeta^*(t))$  is the  $i$ -th component of the vector field  $h(\zeta^*(t))$ .

*PD controller with the Super-Twisting Algorithm* Using the STA, the PD control is selected as

$$u_i(t) = -k_{1,i}(t)e_i(t) - k_{2,i}(t)d_i(t) \quad (18)$$

The gains in the PD controller are determined by

$$\begin{aligned} k_{1,i}(t) &= g_i^{-1}(x_a(t))\bar{k}_{1,i}(t) \\ k_{2,i}(t) &= g_i^{-1}(x_a(t))\bar{k}_{2,i}(t) \end{aligned} \quad (19)$$

with  $\bar{k}_{1,i}$  and  $\bar{k}_{2,i}$  are time varying scalars adjusted by a special tracking error dependent adaptive law described by the following ordinary differential equations:

$$\begin{aligned} \frac{d}{dt}\bar{k}_{1,i}(t) &= -\pi_{1,i}^{-1}e_i(t)M_a^\top P_{2,i}E_i(t) \\ \frac{d}{dt}\bar{k}_{2,i}(t) &= -\pi_{2,i}^{-1}e_{i+n}(t)M_b^\top P_{2,i}E_i(t) \end{aligned} \quad (20)$$

where  $\pi_{1,i}$  and  $\pi_{2,i}$  are free parameters to adjust the velocity of convergence for the adjustable gains. The matrices  $M_a$  and  $M_b$  are given by  $M_a = [1 \ 0]^\top$  and  $M_b = [0 \ 1]^\top$ .

Additionally, the term  $E_i = [e_i \ e_{i+n}]^\top$ . The matrix  $P_{2,i}$  is positive definite and it is presented in the main statement of the main theorem of this article. The following extended system describes the complete dynamics of the error signal in closed-loop with an adequate implementation of (13)

$$\begin{aligned} \frac{d}{dt}e_i(t) &= e_{i+n}(t) \\ \frac{d}{dt}e_{i+n} &= f_i(x(t)) - h_i(x^*(t)) - g_i(x)(k_{1,i}(t)e_i(t) - k_{2,i}(t)d_i(t)) + \eta_i(x(t), t) \\ \frac{d}{dt}\delta_{1,i}(t) &= \delta_{2,i}(t) - \lambda_{1,i}|\delta_{1,i}(t)|^{1/2}\text{sign}(\delta_{1,i}(t)) \\ \frac{d}{dt}\delta_{2,i}(t) &= -\lambda_{2,i}\text{sign}(\delta_{1,i}(t)) - \ddot{e}_i \end{aligned} \quad (21)$$

where  $\delta_{1,i} = \bar{w}_{1,i} - e_i$  and  $\delta_{2,i}(t) = \frac{d}{dt}\delta_{1,i}(t)$ . The term  $d_i(t)$  refers to the approximation of  $e_{i+n}$  which is obtained by means of the second order STA presented in (13). Then, the variable  $\bar{w}_{1,i}(t)$  is the first state of the corresponding STA implemented as differentiator.  $\lambda_{1,i}, \lambda_{2,i}$  are selected according the main theorem introduced below. A set of  $n$  differentiators were used to reconstruct the information of  $e_{i+n}$ .

The following section shows the main result of this paper. The theorem introduced in that section gives a constructive way to adjust the gains of the STA and the adaptive gains of the PD controller.

## 5. ANALYSIS: CONVERGENCE OF THE TRACKING ERROR

The convergence of the tracking error  $e$  is justified by the result presented in the following theorem:

