Disturbance Decoupling in the Monovariable Case by Static Output Feedback.

R. Casas González*,B. del Muro Cuéllar* and J.F. Marquez Rubio*.
*Escuela Superior de Ingeniería Mecánica y Eléctrica, Unidad
Culhuacan, Instituto Politécnico Nacional, Santa Ana 1000, Coyoacán, D.F., 04430, México,
rcasasg0300@hotmail.com,bdelmuro@yahoo.com,jfcomr23@yahoo.com.mx

***Abstract*— It is well known that when a feedback control law is chosen to solve a particular problem, usually there exist some fixed poles in the closed-loop system, i.e. closed-loop fixed poles that do not depend on the choice of the control law but precisely on the fact that this particular problem is being solved. In this paper we present new results concerning the disturbance decoupling by static output feedback.**

Keywords: Disturbance Decoupling; Geometric Aproach; Fixed poles; Pole Placement.

I. Introduction

Since the geometric theory of linear multivariable systems appeared (60-70's), a large number of control problems has been solved using this tool, such as disturbance rejection, input-output decoupling, regulation and failure detection problems (see for instance (Wonham W.M., 1993) and (Basile G. and Marro G., 1992)) This paper deals with the disturbance decoupling problem and their solution using the fixed poles characterization. This approach is very simple to understand and to implement using control software like Matlab.

Exact disturbance decoupling, as a control objective, is a common task in control engineering. In fact, disturbance decoupling is one of the most familiar problems in control theory for which many contributions have already been brought (e.g. (Eldem V. and Ozguler A.B., 1988), (Schumacher J.M., 1980) and (Willems J.C. and Commault C., 1981)). On the other hand, pole placement is a common control strategy (see for instance (T. Kailath, 1980)) for achieving specific closed-loop performances. It is then natural to ask about the connection between exact disturbance decoupling and pole placement. In addition, it is well known that when a feedback control law is chosen in order to solve a particular problem, usually exist some fixed poles in the closed-loop system, i.e. closed-loop fixed poles that do not depend on the choice of the control law but precisely on the fact that this particular problem is being solved. As far as the disturbance decoupling is concerned, the fixed poles have been characterized in geometric and algebraic terms. See for instance (Malabre M., Martínez G.J.C. and Del Muro C.B., 1997) and (Koussiouris T.G. and Tzierakis K.G., 1996), for the static state feedback case, and (Del Muro C.B. and Malabre M., 2001) or (Eldem V. and Ozguler A.B., 1988) for a characterization of the fixed poles when applying dynamic output feedback.

As a consequence of the research concerning disturbance decoupling, fixed poles, and pole placement, it is clear that disturbance decoupling can be achieved through pole placement, under certain structural constraints.

In this paper we present a new results concerning the disturbance decoupling problem in terms of pole placement using a static output feedback in the monovariable case. We discuss too, the static state feedback for monovariable systems, which is a methodology previously developed in (Del Muro C.B. and Martínez G.J.C., 2000), being the base to design a static output feedback achieving disturbance decoupling.

The main interest of this new approach is their simplicity: even if the results are based in geometric tools, we don't need to compute any geometric subspace to use the results here presented.

The paper is organized as follows. In Section 2 we revisit the main results concerning the disturbance decoupling by static output feedback. In particular, we recall the characterization of the fixed poles for this problem.

In Section 3 we present necessary and sufficient solvability conditions for the existence of a solution for the disturbance decoupling problem by state feedback and static output feedback. We present a methodology to decouple disturbance (when possible) via static state feedback and static output feedback which is the main result, under some controllability or observability assumptions. This methodology is based in the *a priory* computation of the "possible" Disturbance Decoupling fixed poles. In Section 4 we present illustrative examples and we conclude in Section 5 with some final comments.

