Preserving Synchronization Under Modifications to Associated Characteristic Polynomials

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Abstract— **In this paper our aim is to show the viability of preserving synchronization of a chaotic system under specific modifications to its Jacobian matrix's associated characteristic polynomial. Furthermore we shall provide evidence to show that linear methods used to achieve said synchronization between master and slave systems, such as passive and active control, assure the preservation of synchronization after undergoing the same transformations. We propose to modify the coefficients of the associated characteristic polynomial by calculating their value to the w-th power, with $w \in \mathbb{R}^+ - \{0\}$. To illustrate the results we present several examples of a well known modified chaotic attractor.**

Keywords: Control, Synchronization, Nonlinear systems, Output feedback.

I. INTRODUCTION

The problem of stability and synchronization preservation has been recently addressed for the case of hyperbolic, nonlinear systems with chaotic dynamics in (Fernández-Anaya *et al.*, 2007) and (Becker-Bessudo *et al.*, 2008). Results reported in these articles deal with strictly linear modifications, i.e. constant term matrix multiplication. Based on these results the goal has been to develop further studies in the field of stability and synchronization preservation for modified dynamical systems. One of the advances we have looked into has been the use of *nonlinear* modifications over the linear part of these systems. The pursuit of this line of thought has involved the development of new criteria as to the extent to which the system's stability, hyperbolic points and synchronizability are preserved under such transformations. This has in turn led to some unusual techniques in the design of state feedback controllers that would allow synchronization in such cases.
For the authors, this line of thought led to the positive results presented in (Becker-Bessudo *et al.*, 2009) where the proposed nonlinear modification consisted in calculating the m-th power, where m is an odd positive integer, of the coefficients of the Jacobian matrix's characteristic polynomial. The intent of this paper is to show some preliminary results derived from ongoing research following these criteria.

II. MATHEMATICAL PRELIMINARIES

A. Suitable Polynomials

Any third degree polynomial function with positive definite coefficients may be defined by either of these equations

$$p_1(s) = (s+a)(s+b)(s+c) \tag{1}$$
$$p_2(s) = (s+d)((s+e)^2+f^2) \tag{2}$$

noting that $\{a, b, c, d, e, f\} \in \mathbb{R}$. Carrying out the products in (1) and (2) and grouping the terms of each power of s we obtain

$$p_1(s) = s^3 + (a+b+c)s^2 + (ab+bc+ac)s + abc \tag{3}$$
$$p_2(s) = s^3 + (d+2e)s^2 + (2de+e^2+f^2)s + d(e^2+f^2) \tag{4}$$

Since the lemma and remarks presented in this paper require that all coefficients of a third degree polynomial function are positive definite and none of their roots posses real parts equal to zero, we must impose several conditions over them. In the case of strictly real roots the conditions are trivial since they must satisfy $a+b+c > 0$, $ab+bc+ac > 0$ and $abc > 0$. For the latter case we may infer the following conditions $d > 0$, $d > -2e$ and $2de > -(e^2+f^2)$.

III. HYPERBOLICITY PRESERVATION

For the following discussion consider the dynamical system described by

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous differentiable function of its argument. Let $A = \left.\frac{\partial f}{\partial x}\right|_{x_0}$ be the Jacobian matrix associated with f evaluated at an equilibrium point x_0.

Lemma 1: If $p(s)$, the characteristic polynomial of our Jacobian matrix, is a real third degree polynomial function, and there are no zeros in the sequence

$$\Delta_1, \Delta_2, \Delta_3$$

of leading principal minors of the corresponding Hurwitz matrix (Lancaster y Tismenetsky, 1985), then $p(s)$ has no pure imaginary zeroes, i.e. the equilibrium point is a hyperbolic point with inertia

$$\nu(p(s)) = k$$
$$\pi(p(s)) = 3 - k$$
$$\delta(p(s)) = 0$$

where the inertia of the matrix is defined as the number of negative (ν), positive (π) and zero (δ) real part of the eigenvalues of the Jacobian matrix.

Now consider the modified $p(s)$ polynomial as

$$p_w(s) = s^3 + a_2^w s^2 + a_1^w s + a_0^w$$

with $w \in \mathbb{R}^+ - \{0\}$. Then the inertia of this new polynomial's associated Hurwitz matrix is the same as the original polynomial's

$$\nu(p_w(s)) = \nu(p(s)) = k$$
$$\pi(p_w(s)) = \pi(p(s)) = 3 - k$$
$$\delta(p_w(s)) = \delta(p(s)) = 0$$

■

Having established all these conditions we may define the sets Ψ and Γ as follows

$$\Gamma = \left\{ p(s) \left| \begin{array}{l} p(s) = s^3 + a_2 s^2 + a_1 s + a_0 \\ \text{where } a_2, a_1, a_0 \in \mathbb{R}^+ - \{0\} \end{array} \right. \right\}$$

$$\Psi = \left\{ w \, \middle| w \in \mathbb{R}^+ - \{0\} \right\}$$

this makes it possible to define the group action Ω of group Ψ over the set Γ as follows

$$\Omega : \Psi \times \Gamma \to \Gamma$$

Remark 1:

1) The group Ψ satisfies the group axioms since the operation does not alter the sign value of the associated coefficients.
2) The action is faithful since no two different elements of Ψ acting on the same set of coefficients will yield the same polynomial.
3) The action is free since our stabilizer (in the group action sense) is trivially the neutral element, i.e. $w = 1$.

Remark 2: We find that in the case where w is an odd integer (presented in (Becker-Bessudo *et al.*, 2009)) the action of a monoid over the set of real polynomials can be shown and when $w \in \mathbb{R}^+ - \{0\}$, as presented here, defines a group action over polynomials with positive definite coefficients.

IV. Preservation of Synchronization in Modified Systems

In this section we show how it is possible to preserve synchronization after the system's eigenvalues have been modified under the action of the class of transformation on the linear part of the nonlinear system described in section III. These transformations may be seen as changes in the parameters of the original system and can lead anywhere from simple scaling of the state outputs to changes in the dynamics of the system itself.

Consider the following *3*-dimensional systems in a master-slave configuration, where the master system and slave systems are given by

$$\dot{x}_m = Ax_m + f(x_m); \quad y_m = Cx_m \tag{5}$$
$$\dot{x}_s = Ax_s + g(x_s) + u(t); \quad y_s = Cx_s \tag{6}$$

where $A \in \mathbb{R}^{3\times3}$ is a constant matrix, x_m and x_s are the states of the master and slave systems, respectively, y_m and y_s are the measurable outputs which are linear combinations of their respective states, $C \in \mathbb{R}^3$, $f, g : \mathbb{R}^3 \to \mathbb{R}^3$ are continuous nonlinear functions, $u \in R^3$ is the control input. The problem of synchronization considered in this section is the complete-state exact synchronization. That is, the master system and the slave system are synchronized by designing an appropriate nonlinear state feedback control $u(t)$ which is attached to the slave system such that

$$\lim_{t\to\infty} \|x_s - x_m\| \to 0$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

Considering the error state vector $e = x_s - x_m \in \mathbb{R}^3$, $g(x_s) - f(x_m) = L(x_m, x_s)$ and an error dynamics equation

$$\dot{e} = Ae + L(x_m, x_s) + u(t).$$

Based in the active control approach (Bai y Lonngren, 2000), to eliminate the nonlinear part of the error dynamics, and choosing $u(t) = v(t) - L(x_m, x_s)$, we obtain $\dot{e} = Ae + v(t)$ then, by choosing $v(t) = KC(y_m - y_s)$, we render the error dynamics equation

$$\dot{e} = (A - KC)e \tag{7}$$

Notice that the synchronization problem is equivalent to the problem of stabilizing the zero-input solution of the last system by a suitable choice of the state feedback.

One such suitable choice for state feedback is a linear quadratic gaussian observer (LQG) (Anderson y Moore, 1990), which is a well-known design technique that provides practical feedback gains. If the pair (A, C) is observable it guarantees the existence of a matrix K, such that $(A - KC)$ is a Hurwitz matrix.

