Adaptive state estimation for uncertain discrete nonlinear systems based on recurrent neural algorithm

I. Salgado
Instituto de Ingeniería-UNAM
Coyoacan DF 04510, México
verde@servidor.unam.mx
Teléfono: (52)-55-56233684

I. Chairez
UPIBI-IPN
G. A. Madero DF 07430, México
jchairez@ctrl.cinvestav.mx
Teléfono: (52)-55-57296000

***Abstract*— State estimation for uncertain systems affected by external noises is an important problem in control theory even for continuous and discrete systems. This paper deals with the state observation problem when the dynamic model of a plant contains uncertainties or is partially unknown and it is oriented to discrete time nonlinear systems because most of the existent results have been developed for continuous time systems. The recurrent neural network (RNN) have shown his advantages to deal with this class of problem. The Lyapunov second method is applied to generate a new learning law, containing an adaptive adjustment rate, implying the stability condition for the free parameters of the neural-observer. A numerical example is given using the RNN in the estimation of a mathematical model of HIV infection with three states.**

Palabras clave: Neural Networks, Discrete Systems, Uncertain Models, Adaptive State Estimation.

I. Introducction

Neural Networks (NN) have shown good identification properties in the presence of mathematical model uncertainties and external disturbances. There exist two main classes of NN: the *static* one, using the well known, backpropagation algorithm (Haykin, 2nd ED, 1999), and *dynamic* neural networks (DNN). The first one deal with the class of global optimization problems trying to adjust the weights of such NN to minimize the identification error. The second approach, exploits the feedback properties to develop a learning process based in an adequate feedback design. Dynamic Neural Observers are studied in (Poznyak *et al.*, 2001).

The design of state observing algorithms is usually dependent on the access to the mathematical description for the nonlinear system. Even that adaptive control theory has resolved the discrete estimation problem if the system is not affected by external perturbations like noise in the output or in the state vector description, there still exists an open problem to describe the discrete state estimation for uncertain systems. The nonparametric modelling and the NN have become into an interesting beginning for the study of this class of systems. Generally speaking, it is difficult to have an adequate representation of the plant under study, modelling this dynamic plants results in the presence of defferential or difference complex equations, and most of the cases, the access to the state vector is limited (Posnyak *et al.*, 1998).

Within the DNN framewrok, the Recurrent Neural Networks (RNN) has became an useful tool in control theory to deal with the problem of identification and control of nonlinear systems (Kumpati *et al.*, 1990). In the study of adaptive Neural Networks (NN) this technique is mostly used as an approximate model for unknown nonlinearities. Besides, they have shown their ability to identify, control and estimate uncertain, nonlinear and complex systems (Sam y Heng, 2008). The RNN have simple structure with a close-loop feedback that may be local or global. Due to this structure these NN have better capabilities to study the dynamic process of discrete nonlinear biological systems.

Comparing to nonlinear continuous-time systems, adaptive control is less developed for nonlinear discrete-time systems. The same concepts in continuous time and discrete time may have different meaning (Khalil, 2002). For this reason, most of the control schemes for continuous-time systems may not be directly suitable for discrete-time systems. For instance Lyapunov design for nonlinear discrete-time systems becomes much more intractable. For instance, consider the linearity property of the Lyapunov function derivative in continuos-time which, in counterpart, it is not present in Lyapunov difference equation in discrete time. However, there are still considerable advances in NN control for discrete-time systems (Bergman *et al.*, 1981), (Pérez Medina, 2002).

Mathematical modells clarify some facts about the dynamic behavior of a wide class of physical systems. In particular, talking about biological systems, finding a model, is really a hard task, due to the chemical, physical and cells interactions inside the human body. However, in many real situations, the modelling rules not always may generate an acceptable reproduction of reality. In this cases, the nonparametric identification (based on NN for example) could be successfully applied to cover the deficiencies or not accuracies of classical methodologies. As result, many methods using NN have been developed and successfully applied in Biomedical Engineering (Hudson y Cohen, 2000), like blood glucose regulation (Bergman *et al.*, 1981), (Logtenberg y Van Bellegooie, 2006), Hepatitis C Estimation (Miranda *et al.*, 2006), cancer schedule chemotherapy (Hao *et al.*, 2005), (Aguilar *et al.*, 2006) etc. Besides, one of the advantages to develop a discrete algorithm is that direct implementation in an electronic device like a DSP, FPGA, etc is feasible, as one can see it in (Deponte *et al.*, 2000).

Appli cation developed in this paper deals with the state estimation for an uncertain HIV model. Usually this illness monitoring implies the analysis of a sample in a period of time, this is naturally linked with discrete-time systems. Other important fact in the application of a discrete-time algorithm is avoiding the use of an online sensor, because for certain illness does not exist this kind of instrument. The rest of this paper is organized as follows: Section II describes the class of nonlinear systems that are studied in this paper and derives the stable learning rules, applying the direct Lyapunov's method. Section III, introduces the mathematical model of the infection of HIV and the results of the RNN application, Section IV concludes the paper and finally section V derives the proof for the main theorem of the work.