II. The disturbance decoupling problem

Consider the linear time-invariant system (A, B, C, D, E) described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ y(t) = Cx(t) \\ z(t) = Ex(t) \end{cases} \tag{1}$$

where:

- $x(\cdot) \in \mathcal{X} \approx \mathbb{R}^n$ denotes the state; $u(\cdot) \in \mathcal{U} \approx \mathbb{R}^m$ the control input; $h(\cdot) \in \mathcal{H} \approx \mathbb{R}^q$ the disturbance

input; $y(\cdot) \in \mathcal{Y} \approx \mathbb{R}^p$ the measurable output; and $z(\cdot) \in \mathcal{Z} \approx \mathbb{R}^r$ the output to be controlled.
- $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $D : \mathcal{H} \to \mathcal{X}$, $E : \mathcal{X} \to \mathcal{Z}$, and $C : \mathcal{X} \to \mathcal{Y}$ are linear applications represented in particular basis by real constant matrices.
- Let us note $\sigma(M)$ the roots of the equation $\det(sI - M) = 0$, for any state matriz M.

The Disturbance Decoupling by (static) Output Feedback (DDOF) problem is defined as follows:

Definition 1: Disturbance Decoupling problem: by static Output Feedback (DDOF): find a static measurement feedback:

$$u(t) = -ky(t),$$

such that:

$$E\left(sI_n - (A - BkC)\right)^{-1} D \equiv 0.$$

When $y(\cdot) = x(\cdot)$, i.e. $C = I_n$, we have the Disturbance Decoupling by (static) State Feedback Problem (DDSF).

As far as DDSF problem is concerned, we shall not use in the sequel k but F, i.e.,when $C = I_n$, $(A + BkC) = (A + BF)$.

A. The Fixed Poles of the DDSF problem

If the DDSF problem is solvable, then exist a familly of feedback $u(t) = Fx(t)$, to decouple the disturbance. Using this family of solutions, some poles can be freely placed in the feedback system except for a set of fixed poles that are always present in any feedback solution. The following result characterizes the only set of poles that is present in any feedback system solution to the DDSF problem, independently the way the solution is obtained, i.e., the DDSF fixed poles.

Theorem 1: (Malabre M., Martínez G.J.C. and Del Muro C.B., 1997) Given the disturbed system (A, B, C, D, E), under the assumption (A, B) controllable. Let us consider that the DDSF problem is solvable, i.e. that there exist a map F such that $E\left(sI_n - (A - BF)\right)^{-1} D \equiv 0$. Then

$$\sigma(A - BF) \supset \sigma_{DDSF}$$

$$\sigma_{DDSF} = \mathcal{Z}(A, B, E) - \mathcal{Z}\left(A, \begin{bmatrix} B & | & D \end{bmatrix}, E\right) \quad (2)$$

where a multiple zero appears as many times its multiplicity order and where $\mathcal{Z}(A, [*], E)$ are the so-called *invariant zeros* of $(A, [*], E)$.

Remark 1: The invariant zeros of an $(A, [*], E)$ system can easily be calculated with Matlab using the instruction $tzero(A, [*], E)$, indeed this set is called "transmission zeros" in Matlab.

III. Decoupling Disturbances in the monovariable case: The role of the fixed poles

Since pole placement is a traditional tool to obtain good closed-loop performance, it is natural to ask about the conditions to achieve disturbance decoupling via pole placement. As far as the monovariable case is concerned, pole placement can in fact be applied to achieve disturbance decoupling in a very simple way, under some controllability and/or observability assumptions. The key idea behind this statement is that of *disturbance decoupling fixed poles*.

A. Disturbance Decoupling by Static State Feedback DDSF

The following result (presented in (Del Muro C.B. and Martínez G.J.C., 2000)) give us necessary and sufficient conditions to solve the DDSF problem in a practical and very easy way:

Lemma 1: Consider the disturbed system (A, B, D, E), under the assumption (A, B) controllable and in the single input case($m = 1$). The DDSF problem is solvable if and only if

$$E\left(sI_n - (A - BF)\right)^{-1} D \equiv 0. \quad (3)$$

where F is any state feedback such that $\sigma(A - BF) \supset \sigma_{DDSF} = \mathcal{Z}(A, B, E) - \mathcal{Z}\left(A, \begin{bmatrix} B & | & D \end{bmatrix}, E\right)$.

proof The proof is almost direct from the characterization of the DDOF fixed poles and noting that the feedback F that place the poles in any particular position is unique in the single input case. See details in (Del Muro C.B. and Martínez G.J.C., 2000).■

Note that the system can be multi disturbance-input (i.e., $q \geq 1$) and multi output.