Now consider $w \in \Psi$ and suppose that the following two 3-dimensional systems are chaotic for some $f, g : R^3 \to$

R^3 continuous nonlinear functions and $\widehat{u} \in R^3$ is the control input.

$$\dot{x}_m = A_w x_m + g(x_m)$$
$$\dot{x}_s = A_w x_s + f(x_s) + \widehat{u}(t)$$

Now, suppose that $\widehat{u}(t) = \widehat{\theta}(t) - L(x_m, x_s)$ stabilizes the zero solution of the error dynamics system, where $\widehat{\theta}(t) = -(KC)_w e$, i.e., the resultant system

$$\dot{e} = A_w e + \widehat{\theta}(t)$$
$$\dot{e} = (A_w - (KC)_w) e$$

is asymptotically stable thus preserving the stability of the error dynamics equations and the synchronization of the modified master/salve pair. The process to find the modified matrices A_w and K_w was based on the the Controllable Canonical Form Theorem procedures presented in (Becker-Bessudo *et al.*, 2009). It is important to note that the modification method and values applied to the original system are also applied to the state feedback matrix KC. Otherwise we cannot insure asymptotical synchronization of the modified system pair.

V. Simulations

A. *Sprott Q System*

The dynamical system used in the following simulations is the well known Sprott Q attractor. It was chosen because its Jacobian matrix's characteristic polynomial, evaluated at the equilibrium point $\mathbf{x}_0 = [0, 0, 0]$, fulfills the required conditions established in section II. The system is described by the following equations

$$\dot{x}_{m1} = -x_{m3}$$
$$\dot{x}_{m2} = x_{m1} - x_{m2}$$
$$\dot{x}_{m3} = 3.1x_{m1} + x_{m2}^2 + 0.5x_{m3}$$

The master system is given by the aforementioned equations and the slave system is a copy of the master system plus a control function $u(t)$ to be determined in order to synchronize the two systems

$$\dot{x}_{s1} = -x_{s3} + u_1(t)$$
$$\dot{x}_{s2} = x_{s1} - x_{s2} + u_2(t)$$
$$\dot{x}_{s3} = 3.1x_{s1} + x_{s2}^2 + 0.5x_{s3} + u_3(t)$$

Considering the errors $e_1 = x_{s1} - x_{m1}$, $e_2 = x_{s2} - x_{m2}$, $e_3 = x_{s3} - x_{m3}$, then the error dynamics for the master/slave system configuration may be written as

$$\dot{e}_1 = -e_3 + u_1(t)$$
$$\dot{e}_2 = e_1 - e_2 + u_2(t)$$
$$\dot{e}_3 = 3.1e_1 + x_{s2}^2 - x_{m2}^2 + 0.5e_3 + u_3(t)$$

The resulting Jacobian (A) and non-linear terms (L) matrices are

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 3.1 & 0 & 0.5 \end{pmatrix}, \; L(x_m, x_s) = \begin{pmatrix} 0 \\ 0 \\ x_{s2}^2 - x_{m2}^2 \end{pmatrix},$$
$$u = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$

We select the matrix C such that (A, C) is observable: $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Now using the dual system (Chen, 1984), and the LQG controller design, with weighting matrices $Q = I$ and $R = 1$, we obtain the matrix $K = \begin{bmatrix} 0.4142 & 0.5012 & 2.454 \end{bmatrix}^\top$, such that

$$KC = \begin{pmatrix} 0 & 0 & 0.4142 \\ 0 & 0 & 0.5012 \\ 0 & 0 & 2.454 \end{pmatrix}$$

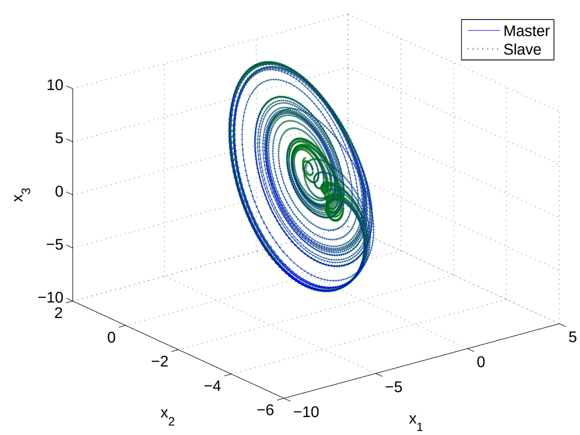

Figure 1. