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# Non-holonomic interpolation motion planning for the car with trailers

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**Abstract:** In this paper we present new results on the path planning problem in the case study of the car with trailers. We formulate the problem in the framework of optimal nonholonomic interpolation and we use standard techniques of nonlinear optimal control theory for deriving hyperelliptic signals as controls for driving the system in an optimal way. The hyperelliptic curves contains as many loops as the number of nonzero Lie brackets generated by the system. We compare the hyperelliptic signals with the ordinary Lissajous-like signals that appear in the literature, we conclude that the former have better performance.

*Keywords:* Optimal control, non-holonomic constraints, sinusoidal and hyperelliptic signals.

## 1. INTRODUCTION

The motion planning problem (MPP) is inspired in situations on which kinematical restrictions are complied at the same time that obstacle avoidance and collision-free navigation are unavoidable issues. A common wisdom suggest a general strategy that starts by proposing a reference trajectory that fulfils the geographic-topological constraints of the workplace, and then use it as Ariadne thread for an approximating procedure by means of paths that satisfy admissibility with respect to the kinematic model of the system, see for instance Li and Canny (1993)

The MPP consists, roughly speaking, in finding a collision-free admissible path for a non-linear control system, that steers the system from an initial state to a goal state; in some cases it might be requested the optimization of a cost functional. The MPP is usually formulated through a fixed controllable control system, together with an arbitrary non-admissible but feasible (collision-free) trajectory, determined, by computational geometric methods such as Vöronoi diagrams or piano movers like strategies, see for instance Berg et al. (2000). The MPP reduces then to the design of control strategies for approximate the reference curve by means of admissible curves within appropriate tubular neighborhoods.

There is an abundant literature in the topic of the MPP and a complete bibliographical recount goes beyond the limits of this paper. To our viewpoint the literature can be separated in two different approaches: the one that proposes methods for motion planning based upon *typical* input signals such as constant controls, polynomial controls, trigonometric controls, etc., for instance Lafferriere and Sussmann (1991), and Tilbury et al. (1995); and the one that pursues the formalization of the motion planning problem through the concepts of complexity, entropy and nilpotent approximations, for instance Jean (2001), Gauthier and Zakalyukin (2005) and Gauthier et al. (2010) just for mentioning some.

The MPP for a car with trailers and the so-called parallel parking problem can be formulate as an optimal control problem given by the energy of admissible trajectories of a driftless control system defined by the so-called Goursat distribution.

In Gauthier and Zakalyukin (2005) the *generic* MPP that involves Lie brackets of order less or equal to three is discussed; by contrast, in the present paper we consider the highly *nongeneric* case of the Goursat flag, but for Lie brackets of any order.

For Lie brackets of order $1, 2$ and $3$ for the generic cases with flags $(2, 3)$, $(2, 3, 4)$, and $(2, 3, 5, 6)$, the projection of the motion along the non-admissible curve on the plane of the distribution, correspond to a circle, a periodic elastica (see Love (1944)), and a closed three loops universal hyperelliptic curve, respectively. The kinematic example associated to the later corresponds to the MPP of *the ball with a trailer* discussed in Boizot and Gauthier (2013).

It is natural to conjecture that the series depicted by closed multi loops curves persists, our intuition points out in that direction and we show in this paper that, at least for Goursat structures, this series continues whatever the number of Lie brackets.

For the system of a car with trailers, and in accordance with the aforementioned literature, two points of view for the MPP become apparent:

(1) The one that puts the Goursat system into the *chained form* and then considers sinusoids for the MPP.

(2) The one that reduces the problem to the successive *nilpotent approximations* of the system along the non-admissible curve to approximate.

For the car with trailers, the nilpotent approximation coincides with the Goursat system, and the system is feedback equivalent to its nilpotent approximation.