II. Recurrent DNN Observer for Uncertain Nonlinear Systems

II-A. Class of nonlinear systems

The class of uncertain discrete time nonlinear SISO systems considered throughout this paper is governed by a set of n nonlinear equations in differences and an algebraic state output mapping given by

$$\begin{array}{c} x_{k+1}=f\left(x_k,u_k\right)+\xi_{1,k} \quad x_o=x^0 \\ y_k=Cx_k+\xi_{2,k} \\ f\left(\cdot,\cdot\right):\Re^{n+p+1}\rightarrow\Re^n \end{array} \tag{1}$$

where $x_k \in \Re^n$ is the state vector of the nonlinear uncertain system at sampled time k, $u_k \in \Re^m$ is the control action, $C \in \Re^{1\times n}$ is an a priory known output matrix, $y_k \in \Re^p$ is the output vector. Nonlinear uncertain function $f\left(\cdot,\cdot\right):\Re^{n+m}\rightarrow\Re^n$ fulfills the following assumptions:

A1. The function $f\left(.\right)$ is enough smooth and satisfy the Lipschitz condition with respect to both arguments, that is:

a) $$\begin{array}{c} \left\|f\left(x_k,u_k\right)-f\left(y_k,v_k\right)\right\| \leq L_1\left\|x_k-y_k\right\|+L_2\left\|u_k-v_k\right\| \\ x,y\in\Re^n; \qquad u_t,v_t\in\Re^m \end{array} \tag{2}$$

which automatically implies the following property

$$b)\ \left\|f\left(x_k,u_k,t\right)\right\|^2 \leq C_1+C_2\left\|x_k\right\|^2 \tag{3}$$

This property guarantees the uniqueness and existence of an unique solution for the original system.

A2. There exists a bounded control function u_k, such that, the system (1) is quadratically stable in closed loop, in other words, exist a Lyapunov function V_k such that the difference between the step in the time $k+1$ and k is bounded by:

$$V_{k+1}-V_k \leq -\lambda\left\|x_k\right\|^2 \tag{4}$$

A3. State and output uncertainties and perturbations $\xi_{1,k}$ and $\xi_{2,k}$ in (1) are bounded as follows

$$\left\|\xi_{j,k}\right\|^2_{\Lambda_{\xi_j}} \leq \Upsilon_j, \quad \Lambda_{\xi_j}\in\Re^{n\times n} \tag{5}$$

where $\Lambda_{\xi_j}=\Lambda^\intercal_{\xi_j}$, $\Lambda_{\xi_j}\succ 0$ and $\Upsilon_j\succ 0$.

II-B. RNN approximation

The uncertain nonlinear system (1) may be allways represented as

$$\begin{array}{c} x_{k+1}=f\left(x_k,u_k,k\right)+\xi_{1,k}= \\ A^*x_k+W_1^*\sigma\left(x_k\right)+W_2^*\varphi\left(x_k\right)u_k+\tilde{f}_k+\xi_{1,k} \\ A^*\in\Re^{n\times n};\ \ x_t\in\Re^n;\ \ u_t\in\Re^m \\ \sigma\left(\cdot\right):\Re^{s_2}\rightarrow\Re^{s_1};\ \ \varphi\left(\cdot\right):\Re^{t_2\times q}\rightarrow\Re^{t_1}; \end{array} \tag{6}$$

where $\tilde{f}_k$ is the error modeling which evidently is defined by

$$\tilde{f}_k := f\left(x_k,u_k,k\right)-\Big(A^*x_k+W_1^*\sigma\left(x_k\right)+W_2^*\varphi\left(x_k\right)u_k+\tilde{f}_k\Big)$$

RNN based observer can be applied just if the following conditions is fulfilled:

Assumption. The modelling error $\tilde{f}_k$ belong to a special sector region, defined by the following inequality

$$\left\|\tilde{f}_k\right\|^2_{\Lambda_{\tilde{f}}} \leq n_1+n_2\left\|x_k\right\|^2_{\Lambda_{\tilde{f}}},\ n_1,n_2\in\Re^+ \tag{7}$$

Here $\Lambda_{\tilde{f}}\in\Re^{n\times n}$, $0<\Lambda_{\tilde{f}}=\Lambda^\top_{\tilde{f}}$.

RNN approximation is based and validated using the Stone-Weistrass theorem (Prolla, 1994) where the sigmoid functions are used as space basis and the Lipschitz property for the nonlinear uncertain system (in fact, quasi-linearity) is assumed. The matrix A is Hurwitz and the pair (A,C) is observable.

