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High Order Sliding Mode Observer for Linear Systems with Unbounded Unknown Inputs

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Abstract: A global observer is designed for strongly detectable systems with unbounded unknown inputs. The design of the observer is based on three steps. First, the system is extended taking the unknown inputs (and possibly some of their derivatives) as a new state; then, using a global high-order sliding mode differentiator, a new output of the system is generated in order to fulfill the Hautus condition which finally allows decomposing the system, in the new coordinates, into two subsystems, the first one being unaffected directly by the unknown inputs, and the state vector of the second subsystem is obtained directly from the original system output. Such decomposition permits of designing a Luenberger observer for the first subsystem. This procedure enables one to estimate the state and the unknown inputs using the least number of differentiations possible. Simulations are given in order to show the effectiveness of the proposed observer.

1. INTRODUCTION

The observation problem for linear systems with unknown inputs has been extensively studied (Zasadzinski et al. (1994), Hui and Zak (2005)). Important works of Molinari and Hautus contributed in establishing the properties that the system should satisfy in order guaranty the existence of an observer, thereby allowing the estimation of the state vector (see, Molinari (1976), Hautus (1983)). In those works, the strong observability property was studied. This property ensures a one-to-one correspondence between the state and the output (this is analogous to the observability property of linear systems without inputs (wo.i.)). Furthermore, the strong detectability property was studied (analogous to detectability for systems wo.i.). These properties have been studied in terms of the weakly unobservable subspace Trentelman et al. (2001), using the theory of invariant subspaces Wonham (1985), Basile and Marro (1992). Molinari gave a recursive algorithm dealing with the construction of a series of matrices that allows for the calculation of the weakly observable subspace in the last step of the algorithm. Such algorithm can be interpreted as a differentiation procedure of the output of the system. Thus, if the system is strongly observable (the weakly observable subspace contains only the zero vector), the state can be obtained through a recursive method of differentiation of the output. On the other hand, Hautus studied the same problem in terms of invariant zeros and gave the conditions under which the estimation of the state vector can be done by means of a linear observer whose input is the output of the original system; those conditions are: the system must be strongly detectable (the invariant zeros of the system must be Hurwitz) and some rank condition, relating the output distribution matrix and the input distribution matrices, must be satisfied. This last condition, which we will refer to as the Hautus condition, might be quite restrictive. In the sliding mode community, several observers have been proposed for systems with unknown inputs. There are works where observers have been designed assuming the Hautus condition is satisfied (Edwards et al. (2002), Hui and Zak (2005)). In other works, a differentiation process has been proposed, mainly based on first and second order sliding mode techniques (Bejarano et al. (2007), Fridman et al. (2007), Floquet and Barbot (2007)). A method for the construction of a new output in order to fulfill the Hautus condition is suggested in Floquet et al. (2007). Nevertheless, a condition related to the relative degree of the original system must be satisfied in order to follow such method, which means the system is required to meet more than just strong detectability. Furthermore, the differentiation procedure is done step-by-step using the super-twisting algorithm (a second order sliding mode, Levant (1998)), which increases the error due to the sample time of sensors or computer calculations. The majority of sliding mode observers allow one to estimate the unknown input vector, for which the unknown input vector is assumed to be uniformly bounded. Here, we propose an observer for strongly detectable systems using the least number of derivatives possible, which are estimated online using an exact differentiator for unbounded high order derivatives Levant (2006). The following notation is used throughout the paper. Let X be a real matrix of dimension $n \times m$. The notation $X^{\perp}$ means a full row rank orthogonal matrix to X, i.e. $X^{\perp}X = 0$ and $\operatorname{rank} X^{\perp} = n - \operatorname{rank} X$. The matrix $X^{\perp\perp}$ must satisfy the conditions $\operatorname{rank} X^{\perp\perp} = \operatorname{rank} X$ and $\det \begin{bmatrix} X^{\perp} \\ X^{\perp\perp} \end{bmatrix} \neq 0$. Meanwhile, $X_{\perp}$ is a matrix whose image spans the null space of X, i.e. $XX_{\perp} = 0$ and $\operatorname{rank} X_{\perp} = m - \operatorname{rank} X$. Let $f(t)$ be a vector function, $f^{[k]}$ represents the k-th anti-differentiator of $f(t)$, i.e. $f^{[k]}(t) = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} f(\tau_k)\, d\tau_k \cdots d\tau_2 d\tau_1$, $f^{[0]}(t) =$

$f(t)$; thus $F^{[k+1]} := \begin{bmatrix} F^{[k]} \\ f^{[k+1]} \end{bmatrix}$, where $F^{[1]} = \begin{bmatrix} f(t) \\ f^{[1]} \end{bmatrix}$. For a matrix A, $\rho_A = \operatorname{rank} A$.

2. PROBLEM STATEMENT

Let Σ be a linear system whose dynamics is governed by the following equations:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Dw(t) \\ y(t) = Cx(t) + Fw(t) \end{cases} \tag{1}$$

The state vector is represented by $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ represents the unknown input vector, and $y(t) \in \mathbb{R}^p$ is the output of the system. The fourfold of constant matrices (A, C, D, F) will be associated to system Σ. Without loss of generality, it is assumed that rank $\begin{bmatrix} D \\ F \end{bmatrix} = m$.

The task is to estimate $x(t)$ and $w(t)$ using only the output values $y(\tau)$ ($\tau \in [0, t]$). Henceforth, the following conditions are assumed to be satisfied.

A1. System Σ is strongly detectable[1] (the set of invariant zeros of (A, C, D, F) lies within the interior of the left half side of the complex plane).