The organization of the paper is as follows: in section 2 we present the model of the system of the car with trailers and formulate the control system by means of a normal form of the Goursat distribution, we also describe the problem of parallel parking which is the prototype of nonholonomic MPP. In section 3 we present general results about the MPP, along with the concepts of non-holonomic nilpotent approximation and entropy for the Goursat case. In section 4 we apply Pontryagin Maximum Principle for deriving necessary conditions for extremals and optimal trajectories, we provide a general description of optimal trajectories, and carry out some numerical experiments that allow us to compare our method with the one that proposes sinusoidal signals.

## 2. THE CAR WITH TRAILERS SYSTEM

The car with trailers is a non-holonomic system defined by a car-like robot pulling certain number of trailers, the system is non-holonomic due to the restriction of both the car and the trailers on rolling without slipping. The system has two degrees of freedom and can be written as follows:

$$\begin{aligned}
\dot{\mathrm{x}} &= \cos\theta_N v_N \\
\dot{\mathrm{y}} &= \sin\theta_N v_N \\
\dot{\theta}_i &= \frac{1}{d_i}(\sin\theta_{i-1} - \theta_i)v_{i-1}, \quad i = 1,\dots,N \\
\dot{\theta}_0 &= \omega,
\end{aligned}$$

where (x,y) is the center of the rear trailer, $\theta_i$ and $v_i$ are the orientation and the tangential velocity of the trailer $i$, and $d_i$ is the distance from the wheels of the trailer $i$ to the trailer $i-1$. The orientation, tangential velocity and angular velocity of the pulling car are given by $\theta_0, v_0$ and $\omega$ respectively, see figure 1.

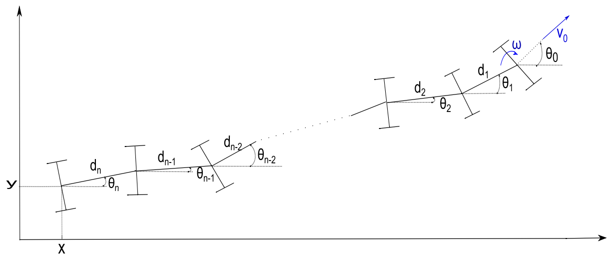


Fig. 1. The car with $N$ trailers.

This system is nonlinear with two controls $v_0$ and $\omega$. It is been shown in Sordalen (1993) that the system can be put in the so-called *chained form*:

$$\begin{aligned}
\dot{\xi}_1 &= u_1 \\
\dot{\xi}_2 &= u_2 \\
\dot{\xi}_j &= \xi_{j-1}u_1, \quad j = 3,\dots,N+3,
\end{aligned}$$

where $\xi_1 = \mathrm{x}$ and $\xi_{N+3} = \mathrm{y}$. By taking coordinates $\xi = (\xi_1,\dots,\xi_{N+3})$, and by considering the vector fields

$$Y_1 = \frac{\partial}{\partial\xi_1} + \sum_{j=2}^{N+2}\xi_j\frac{\partial}{\partial\xi_{j+1}} \quad \text{and} \quad Y_2 = \frac{\partial}{\partial\xi_2},$$

the chained form can be written as the control system

$$\dot{\xi} = u_1Y_1(\xi) + u_2Y_2(\xi), \tag{1}$$

and the only nonzero Lie brackets generated by $Y_1$ and $Y_2$ are the following:

$$Y_j := [Y_{j-1}, Y_1], \quad j = 3,\dots,N+2. \tag{2}$$

It is been shown in Anzaldo-Meneses and Monroy-Pérez (2003) that a further change of coordinates takes the chained form into the following normal form

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_i &= \frac{x_1^{i-2}}{(i-2)!}u_2, \quad i = 3,\dots,n,
\end{aligned}$$

with $x_1 = \mathrm{x}$ and $x_n = \mathrm{y}$.