The vector-functions $\sigma\left(\cdot\right) := \left[\sigma_1\left(\cdot\right),\ldots,\sigma_l\left(\cdot\right)\right]$ and $\varphi\left(\cdot\right) := \left[\varphi_1\left(\cdot\right),\ldots,\varphi_s\left(\cdot\right)\right]$ are usually constructed with sigmoid functions components (following the standard neural networks design algorithms):

$$\begin{array}{c} \sigma_i\left(x\right)=\dfrac{a_i}{1+b_i\exp\left(-c_ix\right)} \\ \varphi_{ij}\left(x\right)=\dfrac{a_{ij}}{1+b_{ij}\exp\left(-c_{ij}x\right)} \end{array} \tag{8}$$

It is quiet usual that nonlinear functions $\sigma_i\left(\lambda\right)$ and $\varphi_{ij}\left(\lambda\right)$ satisfy sector conditions with L_σ and L_φ positive finite constants

$$\begin{array}{c} \left\|\sigma\left(x_1\right)-\sigma\left(x_2\right)\right\|^2 \leq L_\sigma\left\|x_1-x_2\right\|^2 \\ \left\|\varphi\left(x_1\right)-\varphi\left(x_2\right)\right\|^2 \leq L_\varphi\left\|x_1-x_2\right\|^2 \end{array} \tag{9}$$

II-C. Discrete-Time RNN Observer

The introduced discrete-time RNN is classified as a Hopfield observer (Posnyak *et al.*, 1998) which can be use to reproduce the unknown x_{k+1} vector. The RNN based observer follows the standar Luenberger technique is

$$\begin{array}{c}\hat{x}_{k+1} = A\hat{x}_k + (W_{1,k+1} - W_{1,k})\,\sigma(\hat{x}_k) \\ + (W_{2,k+1} - W_{2,k})\,\varphi(\hat{x}_k)\,u_k + K_1\left[y_k - C\hat{x}_k\right]\end{array} \quad (10)$$

where $\hat{x}_k \in \Re^n$ is the state observer, $u_k \in \Re^m$ is the input control signal,$W_{1,k} \in \Re^{n\times k}$ is the weigh matrix for the feedback state, $W_{2,k} \in \Re^{n\times r}$ is the weigh matrix for the input and $A \in \Re^{n\times n}$ is a Hurwitz Matrix.

The weight matrices are updated by nonlinear learning laws

$$W_{j,k+1} = \Phi_j\left(W_{j,k}, \hat{x}_k, y_k, u_k, k \mid \Theta\right) \qquad j = 1, 2. \quad (11)$$

that will be desgined using the thoery of adaptive parameter identification based on Lyapunov method. Here the parameters Θ must be adjusted to minimize the approximation error between the nominal part and the uncertain nonlinear model. The correction matrix K_1 should be selected as usual, that is, in such a way the matrix $A - K_1C$ is stable.

II-D. Problem statement

The principal problem to deal with in this paper is the study of plant's dynamics like the uncertain nonlinear system (1) under the presence of external perturbation in the states and in the output. Therefore, this problem can be formulated as follows:

Under the assumptions A1-A3 for any admissible u_k control injection, select adequate matrices and updating laws (11) (including the selection of $W^0_{j,k}$, $j = 1, 2$) in such way the upper bound for the estimation error:

$$\begin{array}{c}\beta := \overline{\lim_{T\to\infty}}\dfrac{1}{T}\sum\limits_{k=1}^{T}\left(\|\Delta_k\|^2_{Q_0} + \|\Delta_{k+1}\|^2_{Q_1}\right) \le \rho \\ \rho := 2L_\xi\Upsilon_1 + 2n_1\end{array}$$

is bounded.

II-E. Adaptive weights discrete learning law

To adjust the weights of the discrete time neural observer (10), lets apply the following adapting algorithms:

$$\begin{array}{c}\tilde{W}_{1,k+1} = \tilde{W}_{1,k} - k_1^{-1}\left[e_k^\intercal CN_\delta A^\intercal P\right]^\intercal \sigma^\intercal(\hat{x}_k) - \\ k_1^{-1}\left[\delta\sigma^\intercal(\hat{x}_k)\left(\tilde{W}_{1,k+1}+\tilde{W}_{1,k}\right)^\intercal Z_1\right]^\intercal \sigma^\intercal(\hat{x}_k) \\ \tilde{W}_{2,k+1} = \tilde{W}_{2,k} - k_2^{-1}\left[e_k^\intercal CN_\delta A^\intercal P\right]^\intercal \left[\phi(\hat{x}_k)\,u_k\right]^\intercal - \\ k_2^{-1}\left[\delta u_k^\intercal\phi^\intercal(\hat{x}_k)\left(\tilde{W}^\intercal_{2,k+1}+\tilde{W}^\intercal_{2,k}\right) Z_2\right]^\intercal \left[\phi(\hat{x}_k)\,u_k\right]^\intercal \\ Z_1 := PAN_\delta\Lambda_{12}N_\delta A^\intercal P \\ Z_2 := PAN_\delta\Lambda_{11}N_\delta A^\intercal P\end{array} \quad (12)$$