A2. Vector w can be partitioned in the following way: $w^T =: \left[\, w_0^T \; w_1^T \; \cdots \; w_r^T \,\right]$, where $w_0 \in \mathbb{R}^{m_0}$, and none of its derivatives are bounded, but it has an upper-bounded norm, i.e. $\|w_0(t)\| \le w_0^+(t)$; $w_1 \in \mathbb{R}^{m_1}$ has a derivative bounded as $\|\dot{w}_1(t)\| \le w_1^+(t)$, and so forth, until $w_r \in \mathbb{R}^{m_r}$ which does not have a bounded norm for its first $r-1$ derivatives, but it has a bounded norm for its r-th derivative, i.e. $\left\| w_r^{(r)}(t) \right\| \le w_r^+(t)$. Obviously, it must be satisfied that $\sum_{i=0}^{r} m_i = m$.

Notice that assumption A2 is related more with the existence of the bounds of w_i that with the partition itself since, once the bounds are ensured, w can always be partitioned in the required form, perhaps with a change of coordinates. We can partition matrices D and F defining matrices d_i and f_i $(i = 0, \ldots, r)$ by means of the identities $Dw = \sum_{i=0}^{r} d_i w_i$ and $Fw = \sum_{i=0}^{r} f_i w_i$, respectively. Thus, considering assumption A2, system Σ can be rewritten into form (2).

$$\bar{\Sigma}_{\text{ex}} : \begin{cases} \dot{x}_{\text{ex}} = \bar{A}x_{\text{ex}} + \bar{D}w_{\text{ex}} \\ y = \bar{C}x_{\text{ex}} + \bar{F}w_{\text{ex}} \end{cases} \tag{2}$$

We set, by definition

$$x_{\text{ex}}^T = \left[\, x^T \; w_1^T \; \cdots \; w_r^T \; \dot{w}_2^T \; \cdots \; \dot{w}_r^T \; \cdots \; w_r^{(r-1)T} \,\right]$$

and $w_{\text{ex}}^T = \left[\, w_0^T \; \dot{w}_1^T \; w_2^{(2)T} \; \cdots \; w_r^{(r)T} \,\right]$, where $x_{\text{ex}}^T \in \mathbb{R}^{\bar{n}}$ $(\bar{n} = n + \sum_{i=1}^{r} i m_i)$ and $w_{\text{ex}} \in \mathbb{R}^m$.

$$\bar{A} = \operatorname{diag}\left(\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_r\right), \; \bar{A}_1 = \begin{bmatrix} A & d_1 & \cdots & d_r \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\bar{A}_j := \begin{bmatrix} I \\ 0 \end{bmatrix}, \; \bar{A}_j \in \mathbb{R}^{\left(\sum_{i=j-1}^{r} m_i\right) \times \left(\sum_{i=j}^{r} m_i\right)}$$

$$\bar{D} := \operatorname{diag}\left(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_r\right), \; \bar{d}_1 = \begin{bmatrix} d_0 & 0 \\ 0 & I_{m_1} \end{bmatrix}$$

$$\bar{d}_j := \begin{bmatrix} 0 \\ I_{m_j} \end{bmatrix}, \; \bar{C} := \left[\, C \; f_1 \; f_2 \; \cdots \; f_r \; 0 \,\right]$$

$$\bar{d}_j \in \mathbb{R}^{\left(\sum_{i=j-1}^{r} m_i\right) \times m_j}, \; \bar{F} = \left[\, f_0 \; 0 \,\right], \; \bar{F} \in \mathbb{R}^{p \times m}$$

It is easy to verify that the detectability property is maintained for system $\bar{\Sigma}_{\text{ex}}$, i.e. Σ is strongly detectable if, and only if, $\bar{\Sigma}_{\text{ex}}$ is strongly detectable as well. Thus, the problem of estimating $x(t)$ and $w(t)$ is restated as the problem of estimating extended vector x_{ex}.

3. CONSTRUCTION OF A NEW OUTPUT

The identity rank $\begin{bmatrix} CD & F \\ F & 0 \end{bmatrix} = \operatorname{rank} F + m$ will be referred to as the Hautus condition. The Hautus condition is a necessary condition for the design of an observer for system (1), without the need of any output derivatives. Let us define $Q_{\bar{F}} \in \mathbb{R}^{m \times m}$ as a nonsingular matrix $Q_{\bar{F}} := \left[\, \bar{F}_\perp \; Q_1 \,\right]$, which yields the following matrix decomposition:

$$\begin{bmatrix} \bar{D} \\ \bar{F} \end{bmatrix} Q_{\bar{F}} = \begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & \bar{F}_2 \end{bmatrix}, \quad \begin{matrix} \bar{F}_2 \in \mathbb{R}^{p \times \rho_{\bar{F}}} \\ \bar{D}_1 = \bar{D}\bar{F}_\perp \in \mathbb{R}^{n \times (m - \rho_{\bar{F}})} \end{matrix} \tag{3}$$

From (3), rank $\bar{F}_2 = \operatorname{rank} \bar{F}$. From (3), it is easy to verify that the Hautus condition is equivalent to satisfying the identity rank $\bar{F}^\perp \bar{C} \bar{D}_1 = \operatorname{rank} \bar{D}_1$. Thus, this section is devoted to the construction of a new output $y_{n_H} = M_{n_H} x_{\text{ex}}$ so that rank $M_{n_H} \bar{D}_1 = \operatorname{rank} \bar{D}_1$. Therefore, the procedure followed in this section must be taken into account only if rank $\bar{F}^\perp \bar{C} \bar{D}_1 \ne \operatorname{rank} \bar{D}_1$, otherwise this section should be skipped since, in such a case, the procedure given here is needless. Let us give the following lemma, which will help to remove the influence of the unknown inputs in the output injection from the Luenberger-like observer proposed below.