Again, taking the coordinates $g = (x_1,\dots,x_n)$ and the vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad \text{and} \quad X_2 = \sum_{i=2}^{n}\frac{x_1^{i-2}}{(i-2)!}\frac{\partial}{\partial x_i}, \tag{3}$$

the last control system can be written as follows

$$\dot{g} = u_1X_1(g) + u_2X_2(g). \tag{4}$$

In this case the only non-vanishing Lie brackets are the following

$$X_i = [X_1, X_{i-1}], \ i = 3,\dots,n. \tag{5}$$

In this paper we describe the general MPP keeping in mind the particular problem of the *parallel parking*, consisting in approximation of the non-admissible trajectory given by a perpendicular to the line that results when the car and the trailers are aligned, see figure 2. Roughly speaking is a slipping by means of non-slipping. It turns out that this situation is tantamount of moving the system in the direction of the last non-zero Lie bracket.

For this problem in Tilbury et al. (1995) out of phase sinusoidal inputs of the form

$$u_1 = \alpha\sin\,\omega t, \ u_2 = \beta\cos\,(m+1)\omega t, \quad m \in \mathbb{N}, \tag{6}$$

over one period $T = 2\pi/\omega$ are proposed for this MPP. We shall derive another kind of signals that are obtained intrinsically from the system.

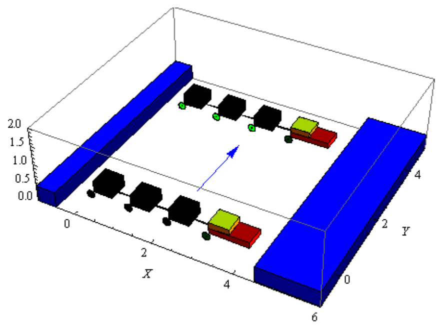


Fig. 2. Motion planning problem for parallel parking.

### 2.1 Goursat structures

To any distribution of vector fields $\Delta$, the following two flags of distributions

$$\begin{aligned} \Delta_i: &\quad \Delta_0 = \Delta,\ \Delta_{i+1} = [\Delta, \Delta_i], \\ D_i: &\quad D_0 = \Delta,\ D_{i+1} = [D_i, D_i] \end{aligned}$$

are well defined.

A bracket generating distribution $\Delta$ is said to be *Goursat* if satisfies the property that all the distributions $D_i, \Delta_i$ have constant rank $r_i$ with $r_0 = 2, r_1 = 3, r_{i+1} = r_i + 1$ as long as $r_i \leq n$. Clearly relations (2) and (5) imply that both $\{Y_1, Y_2\}$ and $\{X_1, X_2\}$ are Goursat distributions

Goursat distributions have been in the literature for long, they were introduced in von Weber (1898) at the beginning of the last century. This class of distributions has been incorporated into the geometric control theory literature, partly because it provides good models for the study of certain kinematic systems such as the one of the car with trailers.

*Proposition 1.* (von Weber (1898)) The dimension $n$ being fixed, there is a single Goursat distribution, up to a (local) diffeomeorphism, generated by the two vector fields (3).

This result together with the generating relations (5) endows $\mathbb{R}^n$ with the structure of $(n-1)-$step nilpotent Lie algebra that we shall denote by $\mathfrak{g}$. The exponential mapping yields the corresponding $(n-1)-$step nilpotent simply connected Lie group that shall be denoted by $G$.

## 3. THE GOURSAT MOTION PLANNING

A rank-2 subriemannian metric over a manifold $M$ is a pair $(\Delta, g)$ where $\Delta$ is a 2-dimensional vector-distribution on $M$, and $g$ is a Riemannian metric over $\Delta$. Equivalently, the metric is specified by the following control system:

$$(\Sigma) \quad \dot{x} = F_1(x)u_1 + F_2(x)u_2, \quad x \in M \tag{7}$$

in such a way that the vector fields $F_1$ and $F_2$ form an orthonormal frame for $g$. Geodesic curves (length minimizing curves) are those which minimize the functional

$$C_1(u) = \int_0^T \sqrt{(u_1(t))^2 + (u_2(t))^2}\, dt, \tag{8}$$

in free time.