The matrix $N_\delta \in \Re^{n\times n}$ is defined as $N_\delta := \left(CC^\top + \delta I_{n\times n}\right)$ with δ a small positive scalar value. The matrices $\tilde{W}_{1,k} \in \Re^{n\times l}$ and $\tilde{W}_{2,k} \in \Re^{2n\times s}$ represent the difference between the current values of the adjustable parameters $W_{1,k}$ and $W_{2,k}$ to some *fitted* values $W^0_{1,k}$ and $W^0_{2,k}$. Δ_k is the output error defined by $\Delta_k := y_k - \hat{y}_k$ and $\Delta_{k+1} := y_{k+1} - \hat{y}_{k+1}$ for instance, $k+1$. Parameters k_1 and k_2 are the learning scalar constants of the neural network. $P = P^\intercal > 0$ is the positive definite solution for the following equations:

$$\begin{array}{c}\tilde{A}^\intercal P\tilde{A} - P + A^\intercal PR_1PA + \bar{Q}_1 = 0 \\ PA + A^\intercal P + PR_2P + \bar{Q}_2 = 0\end{array} \quad (13)$$

satisfying the next condition:

$$\begin{array}{c}\left(\Delta_k^\intercal + \Delta_{k-1}^\intercal A^\intercal\right)P\tilde{W}_{1,k}\sigma(\hat{x}_{k-1}) + \\ \left(\Delta_k^\intercal + \Delta_{k-1}^\intercal A^\intercal\right)P\tilde{W}_{2,k}\phi(\hat{x}_{k-1})\,u_{k-1} \ge 0\end{array} \quad (14)$$

where:

$$\begin{array}{c}\tilde{A}_1 := A - K_1C \\ R_1 := \breve{W}_1^*\Lambda_1^{-1}\left[\breve{W}_1^*\right]^\intercal \\ +\breve{W}_2^*\Lambda_2^{-1}\left[\breve{W}_2^*\right]^\intercal + \Lambda_3^{-1} + \Lambda_4^{-1} \\ \bar{Q}_1 := L_\sigma\left(\Lambda_1 + \Lambda_5 + Q_0\right) \\ +v_0L_\varphi\left(\Lambda_2 + \Lambda_6\right) + \Lambda_9^{-1} + 2n_2I_{nxn} \\ R_2 := \breve{W}_1^*\Lambda_5^{-1}\left[\breve{W}_1^*\right]^\intercal + \breve{W}_2^*\Lambda_6^{-1}\left[\breve{W}_1^*\right]^\intercal \\ +\Lambda_7^{-1} + \Lambda_8^{-1} + K_1C\Lambda_9C^\intercal K_1^\intercal \\ \bar{Q}_2 := Q_1 + \delta\sum\limits_{j=10}^{13}\Lambda_j^{-1} \\ ARA^\intercal \ne 0\end{array}$$

II-F. Main Result

Teorema 1: Considering the structure of the observer given by (10), supplied by the learning law (12) , and assuming the existence of a positive definite matrix Q_{01} such that the algebraic Ricatti equations given by 13 have an unique positive solution, then the estimation state error (Δ_k) is bounded (or equivalently has practical stability) by the following inequality:

$$\begin{array}{c}\overline{\lim_{T\to\infty}}\dfrac{1}{T}\sum\limits_{k=1}^{T}\left(\|\Delta_k\|^2_{Q_0} + \|\Delta_{k+1}\|^2_{Q_1}\right) \le \rho \\ \rho := 2L_\xi\Upsilon_1 + 2n_1\end{array} \quad (15)$$

Demostración: The proof of this theorem is developed in the Apendix. ■

II-G. Training Algorithm

The implementation of the learning algorithms implies a-priory knowledge of nominal matrices W^0_s, $s = 1, 2$ incorporated in $W_{s,k}$, $s = 1, 2$. The, so-called, training process consists in the obtaining of suitable approximation of these values. This process can be realized before the on-line state estimation beggins. This process is conducted choosing the parameters $\Theta := \left[A, W^0_1, W^0_2\right]$ using available experimental data. In this paper the matrix A was chosen taking values inside de unitary circle and in the open left side of the complex plane to guarantee observability for the pair (A, C). The adequate matrices W^0_1 and W^0_2 were obtained by an identification scheme using Neural Networks.

III. Simulation Results

III-A. HIV infection Math Model

The basic modeling of the HIV/AIDS dynamics is described by a 3-D discrete-time model developed in (Adama *et al.*, 2008). This model, given by

$$\begin{array}{l} T_{k+1} = s + (1-\delta)\,T_k - \beta T_k V_k + p\,(T_k, V_k) \\ T^*_{k+1} = \beta T_k V_k + (1-\mu)\,T^*_k \\ V_{k+1} = kT^*_k + (1-c)\,V_k \end{array} \quad (16)$$

includes the dynamics of the no infected CD4+ T-cells, the infected CD4+ T-cells, and the virions. Term

$$p\,(T, V) = rT\left(\frac{V}{KV}\right)$$

is the CD4+ T-cells proliferation term. In the 3-D model, T ($CD4/mm^3$) represents the amount of no infected CD4+ T-cells, T^* ($CD4/mm^3$) represents the amount of the infected CD4+ T-cells, and V (RNA $copies/ml$) represents the free virions. Free virus particles infect healthy cells at a rate proportional to both T and V (βTV). They are removed from the system at the rate c. In (16), it is assume that healthy CD4+ T-cells are produced at a constant rate s. This is the simplest way to model the production of CD4+ T-cells.μ represents the rate at which infected cells are removed from the system. r is the maximal proliferation rate of the process. K is the half saturation constant of the proliferation process (Perelson y Nelson, 1999), (Perelson *et al.*, 1997).