*Theorem 1.* Consider the nonlinear system given in (1), supplied with the control law (18) adjusted with the gains given in (19) and the derivative of the error signal obtained by means of equation (13), if there exists a positive scalar  $\alpha_i$  and if the gains are selected as  $\lambda_{1,i} > 0, \lambda_{2,i} > 0$ , the next Lyapunov inequalities always have a positive definite solution  $P_{1,i}$ ,

$$\begin{aligned} A_{1,i}^\top P_{1,i} + P_{1,i}A_{1,i} &\leq -Q_{1,i} \\ A_{1,i} &= \begin{bmatrix} -\lambda_{1,i} & 1 \\ -2\lambda_{2,i} & 0 \end{bmatrix}, \quad Q_{1,i} = Q_{1,i}^\top > 0, \quad Q_{1,i} \in \mathbb{R}^{2 \times 2} \end{aligned} \quad (22)$$

then for every positive value of  $L_1$  satisfying equation (9) and positive value of  $h^+$  defined in (5), there exist positive gains  $\bar{k}_{1,i}, \bar{k}_{2,i}$  such that if the Riccati equations given by  $P_{2,i}(A_{2,i} + \alpha_i I) + (A_{2,i} + \alpha_i I)^\top P_{2,i} + P_{2,i}R_{2,i}P_{2,i} + Q_{2,i} \leq 0$  (23)

have positive definite solution  $P_{2,i}$  with

$$\begin{aligned} A_{2,i} &= \begin{bmatrix} 0 & 1 \\ -\bar{k}_{1,i} & -\bar{k}_{2,i} \end{bmatrix}, \quad R_{2,i} = \Lambda_{a,i} + \Lambda_{b,i} \\ Q_{2,i} &= 4\lambda_{\max}\{\Lambda_{b,i}^{-1}\}I_{2 \times 2} + \bar{\Lambda}_{a,i}, \quad \bar{\Lambda}_{a,i} = L_i\Lambda_{a,i}, \\ \Lambda_{a,i}, \Lambda_{b,i} &> 0 \text{ and symmetric, } \Lambda_{a,i}, \Lambda_{b,i} \in \mathbb{R}^{2 \times 2} \quad \alpha_i \in \mathbb{R}_+ \end{aligned} \quad (24)$$

and if the adaptive gains of the PD controller are adjusted by (20), thus the trajectories of  $E^\top = [e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}]$  are globally ultimately bounded with bound

$$\overline{\lim}_{t \rightarrow \infty} E^\top(t)P_2E(t) \leq \sum_{i=1}^n \frac{\gamma_i}{\alpha_i} \quad (25)$$

where

$$P_2 = \begin{bmatrix} P_{2,1} & 0_{2 \times 2} & \cdots & 0_{2 \times 2} \\ 0_{2 \times 2} & P_{2,2} & \cdots & 0_{2 \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2 \times 2} & 0_{2 \times 2} & \cdots & P_{2,n} \end{bmatrix} \quad (26)$$

and  $\gamma_i = 2\lambda_{\max} \left\{ \Lambda_{b,i}^{-1} \right\} (h_i^+ + 2c_{+}^* + \eta_{0,i})$

**Proof.** Consider the Lyapunov candidate function given by

$$V(\xi, E, \bar{k}_1, \bar{k}_2) = \sum_{i=1}^n V_i(\xi_i, E_i, \bar{k}_{1,i}, \bar{k}_{2,i})$$

$$V_i(\xi_i, E_i, \bar{k}_{1,i}, \bar{k}_{2,i}) = V_{1,i}(\xi_i) + V_{2,i}(E_i) + V_{3,i}(\bar{k}_{1,i}, \bar{k}_{2,i}) \quad (27)$$

with  $V_{1,i}(\xi_i) = \xi_i^\top P_{1,i} \xi_i$ ,  $V_{2,i}(E_i) = E_i^\top P_{2,i} E_i$  and  $V_{3,i}(\bar{k}_{1,i}, \bar{k}_{2,i}) = \pi_{1,i} \bar{k}_{1,i}^2 + \pi_{2,i} \bar{k}_{2,i}^2$ .

The term labeled  $\xi_i$  is given by  $\xi_i = [|\delta_{1,i}|^{1/2} \text{sign}(\delta_{1,i}) \quad \delta_{2,i}]^\top$ .