The *interpolation entropy* of a path $s \mapsto \Gamma(s)$ transversal to the distribution $\Delta$ is defined as follows:

> For any $\varepsilon > 0$ consider $\ell(\varepsilon)$ the minimum subriemannian length of a $\Sigma-$admissible curve that interpolates $\Gamma$ by means of pieces of sub-riemannian length $\leq \varepsilon$, the function $\ell(\varepsilon)$ tends to infinity when $\varepsilon$ tends to zero. The $\varepsilon-$*interpolation entropy* of $\Gamma$, denoted as $E_\Gamma^\Sigma(\varepsilon)$, is the leading term of $\varepsilon^{-1}\ell(\varepsilon)$, (modulo the equivalence relation $\ell_1(\varepsilon) \approx \ell_2(\varepsilon)$ if and only if $\lim_{\varepsilon\to 0} \frac{\ell_1(\varepsilon)}{\ell_2(\varepsilon)} = 1$).

For a generic pair $(\Sigma, \Gamma)$ with $\Delta$ a $p$-step bracket generating distribution and $\Gamma$ a transversal path to $\Delta$, the entropy has an expression of the form $E_\Gamma^\Sigma(\varepsilon) = \frac{a}{\varepsilon^p}$. However this is true only in the absence of codimension 1 generic singularities, in which case one has $E_\Gamma^\Sigma(\varepsilon) = -\ln(\varepsilon)\frac{a}{\varepsilon^p}$. The generic expression of the constants $a$ has been exhausted for small values of the ranks and co-ranks. For details we refer the reader to Gauthier and Zakalyukin (2005).

Given a generic pair $(\Sigma, \Gamma)$, the behavior of the system along $\Gamma$ is dominated by the *nilpotent approximation of the system along* $\Gamma$, roughly speaking it can be viewed as a one parameter family of nilpotent approximations of the system at the points of $\Gamma$ see Bellaiche (1997). Our concept of nilpotent approximation along a curve relies on *normal coordinates* which are the subriemannian analog of the normal coordinates in Riemannian geometry. In normal coordinates, the curve $\Gamma$ is rectified to become a *vertical* line given by the last coordinate, whereas the distribution along $\Gamma$ is realized by the *horizontal* plane given by the first two coordinates.

### 3.1 The Goursat case

*A Goursat* MPP is a triple $\mathcal{G} = (\Delta, g, \Gamma)$ where $(\Delta, g)$ is a subriemannian structure, $\Delta$ is a Goursat distribution and $\Gamma$ is a curve transversal to $\Delta$. A one parameter family of admissible curves $\gamma_\varepsilon$ realizing the interpolation entropy of the curve $\Gamma$ is the optimal way to approximate $\Gamma$ by $\varepsilon$-close admissible curves.

### 3.2 Nilpotent approximation of $\mathcal{G}$ along $\Gamma$ and entropy.

The data $\mathcal{G} = (\Delta, g, \Gamma)$ is given, with $\Delta = \text{span}\{F_1, F_2\}$ a Goursat distribution and $\Gamma$ a smooth curve tranversal to $\Delta$. The vector fields $F_1, F_2$ define the control system (7), that we shall write simply as $\Sigma = (F1, F2)$.

$\Sigma$ is feedback equivalent to the normal form (3) therefore there exist functions $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$, and local coordinates $x = (x_1, \ldots, x_n)$, such that

$$\begin{aligned} F_1(x) &= \alpha(x)X_1(x) + \beta(x)X_2(x), \\ F_2(x) &= \gamma(x)X_1(x) + \delta(x)X_2(x). \end{aligned}$$

Generically the curve $s \mapsto \Gamma(s) = (\gamma_1(s), \ldots, \gamma_n(s))$ is transversal to $\Delta_{n-3} = D_{n-3}$, therefore $\gamma_n'(s) \neq 0$ and we can make the following change of coordinates:

$$\tilde{x}_i = x_i - \gamma_i \circ \gamma_n^{-1}(x_n), \quad \text{for} \quad i = 1, \ldots, n-1, \quad \text{and}$$
$$\tilde{x}_n = x_n,$$

in these new coordinates the curve $\Gamma$ is *rectified* to become the *vertical* line $\widetilde{\Gamma}(s) = (0, \ldots, 0, \gamma_n(s))$. In what follows we shall omit the tilde symbol in both the coordinates and the curve.