III-B. Simulation Paramaters

The parameters used in the simulation were taking from (Adama *et al.*, 2008):

$$\begin{array}{ccc} s = 10 & \delta = 0{,}01 & \bar{\beta} = 1e^{-7} \\ \mu = 0{,}09 & k = 1000 & c = 0{,}31 \end{array}$$

$$\bar{k} = {,}20 \quad c_2 = {,}05$$

an the initial conditions used, were chosen as:

$$\begin{array}{l} T_0 = 1000\ \mathrm{CD_4/mm^3},\ T^*_0 = 50\ \mathrm{CD_4/mm^3} \\ V_0 = 100\ \mathrm{copies/ml},\ V^*_0 = 100\ \mathrm{copies/ml} \end{array}$$

The parameter in the RNN to realize the training process are:

$$A = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0{,}22 & 0 \\ 0 & 0 & 0{,}255 \end{bmatrix} \quad K_1 = \begin{bmatrix} 1{,}25 \\ 0{,}06 \\ 414 \end{bmatrix}$$

as we can see the matrix A is Hurwitz with his eigenvalues in the left side of the complex plane, and the values are small enough to stay inside the unitary circle. The state estimate for the second state of the system involving the infected TCD4+ cells is depicted in Fig. 1 while the free virions in the human body due the infection of HIV is observed in Fig. 2.The index performance of the Recurrent Neural Network can be seen in Fig. 3.

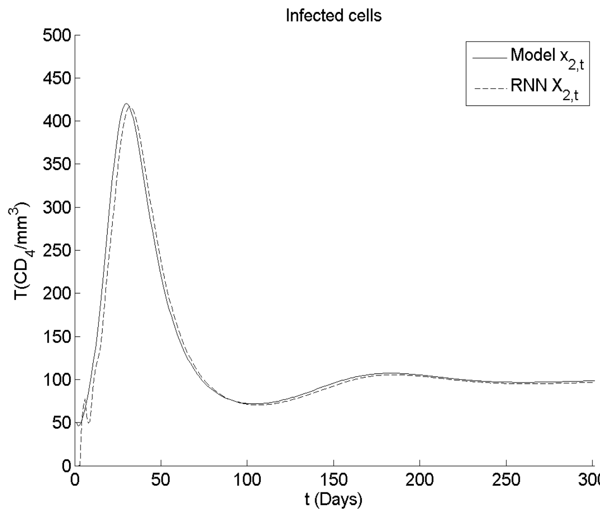

Figura 1. State estimator for the infected TCD4+ cells.

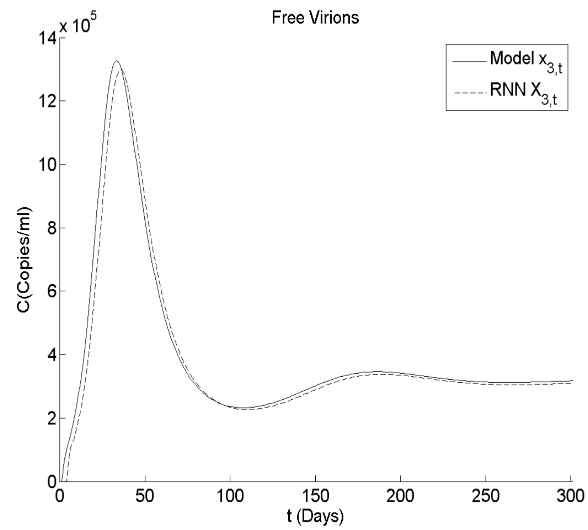

Figura 2. State estimator for the free virions.

IV. Conclusions

This paper has resolved the state estimation for the particular case of nonlinear systems with uncertainties on its states and its output. The practical stability for the observing error has been demonstrated applying the second Lyapunov analysis. Based in this result, it can be possible to generate the corresponding learning laws for the adaptive weights of the RNN. The application of the observer to the HIV infection shows the simulation efficiency for the discrete time RNN learning procedure. It is important to note, that the implementation of this algorithm in a programmable electronic device is direct, and the close-loop control does not need a continuous sensor, due to the sampling rate in the biological systems.

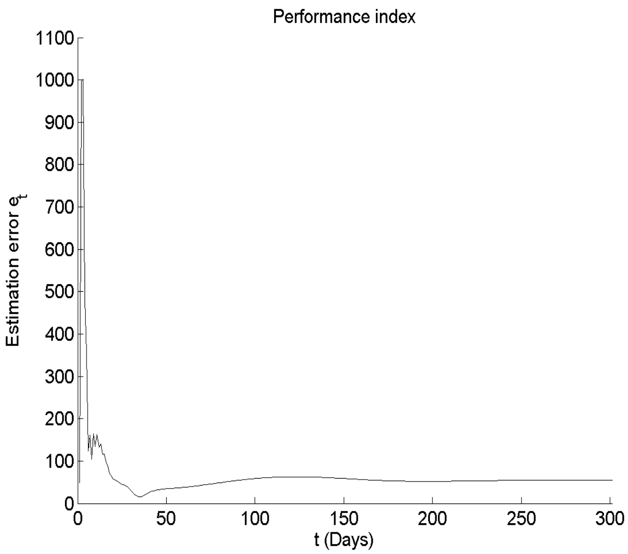

Figura 3. Performance index.