Lemma 1. (A, C, D, F) is strongly detectable if, and only if, $\left(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}, \bar{F}^\perp \bar{C}, \bar{D}\bar{F}_\perp\right)$ is strongly detectable as well.

Proof. Inasmuch as fourfold (A, C, D, F) is strongly detectable if, and only if, fourfold $\left(\bar{A}, \bar{D}, \bar{C}, \bar{F}\right)$ is strongly detectable as well, it is enough to prove the equivalence of the proposition for $\left(\bar{A}, \bar{C}, \bar{D}, \bar{F}\right)$ and $\left(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}, \bar{F}^\perp \bar{C}, \bar{D}\bar{F}_\perp\right)$. Let us define $R_{\bar{\Sigma}}(s)$ as the Rosenbrock matrix associated to $\bar{\Sigma}$, i.e.

$$R_{\bar{\Sigma}}(s) = \begin{bmatrix} sI - \bar{A} & -\bar{D} \\ \bar{C} & \bar{F} \end{bmatrix}.$$

Due to the fact that the identity

$$\begin{bmatrix} sI - \bar{A} + \bar{D}_2 \bar{F}_2^+ \bar{C} & -\bar{D}_1 & -\bar{D}_2 \\ \bar{F}^\perp \bar{C} & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{F}^\perp \\ 0 & \bar{F}_2^+ \end{bmatrix} R_{\bar{\Sigma}}(s) \begin{bmatrix} I & 0 \\ 0 & Q_{\bar{F}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \\ -\bar{F}_2^+ \bar{C} & I \end{bmatrix} \tag{4}$$

always holds, it turns out to be that s_0 is an invariant zero of $\left(\bar{A}, \bar{C}, \bar{D}, \bar{F}\right)$, i.e. rank $R_{\bar{\Sigma}}(s_0) < n + m$, if, and only if,

[1] Recall that Σ is strongly detectable if $y \equiv 0$ implies $x(t) \to \infty$. Some times it is said that the fourfould (A, C, D, F) is strongly detectable meaning that this property is fulfilled for the system Σ associated to this system of matrices.

rank $R_{\bar{\Sigma}_{F^\perp}}(s_0) < n+m-r$ (where the Rosenbrock matrix $R_{\bar{\Sigma}_{F^\perp}}$ for $\bar{\Sigma}_{F^\perp}$ appears in the extended matrix of the left hand side of (4)), which in turn is equivalent to asserting that s_0 is an invariant zero of $\left(\bar{A}-\bar{D}_2\bar{F}_2^{+}\bar{C}, \bar{F}^{\perp}\bar{C}, \bar{D}_1\right)$. ■

Now, selecting $\bar{L}$ so that $\left(\bar{A}-\bar{D}_2\bar{F}_2^{+}\bar{C}-\bar{L}\bar{F}^{\perp}\right)$ be Hurwits, we can design the following Luenberger-like- observer whose trajectories $\tilde{x}$ are governed by the dynamic equations (5).

$$\dot{\tilde{x}} = \left(\bar{A}-\bar{D}_2\bar{F}_2^{+}\bar{C}\right)\tilde{x} + \bar{L}\left(\bar{F}^{\perp}y - \bar{F}^{\perp}\bar{C}\tilde{x}\right) + \bar{D}_2\bar{F}_2^{+}y \quad (5)$$

Defining $\begin{bmatrix}\bar{w}_{\mathrm{ex},1}\\ \bar{w}_{\mathrm{ex},2}\end{bmatrix} := Q_{\bar{F}}^{-1}w_{\mathrm{ex}}$, $\bar{w}_{\mathrm{ex},2}\in\mathbb{R}^{\rho_{\bar{F}}\times m}$, and since $\bar{F}_2^{+}y = \bar{F}_2^{+}\bar{C}x_{\mathrm{ex}} + \bar{F}_2^{+}\bar{w}_{\mathrm{ex},2}$, the dynamics of $e(t) = x_{\mathrm{ex}}(t) - \tilde{x}(t)$ takes the form:

$$\begin{aligned}\dot{e} &= \left(\bar{A}-\bar{D}_2\bar{F}_2^{+}\bar{C}-\bar{L}\bar{F}^{\perp}\right)e + \bar{D}_1\bar{w}_{\mathrm{ex},1}\\ &= \tilde{A}e + \bar{D}_1\bar{w}_{\mathrm{ex},1}\end{aligned} \quad (6)$$

3.1 Output extension via HOSM differentiator

In the sequel, we will construct a new output $y_{\bar{n}_H} = M_{\bar{n}_H}x_{\mathrm{ex}}$ so that Hautus condition can be achieved, i.e. that the identity rank $M_{\bar{n}_H}\bar{D}_1 = \operatorname{rank} D_1$ will be true. For the construction of such an output, we will use a high-order sliding mode differentiator Levant (2006). Firstly, let us write a set of matrices M_k so that M_kx_{ex} can be expressed as a differentiation operator depending on y. Let the matrices M_{k+1} (which can be taken from the Molinari's algorithm Molinari (1976)) be defined by means of the following algorithm:

$$\begin{aligned}M_{k+1} &= N_{k+1}^{\perp\perp}N_{k+1}, \quad M_1 = \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}F^{\perp}\bar{C}\\ N_{k+1} &= T_k\begin{pmatrix}M_k\bar{A}\\ \bar{C}\end{pmatrix}, T_k = \begin{pmatrix}M_k\bar{D}\\ \bar{F}\end{pmatrix}^{\perp}\end{aligned} \quad (7)$$