Obviously, the problem can be solved with stability in a similar way.

Summarizing, if (A, B) is controllable and in the single input case ($m = 1$) we have the following methodology to solve DDSF:

1) Obtain the "possible" Fixed Poles of DDSF, i.e. $\sigma_{DDSF} = \mathcal{Z}(A, B, E) - \mathcal{Z}\left(A, \begin{bmatrix} B & | & D \end{bmatrix}, E\right)$.
2) Built the polynomial $p(\sigma_{DDSF})$ which has as its roots the Fixed Poles of DDSF.
3) Built the desired polynomial as $p(\lambda) = p(\sigma_{DDSF}) \cdot p(\sigma_{free})$, freely assigning the roots of $p(\sigma_{free})$.
4) Built the static state feedback F, such that $\det[\lambda I_n - (A - BF)] = p(\lambda)$. Finally, if

$$E\left(sI_n - (A - BF)\right)^{-1} D \equiv 0$$

then F solves the problem. Otherwise, DDSF is not solvable.

Remark 2: Note that it is easy to modify this methodology to obtain necessary and sufficient conditions to solve the DDSF problem with stability, just asking to σ_{DDSF} to be a stable set.

B. Disturbance Decoupling by Static Output Feedback

It is obvious that the static output feedback is a particular case of static state feedback. Indeed, if $u(t) = -ky(t)$ decouple the disturbance, then $u(t) = -Fx(t)$ with $F = kC$, also decouple the disturbance. Thus, a necessary condition to solve the DDOF problem is the solvability of the DDSF problem. In fact, as it is established in the following proposition, a necessary and sufficient condition can be found if we note that the DDSF fixed poles are contained in the set of closed-loop poles corresponding to the DDOF problem.

Lemma 2: Consider the single input single output ($m = p = 1$) disturbed system (A, B, C, D, E), under the assumption (A, B) controllable and (C, A) observable. The DDOF problem is solvable if and only if there exists a real k such that

$$\sigma(A - BkC) \supset \sigma_{DDSF} \tag{4}$$

and

$$E\left(sI_n - (A - BkC)\right)^{-1} D \equiv 0$$

Note that such k is a solution to the DDOF problem.

proof If: Obvious from $E\left(sI_n - (A - BkC)\right)^{-1} D \equiv 0$.
Only if: Consider that the DDOF problem is solvable. Then the DDSF is also solvable with $F = -kC$, and then $E\left(sI_n - (A - BkC)\right)^{-1} D \equiv 0$. Additionally, from the DDSF fixed poles characterization, under the controllability assumption, $\sigma(A - BkC) \supset \sigma_{DDSF}$.■

If (A, B) is controllable or (C,A) observable, and in the single input single output case we have the following methodology to solve DDOF:

1) Obtain the ”possible” Fixed Poles , i.e.: $\sigma_{fix} = \mathcal{Z}(A, B, E) - \mathcal{Z}\left(A, \left[\begin{array}{ccc} B & | & D \end{array}\right], E\right)$.
2) Using the Root Locus technique (see for instance (Ogata K., 1997)) and if the system is (A, B) controllable, search for a static output feedback k such that $\sigma(A - BkC) \supset \sigma_{DDSF}$. If such k exist and

$$E\left(sI_n - (A - BkC)\right)^{-1} D \equiv 0$$

then k solves the problem. Otherwise, DDOF is not solvable.

Remark 3: This methodologie is easily implemented using computational software as Matlab.

IV. Illustrative Examples

In order to illustrate the methodology previously discussed, we proceed in this section to apply it to a particular monovariable linear time-invariant disturbed systems. We will note the associated Matlab instruction.