V. Acknowledgements

This paper was prepared thanks to the grants assigned for the project SEPI-20082485 finaced by National Polytechnic Institute of Mexico.

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Appendix

To construct the converge algorithm, define the energetic function (Lyapunov-like) as:

$$V_k = \Delta_k^T P \Delta_k + k_1 \mathrm{tr}\left\{\tilde{W}_{1,k}^T \tilde{W}_{1,k}\right\} + k_2 \mathrm{tr}\left\{\tilde{W}_{2,k}^T \tilde{W}_{2,k}\right\} \quad (17)$$

Using the basic principles of Lyapunov method for discrete systems, one has:

$$\begin{gathered} V_{k+1} - V_k = \Delta_{k+1}^\intercal P \Delta_{k+1} - \Delta_k^\intercal P \Delta_k + \\ k_1 \mathrm{tr}\left\{\left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k}\right)^\intercal \left(\tilde{W}_{1,k+1} - \tilde{W}_{1,k}\right)\right\} + \\ k_2 \mathrm{tr}\left\{\left(\tilde{W}_{2,k+1} + \tilde{W}_{2,k}\right)^\intercal \left(\tilde{W}_{2,k+1} - \tilde{W}_{2,k}\right)\right\} \end{gathered} \quad (18)$$

Using the following equation $\delta\Delta_k - \delta\Delta_k + C^\intercal C \Delta_k = C^\intercal e_k$, the following identity holds

$$\begin{gathered} \left(\delta I + C^\intercal C\right)\Delta_k = C^\intercal e_k + \delta\Delta_k \\ \Delta_k = \left(\delta I + C^\intercal C\right)^{-1}\left(C^\intercal e_k + \delta\Delta_k\right) = N_\delta\left(C^\intercal e_k + \delta\Delta_k\right) \end{gathered} \quad (19)$$

Working with the term $\Delta_{k+1}^\intercal P \Delta_{k+1}$ the estimation error has the following structure

$$\begin{gathered} \Delta_{k+1} = A\Delta_k + \breve{W}_1^* \tilde{\sigma}_k + \tilde{W}_{1,k+1}\sigma\left(\hat{x}_k\right) + \\ \breve{W}_2^* \tilde{\varphi}_k u_k + \tilde{W}_{2,k+1}\phi\left(\hat{x}_k\right) u_k - K_1 e_k + \tilde{f}_k + \xi_{1,k} \end{gathered} \quad (20)$$

where $\tilde{W}_{i,k} := \breve{W}_{i,k}^* - \breve{W}_{i,k}$ and $\tilde{\sigma}_k = \sigma\left(x_k\right) - \sigma\left(\hat{x}_k\right)$ $\tilde{\varphi}_k = \varphi\left(x_k\right) - \varphi\left(\hat{x}_k\right)$. Substituting the Δ_k expression

$$\begin{gathered} \Delta_{k+1}^\intercal P \Delta_{k+1} \leq \Delta_k^\intercal \left(A - K_1 C\right)^\intercal P \left(A - K_1 C\right)\Delta_k + \\ \Delta_k^\intercal A^\intercal P \breve{W}_1^* \Lambda_1^{-1}\left[\breve{W}_1^*\right]^\intercal P A \Delta_k \\ + \Delta_k^\intercal A^\intercal P \breve{W}_2^* \Lambda_2^{-1}\left[\breve{W}_2^*\right]^\intercal P A \Delta_k \\ + \Delta_k^\intercal A^\intercal P \Lambda_3^{-1} P A \Delta_k + \Delta_k^\intercal A^\intercal P \Lambda_4^{-1} P A \Delta_k \\ + v_0 L_\varphi \Delta_k^\intercal \left(\Lambda_2 + \Lambda_6\right)\Delta_k + \Delta_k^\intercal \Lambda_9^{-1}\Delta_k + 2L_\xi \Upsilon_1 + \\ + \tilde{f}_k^\intercal \Lambda_3 \tilde{f}_k + \tilde{f}_k^\intercal \Lambda_7 \tilde{f}_k + \Delta_{k+1}^\intercal \left(PA + A^\intercal P\right)\Delta_{k+1} \\ \Delta_{k+1}^\intercal P \breve{W}_1^* \Lambda_5^{-1}\left[\breve{W}_1^*\right]^\intercal P \Delta_{k+1} \\ + \Delta_{k+1}^\intercal P \breve{W}_2^* \Lambda_6^{-1}\left[\breve{W}_1^*\right]^\intercal P \Delta_{k+1} + L_\sigma \Delta_k^\intercal \left(\Lambda_1 + \Lambda_5\right)\Delta_k + \\ \Delta_{k+1}^\intercal P \Lambda_7^{-1} P \Delta_{k+1} + \Delta_{k+1}^\intercal P \Lambda_8^{-1} P \Delta_{k+1} \\ + \Delta_{k+1}^\intercal P K_1 C \Lambda_9 C^\intercal K_1^\intercal P \Delta_{k+1} + \\ \Delta_k^\intercal A^\intercal P \tilde{W}_{1,k}\sigma\left(\hat{x}_k\right) + \Delta_k^\intercal A^\intercal P \tilde{W}_{2,k}\phi\left(\hat{x}_k\right) u_k \\ + \Delta_{k+1}^\intercal P \tilde{W}_{1,k+1}\sigma\left(\hat{x}_k\right) + \Delta_{k+1}^\intercal P \tilde{W}_{2,k+1}\phi\left(\hat{x}_k\right) u_k + \end{gathered}$$