The time derivative of each function  $V_i(\xi_i, e_i, \bar{k}_{1,i}, \bar{k}_{2,i})$  is given by

$$\begin{aligned} \frac{d}{dt} V_i(t) &= 2\xi_i^\top(t) P_{1,i} \frac{d}{dt} \xi_i(t) + 2E_i^\top P_{2,i} \frac{d}{dt} E_i + \\ &2\pi_{1,i} \bar{k}_{1,i} \frac{d}{dt} \bar{k}_{1,i} + 2\pi_{2,i} \bar{k}_{2,i} \frac{d}{dt} \bar{k}_{2,i} \end{aligned} \quad (28)$$

Note that  $V_{1,i}(\xi_i)$  is continuous but not differentiable in  $\delta_{1,i} = 0$ . The ideas given in Moreno and Osorio (2012) are used here to handle this non-classical type of Lyapunov functions. Now, continuing the analysis for the second section  $V_{2,i}(E_i)$ . Taking the full time derivative of this second function, one has  $\frac{d}{dt} V_{2,i}(t) = 2E_i^\top(t) \frac{d}{dt} E_i(t)$ . The substitution of  $\frac{d}{dt} \xi_i$  and  $\frac{d}{dt} E_i$  in the full time derivative of  $V_i(t)$  yields to

$$\begin{aligned} \frac{d}{dt} V_i(t) &\leq \xi_i^\top(t) [P_{1,i} A_{i,1} + A_{i,1}^\top P_{1,i}] + \\ &E_i^\top(t) [P_{2,i} (A_{2,i} + \alpha_i I) + (A_{2,i} + \alpha_i I)^\top P_{2,i}] E_i(t) + \\ &E_i^\top(t) [P_{2,i} R_{2,i} P_{2,i} + Q_{2,i}] E_i(t) - \alpha_i V_i(t) + \\ &\gamma_i + 2\bar{k}_{1,i}(t) \left\{ \pi_{1,i} \frac{d}{dt} \bar{k}_{1,i}(t) + e_i(t) M_a^\top P_{2,i} E_i(t) \right\} + \\ &2\bar{k}_{2,i}(t) \left\{ \pi_{2,i} \frac{d}{dt} \bar{k}_{2,i}(t) + e_{i+n}(t) M_b^\top P_{2,i} E_i(t) \right\} \end{aligned} \quad (29)$$

The previous differential inclusion was obtained after applying the so-called lambda inequality  $X^\top Y + Y^\top X \leq X^\top \Lambda^{-1} X + Y^\top \Lambda Y$ ,  $X, Y \in \mathbb{R}^{s \times q}$  with  $0 < \Lambda = \Lambda^\top \in \mathbb{R}^{s \times s}$  Poznyak (2008).

Taking the assumption on the existence of positive definite solutions for both the Lyapunov equation and the Riccati equation presented in the theorem statement and considering the class of adjustment methods for the gains  $\bar{k}_{1,i}(t)$  and  $\bar{k}_{2,i}(t)$  presented in (20), one gets that last inclusion can be transformed to:  $\frac{d}{dt} V_{2,i}(t) \leq -\alpha_i V_{2,i}(t) + \gamma_i$ . Taking the solution of the previous inclusion, and by the comparison Lemma, it can be proved that  $V_{2,i}(t) \leq V_{2,i}(0) e^{-2\alpha_i t} + \frac{\gamma_i}{\alpha_i} (1 - e^{-2\alpha_i t})$ . Taking just the case when the previous inclusion became into an ordinary differential equation and redefining  $V_i(t)$  as  $V_i^{eq}(t)$ , one can prove that

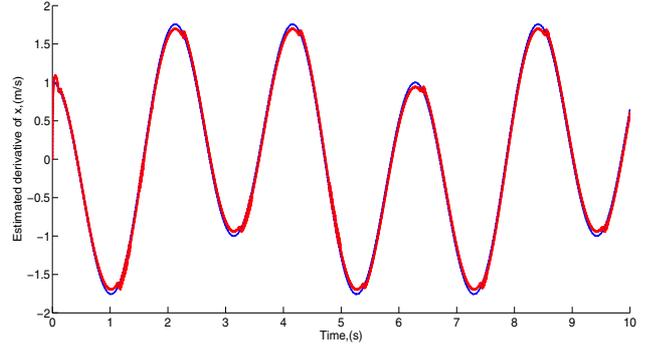


Fig. 1. Estimation of velocity for the coordinate  $x(t)$  based on the STA applied as RED. The estimated velocity is compared with the measured velocity obtained from the model exported to Matlab.