For an arbitrary but fixed point on the curve $\Gamma$, the coordinates $x_1, \ldots, x_{n-1}$ are centered at zero but $x_n$ is not. Assuming that the last coordinate is small, we consider the gradation in the formal power series in the variables $x_1, \ldots, x_{n-1}, x_n - \gamma_n(s)$ obtained by assigning weight 1 to both $x_1$ and $x_2$, weight 2 to $x_3$, weight 3 to $x_4, \ldots$, weight $(n-2)$ to $x_{n-1}$ and weight $(n-1)$ to $x_n - \gamma_n(s)$. This gradation induces a gradation in the formal vector fields in such way that both $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ have weight $-1$, $\frac{\partial}{\partial x_3}$ has weight $-2, \ldots, \frac{\partial}{\partial x_{n-1}}$ has weight $-(n-2)$ and $\frac{\partial}{\partial x_n}$ has weight $-(n-1)$. This gradation is homogeneous with respect to the Lie bracket operation along the curve $\Gamma$.

We define the *Goursat nilpotent approximation* $\widehat{\Sigma} = (\widehat{F_1}, \widehat{F_2})$ of $\Sigma$ along $\Gamma$ to be the homogeneous term of order $-1$ of the gradation, that is,

$$\begin{aligned} F_1 &= \alpha(\Gamma(s))X_1(x_1) + \beta(\Gamma(s))X_2(x_1) + O^0 \\ &= \widehat{F}_1\Big|_{x=\Gamma(s)} + O^0, \\ F_2 &= \gamma(\Gamma(s))X_1(x_1) + \delta(\Gamma(s))X_2(x_1) + O^0 \\ &= \widehat{F}_2\Big|_{x=\Gamma(s)} + O^0. \end{aligned}$$

Following estimates similar to the ones in Gauthier and Zakalyukin (2005) we obtain

$$E_{\Gamma}^{\widehat{\Sigma}}(\varepsilon) \approx E_{\Gamma}^{\Sigma}(\varepsilon) \tag{9}$$

We have the following general result:

*Theorem 2.* For $\widehat{\Sigma}$ a length one extremal of the interpolation problem between the origin and a point of the $x_n$ axis, maximizing the endpoint coordinate $x_n$ can be explicitly calculated. Its projection on the plane $(x_1, x_2)$ is a closed hyperelliptic curve, smooth-periodic, with $n-2$ loops. The curve that interpolates with length $\varepsilon$ is obtained from this one by homogeneity.

*Remark 3.*

Given a Goursat MPP $\mathcal{G} = (\Delta, g, \Gamma)$ first it is put via feedback and change of coordinates under the canonical form (3). After that, one chooses to apply the interpolation entropy strategy that provides an exact $\varepsilon$-interpolation control strategy, but with a non-natural cost (due to the preliminary feedback). This is equivalent, after feedback, to solve the problem of finding admissible $\varepsilon$-interpolating curves that have an arbitrary but fixed subriemannian length given by (8), and that maximize the distance on $\Gamma$ between two successive interpolated points. Under this strategy the actual size of $\varepsilon$ is irrelevant.

## 4. HYPERELLIPTIC SIGNALS FOR $\mathcal{G} = (\Delta, G, \Gamma)$

What remains to be done is to find the explicit equations for the extremals and its corresponding projections. The key point for this is the fact that the Hamiltonian system of geodesic equations is Liouville integrable whatever the dimension $n$.