Using the upper bound value for $\tilde{f}_k^\intercal \Lambda_3 \tilde{f}_k$ and $\tilde{f}_k^\intercal \Lambda_7 \tilde{f}_k$ and the equation given in (19), one has

$$
\begin{gathered}
V_{k+1} - V_k \leq \Delta_k^\intercal \left[(A - K_1 C)^\intercal P (A - K_1 C) - P \right] \Delta_k + \\
\Delta_k^\intercal A^\intercal P \breve{W}_1^* \Lambda_1^{-1} \left[\breve{W}_1^* \right]^\intercal P A \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \breve{W}_2^* \Lambda_2^{-1} \left[\breve{W}_2^* \right]^\intercal P A \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \Lambda_3^{-1} P A \Delta_k + \Delta_k^\intercal A^\intercal P \Lambda_4^{-1} P A \Delta_k + \\
L_\sigma \Delta_k^\intercal (\Lambda_1 + \Lambda_5) \Delta_k + v_0 L_\varphi \Delta_k^\intercal (\Lambda_2 + \Lambda_6) \Delta_k \\
+ \Delta_k^\intercal \Lambda_9^{-1} \Delta_k + 2 L_\xi \Upsilon_1 + \\
2 n_1 + 2 n_2 \left\| \Delta_k \right\|^2 + \Delta_{k+1}^\intercal (P A + A^\intercal P) \Delta_{k+1} + \\
\Delta_{k+1}^\intercal P \breve{W}_1^* \Lambda_5^{-1} \left[\breve{W}_1^* \right]^\intercal P \Delta_{k+1} \\
+ \Delta_{k+1}^\intercal P \breve{W}_2^* \Lambda_6^{-1} \left[\breve{W}_1^* \right]^\intercal P \Delta_{k+1} + \\
\Delta_{k+1}^\intercal P \Lambda_7^{-1} P \Delta_{k+1} + \Delta_{k+1}^\intercal P \Lambda_8^{-1} P \Delta_{k+1} \\
+ \Delta_{k+1}^\intercal P K_1 C \Lambda_9 C^\intercal K_1^\intercal P \Delta_{k+1} + \\
\delta \Delta_k^\intercal \Lambda_{10}^{-1} \Delta_k + \delta \Delta_{k+1}^\intercal \Lambda_{12}^{-1} \Delta_{k+1} \\
+ \delta \Delta_k^\intercal \Lambda_{11}^{-1} \Delta_k + \delta \Delta_{k+1}^\intercal \Lambda_{12}^{-1} \Delta_{k+1} + \\
e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k} \sigma(\hat{x}_k) \\
+ e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k+1} \sigma(\hat{x}_k) + \\
\delta \sigma^\intercal(\hat{x}_k) \left(\tilde{W}_{1,k+1}^\intercal + \tilde{W}_{1,k}^\intercal \right) P A N_\delta \Lambda_{12} \\
\div N_\delta A^\intercal P \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \sigma(\hat{x}_k) + \\
e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k} \phi(\hat{x}_k) u_k \\
+ e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k+1} \phi(\hat{x}_k) u_k + \\
\delta u_k^\intercal \phi^\intercal(\hat{x}_k) \left(\tilde{W}_{2,k+1}^\intercal + \tilde{W}_{2,k}^\intercal \right) P A N_\delta \Lambda_{11} \\
\times N_\delta A^\intercal P \left(\tilde{W}_{2,k+1}^\intercal + \tilde{W}_{2,k}^\intercal \right) \phi(\hat{x}_k) u_k + \\
k_1 \operatorname{tr} \left\{ \left(\tilde{W}_{1,k+1} - \tilde{W}_{1,k} \right)^\intercal \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \right\} \\
+ k_2 \operatorname{tr} \left\{ \left(\tilde{W}_{2,k+1} - \tilde{W}_{2,k} \right)^\intercal \left(\tilde{W}_{2,k+1} + \tilde{W}_{2,k} \right) \right\}
\end{gathered}
$$

So for the updating stage $e_{k-1} = e_k$, it is obtained

$$
\begin{gathered}
e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k} \sigma(\hat{x}_k) \\
+ e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k+1} \sigma(\hat{x}_k) \\
= e_k^\intercal C N_\delta A^\intercal P \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \sigma(\hat{x}_k) \\
e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k} \phi(\hat{x}_k) u_k \\
+ e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k+1} \phi(\hat{x}_k) u_k \\
= e_k^\intercal C N_\delta A^\intercal P \left(\tilde{W}_{2,k+1} + \tilde{W}_{2,k} \right) \phi(\hat{x}_k) u_k
\end{gathered}
$$