Example 1
Consider the disturbed linear time-invariant system (A, B, C, D, E) described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ y(t) = Cx(t) \\ z(t) = Ex(t) \end{cases}$$

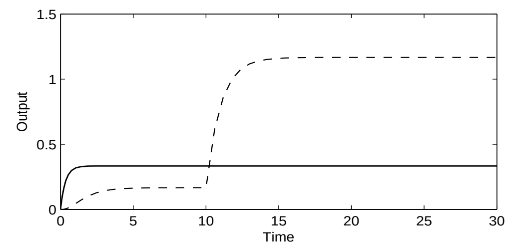

Figure 1. Disturbance Decoupling by Static Output Feedback

with:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 7 & 0 & -3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; E = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

where: $u(t)$ denotes the control input; $h(t)$ the disturbance; $y(t)$ denotes the measurable output and $z(t)$ the output to be controlled.

- Let us first compute the possible DDSF fixed poles σ_{DDSF} : $\mathcal{Z}(A, B, E) = \{-2, -1\}$, $\mathcal{Z}\left(A, \left[\begin{array}{ccc} B & | & D \end{array}\right], E\right) = \{-2\}$, then $\sigma_{fix} = \{-1\}$. (Use the Matlab instruction $tzero(A, B, E, 0)$).
- From the Root Locus technique, $k = 7$ is such that:

$$\sigma(A - BkC) = \{-1, -2, -3\} \supset \sigma_{DDSF} = \{-1\}$$

(You can use the Matlab instructions $G = ss2tf(A, B, E, 0)$, $rlocus(G)$, $rlockfind(G)$).
- And finally (using the Matlab instruction $ss2tf(A - BkC, D, E, 0)$).

$$E\left(sI_n - (A - BkC)\right)^{-1} D = 0.$$

then the DDOF problem is solvable and the closed loop system solution is stable. You can appreciate this in Figure 1, where a step disturbance is applied at $t = 10$ seconds. The decoupled output $z(t)$ is the continuous line and the dashed line corresponds with the measured output.

Example 2
Consider the disturbed linear time-invariant system (A, B, C, D, E) described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ y(t) = Cx(t) \\ z(t) = Ex(t) \end{cases}$$

with:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 7 & 0 & -3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; E = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

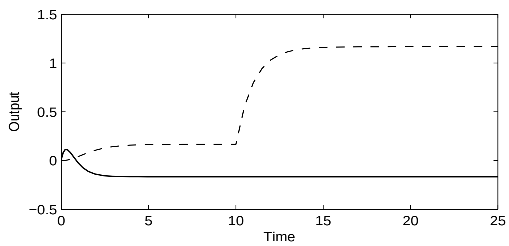

Figure 2. Disturbance Decoupling by Static Output Feedback

where: $u(t)$ denotes the input; $h(t)$ the disturbance; $y(t)$ denotes the measurable output and $z(t)$ the output to be controlled.

- Let us first compute the DDSF fixed poles σ_{DDSF} : $\mathcal{Z}(A,B,E) = \{1,-1\}$, $\mathcal{Z}\left(A, \left[\begin{array}{ccc} B & | & D \end{array}\right], E\right) = \{1\}$, then $\sigma_{fix} = \{-1\}$ (Use the Matlab instruction $tzero(A,B,E,0)$).
- From the Root Locus technique, $k = 7$ is such that:

$$\sigma(A - BkC) = \{-1,-3,1\} \supset \sigma_{DDSF} = \{-1\}$$

(Use the Matlab instructions $G = ss2tf(A,B,E,0)$, $rlocus(G)$, $rlockfind(G)$).
- Cheking transfer function:

$$E\left(sI_n - (A - BkC)\right)^{-1} D = 0.$$

(Use the Matlab instruction $ss2tf(A-BkC,D,E,0)$). Note than the DDOF problem is solvable, but without stability because $\sigma(A - BkC) = \{-1,-3,1\}$
- Now we can solve the previous problem using DDSF seeking stability.
- We can built the desired polynomial as $p(\lambda) = p(\sigma_{fix}) \cdot p(\sigma_{free})$, considering for instance $\sigma_{free} = -2,-3$. Then $p(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3)$.
- We compute F such that $p(A - BF) = (\lambda + 1)(\lambda+2)(\lambda+3)$. In this case we easily get $F = \left[\begin{array}{ccc} 7 & 12 & 3 \end{array}\right]$ (Use the Matlab instruction $F = place(A,B,[-1,-2,-3])$.
- We can verify that for the current example, the DDSF problem is solvable:

$$E\left(sI_n - (A - BF)\right)^{-1} D \equiv 0.$$

(Use the Matlab instruction $ss2tf(A-BF,D,E,0)$).