At the level of the nilpotent approximation we deal with the optimal control problem on $G$ that consists in finding among the admissible trajectories of the system

$$\dot{g} = u_1(\alpha X_1(g) + \beta X_2(g)) + u_2(\gamma X_1(g) + \delta X_2(g)), \tag{10}$$

with $\alpha\delta - \beta\gamma \neq 0$, the one that minimizes

$$\int (u_1(t)^2 + u_2(t)^2)\ dt \tag{11}$$

Furthermore, as explained before, we can choose coordinates along $\Gamma$ and orthonormal vector fields $F_1, F_2$ generating $\Delta$, such that the nilpotent approximation along $\Gamma$ writes

$$\dot{g} = u_1 X_1(g) + u_2 X_2(g), \tag{12}$$

and $\Gamma(s) = (0, \ldots, 0, \varphi(s))$ for some smooth function $\varphi(s)$.

### *4.1 Application of the Pontryagin Maximum Principle.*

We have the optimal control problem on $G$ consisting of the minimization of (11) among the admissible trajectories of (12). The Pontryagin Maximum Principle provides a standard geometric tool for the description of extremals by establishing necessary conditions for optimality, for details we refer the reader to Agrachev and Sachkov (2004).

If $p$ denotes the dual variable in $\mathfrak{g}^*$, then for each vector field $X_i$ we have the corresponding Hamiltonian $H_i = \langle p, X_i \rangle, i = 1, \ldots n$ with Poisson brackets satisfying commuting relations dual to those of (5), that is,

$$H_i = \{H_1, H_{i-1}\}, \quad i = 3, \ldots n. \tag{13}$$

Maximality condition of the Pontryagin Maximum Principle yields

$$u_1 = H_1, \quad \text{and} \quad u_2 = H_2,$$

and the system Hamiltonian becomes quadratic

$$\mathcal{H} = \frac{1}{2}(H_1^2 + H_2^2),$$

the associated adjoint equations are obtained by differentiating along the extremal as customary: $\dot{H}_i = \{H_i, \mathcal{H}\}$. In consequence, the commuting relations (13) clearly yield

$$\dot{H}_1 = H_2 H_3 \tag{14}$$

$$\dot{H}_i = -H_1 H_{i+1}, i = 1, \ldots, n-1 \tag{15}$$

$$\dot{H}_n = 0 \tag{16}$$

Therefore $H_n$ is constant along extremals and shall be denoted $c_1 := H_n = H_n(0)$.

Following the iterative integration process in Anzaldo-Meneses and Monroy-Pérez (2003) a sufficient number of constant of motion can be found for integrating the adjoint system.

*Proposition 4.* The adjoint system given by equations (14) to (16) is Liouville integrable and the extremal curves lie in the intersection of the energy cylinder $\mathcal{H} = \frac{1}{2}$ and the cylinder with generatrix

$$\frac{1}{k!} H_{n-1}^k - c_1^{k-1} H_2 - \sum_{j=2}^{k} \frac{c_j}{(k-j)!} H_{n-1}^{k-j} = 0 \qquad (17)$$

This fact is very useful and it is illustrated in the figure (3), where the intersection of these two cylinders is shown.

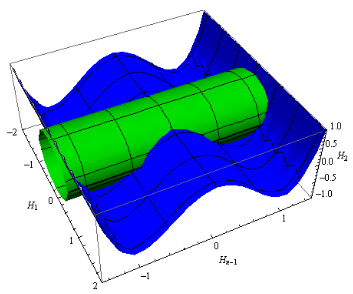


Fig. 3. Intersection of the cylinders $\mathcal{H} = \frac{1}{2}$ and (17)

*4.2 General description of the optimal trajectories*

We write now the dimension as $n = 2 + k$, and the coordinates as $(x, y, \eta_1, \ldots, \eta_k)$. The corank $k$ distribution is given by the normal form (3), that in these coordinates writes as follows:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + \sum_{i=1}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \eta_i}.$$

The Pontryagin maximum principle applies in the same lines as before, to obtain that along extremals one has

$$\dot{x} = H_1, \quad \dot{y} = H_2 \quad \text{and} \quad \dot{\eta}_j = H_2 \frac{x^j}{j!}$$

*Lemma 5.* $H_{n-1}$ is a linear function of $x$. Moreover $x$ and $\dot{y}$ are periodic functions of time.

**Proof.** In fact equation (15) with $i = n-1$ writes

$$\dot{H}_{n-1} = -c_1 H_1 = -c_1 \dot{x}.$$

In consequence $H_{n-1} = H_{n-1}(0) - c_1(x - x(0))$, as claimed.

Without loss of generality we can assume that $c_1 = H_n = H_n(0) = -1$ and that $H_{n-1}(0) = x(0) = 0$, in such a way that $H_{n-1}$ may be identified with $x$. Furthermore, assuming that $c_1 = -1$ we have

$$H_3 = -\frac{1}{(k-1)!} x^{k-1} + \sum_{j=2}^{k-1} \frac{c_j}{(k-1-j)!} x^{k-1-j} =: p_{k-1}(x),$$

$$H_2 = -\frac{1}{k!} x^k + \sum_{j=2}^{k} \frac{c_j}{(k-j)!} x^{k-j} =: p_k(x).$$

A further derivation of equation (15) with $i = n-1$, together (14) yields $\ddot{x} = \dot{H}_1 = H_2 H_3 = p_k(x) p_{k-1}(x)$, and as a consequence $x$ can be explicitly integrated by inverting the corresponding hyperelliptic integral. In conclusion the optimal trajectories are given as follows

$$\dot{x} = \sqrt{1 - p_k^2(x)}, \qquad (18)$$
$$\dot{y} = p_k(x), \quad \text{and} \qquad (19)$$
$$\dot{\eta}_j = \frac{x^j}{j!} p_k(x), \quad \text{for} \quad j = 1, \ldots k. \qquad (20)$$

*4.3 Explicit calculation of the hyperelliptic curves.*

We consider the three-dimensional space with coordinates $(x, u_1, z) = (x, u_1, u_2) = (H_{n-1}, H_1, H_2)$ and the cylinders: $\mathcal{C}_1 = \{(x, u_1, z) \mid u_1^2 + z^2 = 1\}$ and $\mathcal{C}_2 = \{(x, u_1, z) \mid z = p_k(x)\}$. We can assume that the $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ and by choosing the initial conditions properly, we can assume that this intersection is a smooth, connected, closed and simple curve. We denote by $\mathcal{C}$ the parametrized curve that is the projection of $\mathcal{C}_1 \cap \mathcal{C}_2$ to the plane $\{(x, u_1)\}$, by taking equation (18) into consideration we have:

$$\mathcal{C} = \{(x, u_1) \mid u_1 = (1 - p_k^2(x))^{\frac{1}{2}}\}.$$

This curve is smooth, closed and simple and has both, vertical and horizontal symmetries, and it can be assumed that it is centered at the origin, see figure (3).

Let $\mathcal{E}$ be the corresponding arc-length parametrized extremal curve in the plane $\{(x, y)\}$ taking equation (19) into account we have:

$$\mathcal{E} = \{(x, y) \mid \dot{y} = p_k(x)\}.$$

We assume $\mathcal{E}$ to be centered at the origin, figure 4 illustrates curves $\mathcal{C}$ and $\mathcal{E}$, for Lie brackets of length 3 and 4. Observe that the curve $(x, y)$ has as many loops as the number of non-zero Lie brackets.