$$V(t) \leq \sum_{i=1}^n \sqrt{V_{1,i}(0)} - 2t \sum_{i=1}^n \kappa_{1,i} + \sum_{i=1}^n \left( V_{2,i}(0) e^{-\alpha_i t} + \frac{\gamma_i}{\alpha_i} (1 - e^{-\alpha_i t}) \right) \quad (30)$$

Clearly, when  $t \geq T$  with  $T = \frac{\sum_{i=1}^n \sqrt{V_{1,i}(0)}}{2 \sum_{i=1}^n \kappa_{1,i}}$  the following inequality is valid

$$V(t) \leq \sum_{i=1}^n \left( V_{2,i}(T) e^{-\alpha_i(t-T)} \right) + \sum_{i=1}^n \frac{\gamma_i}{\alpha_i} (1 - e^{-\alpha_i(t-T)}) \quad (31)$$

And if we take the upper limit when  $t \rightarrow \infty$  in the previous inequality, one gets  $\overline{\lim}_{t \rightarrow \infty} V(t) \leq \sum_{i=1}^n \frac{\gamma_i}{\alpha_i}$ . This result finished the proof of this Theorem.

■

## 6. NUMERICAL RESULTS

When the adaptive PD controller is computed for the system, the derivative obtained by the STA brings some advantages. The robustness of STA applied as differentiator forced a better performance for any controller applied on second order systems when the only available information is the output signal.

Then, the first part of the numerical simulations is devoted to evaluate the performance of the STA as RED. The derivative of tracking error is compared with the information provided by the measurements obtained directly from the simulation of the quadrotor and the derivative of the reference signal. The differentiation of the error signal is shown in Figure 1. In this figure, one can observe how the estimation process is accurate when the STA is applied to the velocity estimation of the  $x$  coordinate of the quadrotor system.

The simulation results for the tracking performance in a three dimensional space showed the better performance of the controller proposed in this study. This figure is useful to demonstrate the convergence of all the quadrotor

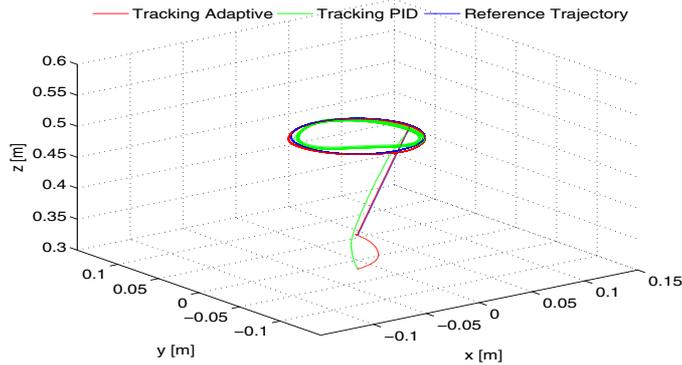


Fig. 2. Trajectory tracking in a three dimensional space of a quadrotor system.

states. These states track the reference trajectories formulated as result of a preliminary study. Moreover, the fast convergence obtained of the tracking error ensures the solution of regulation for the quadrotor path tracking. The previous study served to construct a reference trajectory that corresponds to a circular trajectory on a predefined zeta coordinate.

## 7. CONCLUSION

An adaptive output based controller based on the proportional derivative controller was implemented to regulate the path tracking process of a quadrotor robot. The controller used the velocity information estimated by a RED implementing the STA. The controller used a class of simple indirect control algorithm. The closed loop controller forced the ultimate boundedness of the tracking errors to a region around the origin. This condition was proved using a special class of Lyapunov function that was adequate to produce the adaptive gains of the PD controller as well as the convergence of the STA used as RED. The adjustable gains are quadratic functions of the tracking errors. The reference trajectories were generated as result of predefined flying (path tracking) sequence. The controller was successfully implemented to force the quadrotor to track a circular trajectory.

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