Notice that $N_{k+1}^{\perp\perp}$ excludes the linearly dependent rows of N_{k+1}, so M_{k+1} has full row rank. Defining $\Phi_1 := J_1y$, where $J_1 := \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}\bar{F}^{\perp}$, leads to

$$\Phi_1 = \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}\bar{F}^{\perp}\bar{C}x_{\mathrm{ex}} = M_1x_{\mathrm{ex}} \quad (8)$$

Now with $\Phi_2 := N_2^{\perp\perp}T_1\begin{bmatrix}\frac{d}{dt}M_1x_{\mathrm{ex}}\\ y\end{bmatrix}$, and moving the differentiation operator outside of the parenthesis, the following identity is obtained

$$\Phi_2 = M_2x_{\mathrm{ex}} = \frac{d}{dt}N_2^{\perp\perp}T_1\begin{bmatrix}J_1 & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}y\\ y^{[1]}\end{bmatrix} = \frac{d}{dt}J_2\begin{bmatrix}y\\ y^{[1]}\end{bmatrix}$$

The matrix J_2 is defined by the previous identity. Then, we can generalize the procedure as follows: defining $\Phi_k := N_{\bar{n}}^{\perp\perp}T_{\bar{n}-1}\begin{bmatrix}\frac{d}{dt}M_{\bar{n}-1}x_{\mathrm{ex}}\\ y\end{bmatrix}$ $(k = 2,\ldots,\bar{n}-\rho_{M_1})$, the identity

$$\begin{aligned}\Phi_{k+1} &= M_{k+1}x_{\mathrm{ex}} = \frac{d^k}{dt^k}N_{k+1}^{\perp\perp}T_k\begin{bmatrix}J_k & 0\\ 0 & I\end{bmatrix}Y^{[k]}\\ &= \frac{d^k}{dt^k}J_{k+1}Y^{[k]}\end{aligned} \quad (9)$$

holds, where $J_{k+1} = N_{k+1}^{\perp\perp}T_k\begin{bmatrix}J_k & 0\\ 0 & I\end{bmatrix}$.

The equality rank $M_i = \operatorname{rank} M_{i-1}$ implies rank $M_{i+1} = \operatorname{rank} M_i$. From the previous impilication we can deduce that rank $M_{\bar{n}-\rho_{M_1}+2} = \operatorname{rank} M_{\bar{n}-\rho_{M_1}+1}$, i.e. at most $\bar{n}-\rho_{M_1}$ differentiations are needed, which means that, for a strongly observable Σ (see, e.g. Molinari (1976), Trentelman et al. (2001)), i.e. $\det M_{\bar{n}-\rho_{M_1}+1}^{-1} \neq 0$, the state vector x_{ex} can be expressed by the identity $x_{\mathrm{ex}} = \frac{d^k}{dt^k}M_{\bar{n}-\rho_{M_1}+1}^{-1}J_{\bar{n}-\rho_{M_1}+1}Y^{[\bar{n}-\rho_{M_1}]}$. Nevertheless, two drawbacks appear in order to reconstruct x_{ex} in such a manner. The first drawback has to do with the assumption that the system is strongly detectable but not strongly observable; therefore, a transformation must first be done in order to decompose the system into the strongly observable part and the detectable part (see, e.g. Bejarano et al. (2009)). The second drawback has to do with the fact that, during the implementation of the differentiator, some errors appear due to the computation sample time and noises in the sensors.

It was proven in Bejarano et al. (2009) that a necessary requirement to $\left(\bar{A},\bar{D},\bar{C},\bar{F}\right)$ be strongly detectable is that rank $\begin{bmatrix}M_{\bar{n}}\bar{D}\\ \bar{F}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\bar{D}\\ \bar{F}\end{bmatrix}$; therefore,

$$\operatorname{rank}\begin{bmatrix}M_{\bar{n}}\bar{D} & \bar{F}\\ \bar{F} & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\bar{D} & \bar{F}\\ \bar{F} & 0\end{bmatrix} = \operatorname{rank}\bar{F} + m$$

Hence, the Hautus condition might be satisfied taking $M_{\bar{n}}x_{\mathrm{ex}}$ as a new output of the system. Furthermore, it is easy to check that rank $M_{\bar{n}}\bar{D}_1 = \operatorname{rank}\bar{D}_1$. Thus, we can define $\bar{n}_H$ $(2 \le \bar{n}_H \le \bar{n}-\rho_{M_1}+1)$ as a the least natural number such that the matrix $M_{\bar{n}_H}$, calculated by (7), satisfies the rank condition rank $M_{\bar{n}_H}\bar{D}_1 = \operatorname{rank}\bar{D}_1$, and define the extended output $y_{\bar{n}H} = M_{\bar{n}_H}x_{\mathrm{ex}}$ as the new output of the system. The new output $y_{\bar{n}H}$ will be calculated using the differentiation procedure described above.

Remark 1. Notice that, from (9), the construction of $y_{\bar{n}_H} = M_{\bar{n}_H}x_{\mathrm{ex}}$ can be guaranteed by means of a derivative of order $\bar{n}_H-1$; however, some terms of the vector $y_{\bar{n}_H}$ might exist that could be estimated with a derivative of order less than $\bar{n}_H-1$. Let us exemplify this point, suppose that $y_{\bar{n}_H,j}$, the j-th term of $y_{\bar{n}_H}$, is calculated using (9) and the j-th row of $J_{\bar{n}_H}$ has the form $J_{\bar{n}_H,j} = [\,0\ J_*\,]$, i.e. the first term of $J_{\bar{n}_H,j}$ is zero. Then $y_{\bar{n}_H,j}$ becomes $y_{\bar{n}_H,j} = \frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}}J_{\bar{n}_H,j}Y^{[\bar{n}_H-1]} = \frac{d^{\bar{n}_H-2}}{dt^{\bar{n}_H-2}}J_*Y^{[\bar{n}_H-2]}$, whence we conclude $y_{\bar{n}_H,j}$ can be calculated using a derivative of order $\bar{n}_H-2$ instead of one of order $\bar{n}_H-1$.

In the sequel, we will give a method for calculating a derivative of order $\bar{n}_H-1$; however, it is advised to take into account Remark 1. From (9), the vector $M_{\bar{n}_H}e$ can be expressed as:

$$M_{\bar{n}_H}e = \frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}}\left[J_{\bar{n}_H}Y^{[\bar{n}_H-1]} - \left(M_{\bar{n}_H}\tilde{x}\right)^{[\bar{n}_H-1]}\right]$$

Defining the vector $H(t) = J_{\bar{n}_H}Y^{[\bar{n}_H-1]} - M_{\bar{n}_H}\tilde{x}^{[\bar{n}_H-1]}$, the j-th term of $M_{\bar{n}_H}e = \frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}}H(t)$ is calculated as $\frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}}H_j(t)$ $(j = 1,\ldots,l$; where l is the dimension of $y_{\bar{n}_H})$. Thus, the j term of $M_{\bar{n}_H}e$ can be estimated by means of a sliding mode differentiator of high order which has the form (for more details see Levant (2003)):

$$
\begin{aligned}
\dot{z}_{j,0} &= \lambda_0 \left|z_{j,0} - H_j\right|^{\frac{\bar{n}_H - 1}{\bar{n}_H}} \operatorname{sign}\left(z_{j,0} - H_j\right) + z_{j,1} \\
\dot{z}_{j,1} &= \lambda_1 \left|z_{j,1} - \dot{z}_{j,0}\right|^{\frac{\bar{n}_H - 2}{\bar{n}_H - 1}} \operatorname{sign}\left(z_{j,1} - \dot{z}_{j,0}\right) + z_{j,2} \\
&\vdots \\
\dot{z}_{j,\bar{n}_H-2} &= \lambda_{\bar{n}_H-2} \left|z_{j,\bar{n}_H-2} - \dot{z}_{j,\bar{n}_H-3}\right|^{1/2} \times \\
&\quad \operatorname{sign}\left(z_{j,\bar{n}_H-2} - \dot{z}_{j,\bar{n}_H-3}\right) + z_{j,\bar{n}_H-1} \\
\dot{z}_{j,\bar{n}_H-1} &= \lambda_{\bar{n}_H-1} \operatorname{sign}\left(z_{j,\bar{n}_H-1} - \dot{z}_{j,\bar{n}_H-2}\right)
\end{aligned}
$$

It was shown in Levant (2003) that, with the proper choice of constants λ_i $(i = 0, \cdots, \bar{n}_H - 1)$, there exists a finite time t_j such that the identity $z_{j,\bar{n}_H-1}(t) = \frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}} H_j(t)$ is achieved for all $t \geq t_j$. Thus, every term of $y_{\bar{n}_H}$ can be calculated using a sliding mode differentiator of high order. λ_i can be calculated in the following form, $\lambda_i = \lambda_{0i} K^{1/(\bar{n}_H - i)}(t)$, where $K(t)$ is a local Lipschitz constant for $\frac{d^{\bar{n}_H-1}}{dt^{\bar{n}_H-1}} H_j(t) = M_{\bar{n}_H} e$ and λ_{0i} is calculated for the case when $K(t) = 1$ (λ_{0i} can be calculated using simulations). A value of λ_{0i} $(i = 0, \cdots, \bar{n}_H - 1)$ for a fifth order differentiator was given in Levant (2006), with $\lambda_{00} = 12$, $\lambda_{01} = 8$, $\lambda_{02} = 5$, $\lambda_{03} = 3$, $\lambda_{04} = 1.5$, and $\lambda_{05} = 1.1$.

Thus, defining the vector $z_{\bar{n}_H-1} = \begin{bmatrix} z_{1,\bar{n}_H-1} & \cdots & z_{l,\bar{n}_H-1} \end{bmatrix}^T$, we achieve the identity $z_{\bar{n}_H-1} = M_{\bar{n}_H} e$, and consequently, the identity

$$
\hat{y}_{\bar{n}_H} := z_{\bar{n}_H-1} + M_{\bar{n}_H} \tilde{x} = M_{\bar{n}_H} x_{\text{ex}} = y_{\bar{n}_H} \tag{10}
$$

holds for all $t \geq \max_{i=\overline{1,l}} T_j > T$.

A function $K(t)$ required by the differentiator might be calculated in the following manner.

Lemma 2. Under assumptions A1 and A2, there exist a time $\tilde{t}$ and known positive constants γ, φ, ξ, and δ such that, for $t > \tilde{t}$,

$$
\begin{aligned}
\left\| \frac{d}{d} M_{\bar{n}_H} e(t) \right\| &\leq K(t) = \gamma + \xi w_M^+(t) \\
&\quad + \varphi \int_0^t e^{-\delta(t-\tau)} w_M^+(\tau)\, d\tau
\end{aligned}
$$

with $w_M^+(t) = \max_{i=\overline{0,r}} \left(w_i^+(t)\right)$.

Proof. Since $\tilde{A}$ is Hurwitz, the exponential matrix $e^{\tilde{A}(t-t_0)}$ $(t \geq t_0)$ has a bounded norm, i.e. $\left\|e^{\tilde{A}(t-t_0)}\right\| \leq \bar{\varphi} e^{-\delta(t-t_0)}$ for known positive constants $\bar{\varphi}$, δ. Thus, from the solution of (6), we have that

$$
\begin{aligned}
\|e(t)\| &\leq \left\|e^{\tilde{A}(t-t_0)}\right\| \|e(0)\| \\
&\quad + \left\|\bar{D}_1\right\| \left\|Q_{\bar{F}}^{-1}\right\| \int_0^t \left\|e^{\tilde{A}(t-\tau)}\right\| w_M^+(t) \\
&\leq \bar{\varphi} e^{-\delta(t-t_0)} \|e(0)\| \\
&\quad + \bar{\varphi} \left\|\bar{D}_1\right\| \left\|Q_{\bar{F}}^{-1}\right\| \int_0^t e^{-\delta(t-t_0)} w_M^+(t)
\end{aligned}
$$

Thus, after a finite time $\tilde{t}$, the term $\bar{\varphi} e^{-\delta(t-t_0)} \|e(0)\|$ will be less than any constant $\bar{\gamma} = \bar{\gamma}\left(\tilde{t}\right)$. Hence, we obtain

$$
\|e(t)\| < \gamma + \tilde{\varphi} \int_0^t e^{-\delta(t-\tau)} w_M^+(t)\, d\tau
$$

with $\tilde{\varphi} = \bar{\varphi} \left\|\bar{D}_1\right\| \left\|Q_{\bar{F}}^{-1}\right\|$. Which in turn yields the inequality

$$
\begin{aligned}
\left\| \frac{d}{d} M_{\bar{n}_H} e \right\| &\leq \left\| M_{\bar{n}_H} \tilde{A} \right\| \left(\bar{\gamma} + \tilde{\varphi} \int_0^t e^{-\delta(t-\tau)} w_M^+(t)\, d\tau \right) \\
&\quad + \left\|M_{\bar{n}_H}\right\| \left\|\bar{D}_1\right\| \left\|Q_{\bar{F}}^{-1}\right\| w_M^+(t)
\end{aligned}
$$

Thus, the lemma is proven with $\gamma = \bar{\gamma} \left\| M_{\bar{n}_H} \tilde{A} \right\|$, $\varphi = \tilde{\varphi} \left\| M_{\bar{n}_H} \tilde{A} \right\|$, and $\xi = \left\|M_{\bar{n}_H}\right\| \left\|\bar{D}_1\right\| \left\|Q_{\bar{F}}^{-1}\right\|$. ■

4. ASYMPTOTIC OBSERVER

With $y_{\bar{n}_H}$ estimated exactly by means of $\hat{y}_{\bar{n}_H}$. The system with its new output takes the form

$$
\begin{aligned}
\dot{x}_{\text{ex}} &= \left(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}\right) x_{\text{ex}} + \bar{D}_1 \bar{w}_{\text{ex},1} + \bar{D}_2 \bar{F}_2^+ y \\
\hat{y}_{\bar{n}_H} &= M_{\bar{n}_H} x_{\text{ex}}
\end{aligned}
$$

Since $M_1 = \left(\bar{F}^\perp \bar{C}\right)^{\perp\perp} \bar{F}^\perp \bar{C}$, it means that $M_{\bar{n}_H} x_{\text{ex}} = 0$ implies $\bar{F}^\perp \bar{C} x_{\text{ex}} = 0$, and since $(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}, \bar{F}^\perp \bar{C}, \bar{D} \bar{F}_\perp)$ is strongly detectable (the strong detectability is not lost by output injection), the triple $(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}, M_{\bar{n}_H}, \bar{D} \bar{F}_\perp)$ is strongly detectable as well. In fact, it can be proven that the converse is true as well.

Now let us make a change of coordinates to design an asymptotic observer. Let the state and output transformations be given by means of the matrices T and U, respectively, defined as follows:

$$
T = \begin{bmatrix} \bar{D}_1^\perp \\ \left(M_{\bar{n}_H} \bar{D}_1\right)^+ M_{\bar{n}_H} \end{bmatrix}, \; U = \begin{bmatrix} \left(M_{\bar{n}_H} \bar{D}_1\right)^\perp \\ \left(M_{\bar{n}_H} \bar{D}_1\right)^+ \end{bmatrix} \tag{11}
$$

where the inverse matrices are

$$
\begin{aligned}
T^{-1} &= \left[\left[I - \bar{D}_1 \left(M_{\bar{n}_H} \bar{D}_1\right)^+ M_{\bar{n}_H} \right] \bar{D}_1^{\perp+} \;\; \bar{D}_1 \right] \\
U^{-1} &= \left[\left(I - M_{\bar{n}_H} \bar{D}_1 \left(M_{\bar{n}_H} \bar{D}_1\right)^+ \right) \left(M_{\bar{n}_H} \bar{D}_1\right)^{\perp+} \;\; M_{\bar{n}_H} \bar{D}_1 \right]
\end{aligned}
$$

Thus, the equations governing the dynamics of the system, in the new coordinates $z = T x_{\text{ex}}$, and $\bar{y} = U y_{\bar{n}_H} = U M_{\bar{n}_H} x_{\text{ex}}$, take the form:

$$
\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{w}_{\text{ex},1} \end{bmatrix} \\
&\quad + \begin{bmatrix} \bar{D}_1^\perp \\ \left(M_{\bar{n}_H} \bar{D}_1\right)^+ M_{\bar{n}_H} \end{bmatrix} \bar{D}_2 \bar{F}_2^+ y
\end{aligned} \tag{12a}
$$

$$
\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} C_1 z_1 \\ z_2 \end{bmatrix} \tag{12b}
$$

where $z_2 \in \mathbb{R}^m$.