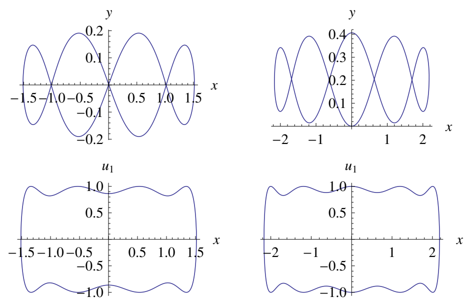


Fig. 4. Curves $\mathcal{C}$ and $\mathcal{E}$

For depicting the curve $\mathcal{E}$ we have the freedom of choosing the coefficients of the polynomial $p_k(x)$ and the initial condition $y(0)$. Also we have to take into consideration the following *interpolation conditions* (which are independent of any translation of coordinates):

(1) The coordinate $y$ is periodic, $y(1) = y(0)$.
(2) The *moments*

$$m_i = \int_{\mathcal{E}} x^i dy$$

are all zero for $i = 1, \ldots, n-3$, which corresponds to the fact that the coordinates $\eta_1, \ldots, \eta_{n-3}$, given by (19) are all periodic. Observe that the first moment $m_1$ is the area swept out by the curve, whereas the last moment $m_{n-2}$ is not only non vanishing but also the one to be maximized.

The curves $\mathcal{E}$ are symmetric with respect to the $x$-axis but we do not know if they are symmetric with respect to the $y$-axis, however we make this assumption, a priori reasonable. Under these symmetry considerations and depending on the parity of $n$, (which is the same parity of $k$) certain moments are automatically zero and the description of $\mathcal{E}$ can always be completed.

- If $n$ is even, the odd moments are zero. The polynomial $p_k(x)$ has even degree and by the symmetry considerations, it has no terms of odd degree. Then, if we chose a monic polynomial, it remains $\frac{k}{2}$ free coefficients, that have to be used to vanish $\frac{k}{2} - 1$ moments (plus the zero-moment $y$).
- if $n$ is odd, the even moments are zero. The polynomial $p_k(x)$ has odd degree and by the symmetry considerations, it has no terms of even degree. Then, if we chose a monic polynomial, it remains $\frac{k-1}{2}$ free coefficients, that have to be used to vanish $\frac{k-1}{2}$ moments and the value $y(0)$ to make $y$ periodic, which can be done as an independent (trivial) step.

From this analysis we can conclude that, at the end we have as many free parameters (plus one that accounts for the initial condition $y(0)$) as the number of moments that have to vanish.

### *4.4 Numerical experiments*

We consider the car with 3 trailers (N=6) with $d_1 = d_2 = d_3 = 1$, the interpolation conditions allow to find the coefficients of $p_6(x)$ and to calculate the hyperelliptic inputs. We carried out the comparation of the two methods by fixing first the subriemannian length, and using the sinusoidal inputs : $u_1(t) = \sin(\omega t)$ and $u_2(t) = \cos(4\omega t)$.

In the figure 5, the trajectories in red correspond to the last trailer with hyperelliptic inputs whereas the ones in blue correspond to trajectories of the last trailer with sinusoidal inputs. We can see that, for the same subriemannian length, our inputs are better compared to the corresponding displacements in the direction of $y$, as illustrated in table 1.

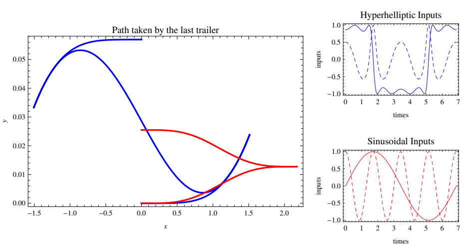


Fig. 5. Path and inputs

## 5. CONCLUSIONS AND PERSPECTIVES

Utilizing nilpotent approximations and geometric optimal control techniques, hyperelliptic signal are intrinsically derived for the Goursat system. In the case of a car with trailers, such signals present better performance that the classical sinusoidal ones. It is expected that similar behavior shall prevail in systems with higher nilpotency order.

Table 1. Sinusoids vs. Hyperelliptic

| Length | Displacement along $y$ with Sinusoidal inputs | Displacement along $y$ with Hyperelliptic inputs |
|---|---|---|
| 6.865 | 0.03100 | 0.05690 |
| 5.492 | 0.01010 | 0.01860 |
| 4.805 | 0.00440 | 0.00945 |
| 4.119 | 0.00241 | 0.00443 |

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