As can be noted, the DDSF problem is solved with stability because $\sigma(A - BF) = \{-1,-2,-3\}$.You can appreciate this result in Figure 2, where a step disturbance is applied at $t = 10$ seconds. The decoupled output $z(t)$ is the continuous line and the dashed line corresponds with the measured output.

V. FINAL COMMENTS

We have present in this paper new results concerning the disturbance decouple by static output feedback and a methodology to decouple the disturbance (when possible) acting in linear time-invariant systems. Our proposed methodology is based on a pole placement strategy considering the *a priory* computation of the disturbance decoupling fixed poles, under some controllability or observability assumptions. The proposed methodology is a tool to verify the solvability of the problem in a very easy way.

We have also discussed the relationship between disturbance decoupling by static state feedback (Malabre M., Martínez G.J.C. and Del Muro C.B., 1997) and by static output feedback. The solvability conditions here presented ask for the possibility to place some poles by a static output feedback exactly in the positions corresponding to the disturbance decoupling by state feedback fixed poles.

The main interest of the above results is their simplicity. Indeed, as far as the monovariable case is concerned, we do not need any (geometric or structural) sophisticated tool to verify the Disturbance Decoupling problem by static State (or by Output) Feedback solvability; it is enough to compute the position of some invariant zeros and obtain a control law that places the poles in that positions. Moreover, the proposed methodology let us to directly compute a particular solution (if the solvability conditions are verified). Unfortunately, a direct extension to the general MIMO case is not possible because the non uniqueness of the solution in the pole placement by static state feedback.

VI. ACKNOWLEDGMENTS

This work was supported in part by CONACyT-México, Project: 82944 y 61713. And by Institutional Program of Researchers Formation from IPN, Project: 20081289.

REFERENCES

Basile G. and Marro G. (1992). Controlled and Conditioned Invariant Subspaces in Linear System Theory. *Prentice-Hall* New Jersey, USA.

Del Muro C.B. and MartínezG.J.C. (July 2000). Rejecting Disturbance via Pole Placement: The Monovariable Case.*American Control Conference ACC'2000* Chicago, USA.

Del Muro C.B. and Malabre M. (2001). Fixed poles of disturbance rejection by dynamic measurement feddback: a geometric approach. *Automatica (2001)***37(2)**, 231-238.

Eldem V. and Ozguler A.B. (January 1988). Disturbance Decoupling by Measurement Feedback: A Characterization of all Solutions and Fixed Modes. *SIAM J. Contr. and Optimization***26(1)**

T. Kailath (1980). Linear Systems. *Prentice-Hall* New Jersey, USA.

Koussiouris T.G. and Tzierakis K.G. (1996). Frequency-domain Conditions for Disturbance Rejection and Decoupling with Stability or Pole Placement.*Automatica***32(2)**, 229-234.

Malabre M., Martínez G.J.C. and Del Muro C.B. (1997). On the fixed poles for Disturbance Rejection. *Automatica***33(6)**, 1209-1211.

Ogata K. (1997).Modern Control Engineering *Third Edition,Prentice Hall*

Schumacher J.M. (December 1980). Compensator Synthesis using (C, A, B)-Pairs. *IEEE Transactions on Automat. Contr***AC-25(6)**, 1133-1138.

Willems J.C. and Commault C. (July 1981). Disturbance Decoupling by Measurement Feedback with Stability or Pole Placement. *SIAM J. Contr. and Optimization***19(4)**, 409-504.

Wonham W.M. (1993). Linear Multivariable Control: A Geometric Approach *Third Edition, Springer Verlag*,New York, USA