Lemma 3. $\left(\bar{A}, \bar{D}, \bar{C}, \bar{F}\right)$ is strongly detectable if, and only if, the pair (A_1, C_1) is detectable.

Proof. Let $\Sigma_{M_{n_H}}$ be the system associated to the triple $\left(\bar{A} - \bar{D}_2 \bar{F}_2^+ \bar{C}, M_{n_H}, \bar{D} \bar{F}_\perp\right)$. It is easy to check the following identity,

$$
\begin{aligned}
&\begin{bmatrix} sI - A_1 & -A_2 & 0 \\ -A_3 & sI - A_4 & I \\ C_1 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \\
&= \operatorname{diag}(I, U) \operatorname{diag}(T, I) R_{\Sigma_{M_{n_H}}}(s) \operatorname{diag}\left(T^{-1}, I\right).
\end{aligned}
$$

Thus, $\operatorname{rank} R_{\Sigma_{M_{n_H}}}(s_0) < \bar{n} + m - r$ if, and only if, $\operatorname{rank} H_{\mathrm{T}}(s_0) < \bar{n} - m$, where $H_{\mathrm{T}}(s) = \begin{bmatrix} sI - A_1 \\ C_1 \end{bmatrix}$. Therefore, since $\left(\bar{A} - \bar{D}_2 \bar{F}_2^{+} \bar{C}, M_{\bar{n}_H}, \bar{D}\bar{F}_{\perp}\right)$ is strongly detectable if, and only if, $\left(\bar{A}, \bar{C}, \bar{D}, \bar{F}\right)$ is strongly detectable, we have that $\operatorname{rank} R_{\bar{\Sigma}}(s_0) < \bar{n} + m$ if, and only if, $\operatorname{rank} H_{\mathrm{T}}(s_0) < \bar{n} - m$, and the lemma is proven. ■

Thus, the design of the asymptotic observer can be formulated in two steps:

(1) If $\operatorname{rank} \bar{F}^{\perp}\bar{C}\bar{D}_1 = \operatorname{rank} \bar{D}_1$ set $\bar{n}_H = 1$ and $\hat{y}_{\bar{n}_H} = \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp} \bar{F}^{\perp}\bar{C}x = J_1 y$, and go to step 2; otherwise, generate the extended output $\hat{y}_{\bar{n}_H} = M_{\bar{n}_H} x$ by means of (10).

(2) For the system

$$\dot{x}_{\mathrm{ex}} = \left(\bar{A} - \bar{D}_2 \bar{F}_2^{+} \bar{C}\right) x_{\mathrm{ex}} + \bar{D}_1 \bar{w}_1 + \bar{D}_2 \bar{F}_2^{+} y$$
$$\hat{y}_{\bar{n}_H} = M_{\bar{n}_H} x_{\mathrm{ex}},$$

which corresponds to triple $(\bar{A} - \bar{D}_2 \bar{F}_2^{+} \bar{C}, M_{\bar{n}_H}, \bar{D}_1)$, construct an asymptotic observer based on the design described in this subsection, that is,

(a) with the change of coordinates $z = T x_{\mathrm{ex}}$ and $\bar{y} = U M_{\bar{n}_H} x_{\mathrm{ex}}$, system (12) is obtained;

(b) since $\bar{y}_2 \equiv z_2$, state z_1 can be observed from output $\bar{y}_1$ using a usual Luenberger observer:

$$\dot{\hat{z}}_1 = A_1 \hat{z}_1 + L_1 \left(\bar{y}_1 - C_1 \hat{z}_1\right) + A_2 \bar{y}_2 + \bar{D}_1^{\perp} \bar{D}_2 \bar{F}_2^{+} y \tag{13}$$

and the observer for the original system is designed as

$$\hat{x}_{\mathrm{ex}} := T^{-1} \begin{bmatrix} \hat{z}_1 \\ \bar{y}_2 \end{bmatrix}. \tag{14}$$

Thus, the main result can be summarized through the following theorem.

Theorem 4. Assuming that (A, C, D, F) is strongly detectable and $\bar{L}$ is designed so that $(A_1 - L_1 C_1)$ is Hurwitz, then observer $\hat{x}_{\mathrm{ex}}$, designed using the two-step procedure given above, converges asymptotically to the state vector x_{ex}.

Proof. It comes directly from lemma (3) and comparing (12a) with (13). ■

Remark 2. For the case when $K(t) = K$, we can compare the precision of the observer proposed in this paper with respect to already published observers. In the observer designed in Bejarano et al. (2007) the precision of the observer with respect to a Lebesgue measurable and bounded noises appearing in the system output $\|\eta(t)\| \leq \epsilon$ is of order $O\left(\epsilon^{\frac{1}{2^k}}\right)$ where k is the number of times to derive to reconstruct the extended state vector (x and w), i.e., $\operatorname{rank}(M_k) = \bar{n}$. Meanwhile, the precision with respect to a sampling time τ is of order $O\left(\tau^{\frac{1}{2^{k-1}}}\right)$. This error is due to the proper design of the observer which is based in the consecutive use of a second order differentiator (super-twisting algorithm).

The observer designed in Fridman et al. (2007) has a precision of order $O\left(\left(\epsilon^{\frac{1}{r}}\right)^{\frac{1}{2^{k-r}}}\right)$ w.r.t. noise and of order $O\left(\tau^{\frac{1}{2^{k-r}}}\right)$ w.r.t. the sample time τ, where r is the maximum of the terms of the vector of relative degrees of the system output with respect to the unknown inputs.

At difference with the previous two observer, the observer design proposed here only needs to use once an exact sliding mode differentiator of order $(\bar{n}_H - 1)$ which in general is less than k. Thus according to Levant (2003), the precision of the observer w.r.t. noise is of order $O\left(\epsilon^{\frac{1}{\bar{n}_H - 1}}\right)$ and w.r.t. sample time is $O(\tau)$.

5. SIMULATIONS

Consider that the matrices of the system are the following.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D_{11} = D_{51} = D_{91} = D_{82} = 1, \text{ otherwise } D_{ij} = 0$$
$$C_{11} = C_{25} = C_{39} = C_{48} = 1, \text{ otherwise } C_{ij} = 0$$
$$F_{41} = 1, \text{ otherwise } F_{ij} = 0$$

The components of $w(t)$ are: $w_1 = -0.2t + \sin(2t+3) + 0.3\cos(2t) + 0.1t^2 - 6$, $w_2 = 0.6t\sin(0.6t) - 0.4\sin\left(0.4t^2\right) - 0.5t + 4$, $w_3 = 0.1 + 0.5t - 0.5cos(2t) - 0.2t^2$. The extended state vector is $x_{\mathrm{ex}} = \begin{pmatrix} x^T & w^T \end{pmatrix}^T$. Since $\operatorname{rank} M_{12} = 11$, and M_1 to M_5 have 4, 7, 9, 10, and 11 rows, respectively, a derivative of fourth order should be estimated online if a differentiation process would be followed to estimate the state vector $x_{\mathrm{ex}}(t)$. However, to satisfy the Hautus condition, only a second order derivative must be carried out, and so the observer proposed in (14) can be designed. The sample time used was 10^{-4}. Figs. 1 to 3 show the observation error for the vector state $x_{\mathrm{ex}}(t)$. The unknown inputs and their respective estimation are shown in fig. 4.

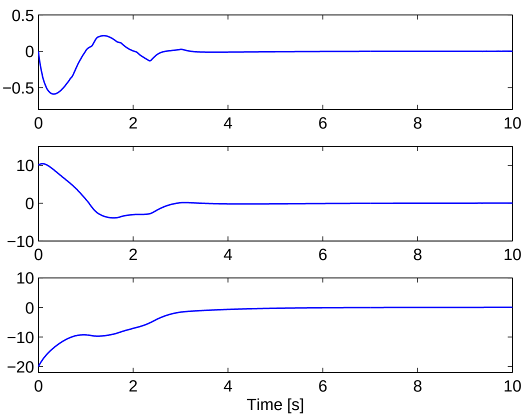

Fig. 1. Estimation error $e_i = x_{\mathrm{ex},i} - \hat{x}_{\mathrm{ex},i}$ $(i = 1, 2, 3)$.

6. CONCLUSIONS

In this paper it is proposed a new observer for linear systems with unknown inputs. Here we assumed the struc-

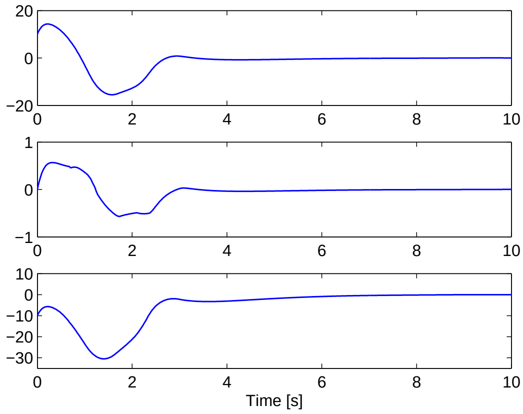

Fig. 2. Estimation error $e_i = x_{\text{ex},i} - \hat{x}_{\text{ex},i}$ $(i = 4, 5, 6)$.

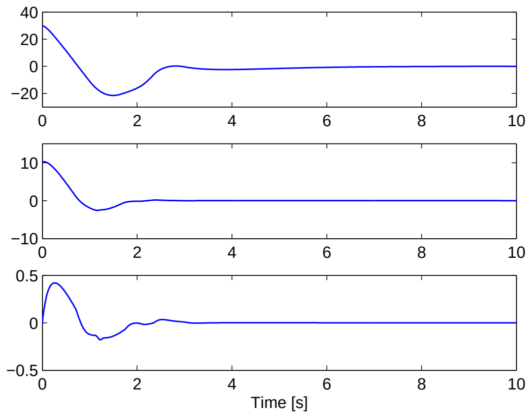

Fig. 3. Estimation error $e_i = x_{\text{ex},i} - \hat{x}_{\text{ex},i}$ $(i = 7, 8, 9)$.

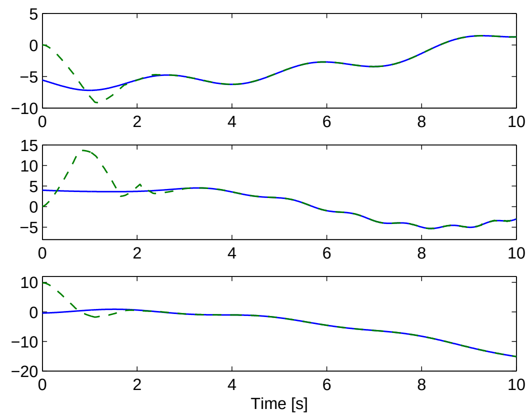

Fig. 4. Unknown inputs and their respective estimation (dashed).

tural condition of strongly detectability and the assumption that each unknown inputs has a high order derivative that is bounded by a real valued function. We proposed to use a differentiation process that is stopped when the Hautus condition is satisfied. Thus, an asymptotic observer can be designed. Using the proposed method one should differentiate just the number of times needed for the estimation of the state and the unknown inputs.

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