Now, working in the adaptive stage $\tilde{W}_{1,k+1} = \tilde{W}_{1,k}$, $\tilde{W}_{2,k+1} = \tilde{W}_{2,k}$ This implies that

$$
\begin{gathered}
\left(e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k} + e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{1,k+1} \right) \sigma(\hat{x}_k) = 0 \\
\left(e_k^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k} + e_{k+1}^\intercal C N_\delta A^\intercal P \tilde{W}_{2,k+1} \right) \phi(\hat{x}_k) u_k = 0
\end{gathered}
$$

Leading to $\tilde{W}_{i,k} = \breve{W}_{i,k} - \breve{W}_{i,k}^* = 0$. Using the couple of results just recently developed, we have

$$
\begin{gathered}
V_{k+1} - V_k \leq \Delta_k^\intercal \left[(A - K_1 C)^\intercal P (A - K_1 C) - P \right] \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \breve{W}_1^* \Lambda_1^{-1} \left[\breve{W}_1^* \right]^\intercal P A \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \breve{W}_2^* \Lambda_2^{-1} \left[\breve{W}_2^* \right]^\intercal P A \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \Lambda_3^{-1} P A \Delta_k \\
+ \Delta_k^\intercal A^\intercal P \Lambda_4^{-1} P A \Delta_k + \\
L_\sigma \Delta_k^\intercal (\Lambda_1 + \Lambda_5 + Q_0) \Delta_k \\
+ v_0 L_\varphi \Delta_k^\intercal (\Lambda_2 + \Lambda_6) \Delta_k + \Delta_k^\intercal \Lambda_9^{-1} \Delta_k + \\
2 L_\xi \Upsilon_1 + 2 n_1 + 2 n_2 \left\| \Delta_k \right\|^2 \\
+ \Delta_{k+1}^\intercal (P A + A^\intercal P + Q_1) \Delta_{k+1} + \\
\Delta_{k+1}^\intercal P \breve{W}_1^* \Lambda_5^{-1} \left[\breve{W}_1^* \right]^\intercal P \Delta_{k+1} \\
+ \Delta_{k+1}^\intercal P \breve{W}_2^* \Lambda_6^{-1} \left[\breve{W}_1^* \right]^\intercal P \Delta_{k+1} + \\
\Delta_{k+1}^\intercal P \Lambda_7^{-1} P \Delta_{k+1} + \Delta_{k+1}^\intercal P \Lambda_8^{-1} P \Delta_{k+1} \\
+ \Delta_{k+1}^\intercal P K_1 C \Lambda_9 C^\intercal K_1^\intercal P \Delta_{k+1} + \\
\delta \Delta_k^\intercal \Lambda_{10}^{-1} \Delta_k + \delta \Delta_{k+1}^\intercal \Lambda_{12}^{-1} \Delta_{k+1} \\
+ \delta \Delta_k^\intercal \Lambda_{11}^{-1} \Delta_k + \delta \Delta_{k+1}^\intercal \Lambda_{12}^{-1} \Delta_{k+1} + \\
- \left\| \Delta_k \right\|_{Q_0}^2 - \left\| \Delta_{k+1} \right\|_{Q_1}^2 \\
e_k^\intercal C N_\delta A^\intercal P \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \sigma(\hat{x}_k) + \\
\delta \sigma^\intercal(\hat{x}_k) \left(\tilde{W}_{1,k+1}^\intercal + \tilde{W}_{1,k}^\intercal \right) P A N_\delta \Lambda_{12} \\
\times N_\delta A^\intercal P \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \sigma(\hat{x}_k) + \\
e_k^\intercal C N_\delta A^\intercal P \left(\tilde{W}_{2,k+1} + \tilde{W}_{2,k} \right) \phi(\hat{x}_k) u_k + \\
\delta u_k^\intercal \phi^\intercal(\hat{x}_k) \left(\tilde{W}_{2,k+1}^\intercal + \tilde{W}_{2,k}^\intercal \right) P A N_\delta \Lambda_{11} \\
\times N_\delta A^\intercal P \left(\tilde{W}_{2,k+1}^\intercal + \tilde{W}_{2,k}^\intercal \right) \phi(\hat{x}_k) u_k + \\
k_1 \operatorname{tr} \left\{ \left(\tilde{W}_{1,k+1} - \tilde{W}_{1,k} \right)^\intercal \left(\tilde{W}_{1,k+1} + \tilde{W}_{1,k} \right) \right\} \\
+ k_2 \operatorname{tr} \left\{ \left(\tilde{W}_{2,k+1} - \tilde{W}_{2,k} \right)^\intercal \left(\tilde{W}_{2,k+1} + \tilde{W}_{2,k} \right) \right\}
\end{gathered}
$$

Considering the condition given in (14), in view of the assumption on the positiveness of the solution for the Riccati equation and using the adaptive laws for the weights, one finally leads to $V_k - V_{k-1} \leq \rho - \left\| \Delta_k \right\|_{Q_0}^2 - \left\| \Delta_{k+1} \right\|_{Q_1}^2$. Summing both sides in the last inequality and taking the upper limit when $T \to \infty$, one gets $\overline{\lim_{T \to \infty}} \frac{1}{T} \sum_{k=1}^{T} \left(\left\| \Delta_k \right\|_{Q_0}^2 + \left\| \Delta_{k+1} \right\|_{Q_1}^2 \right) \leq \rho$.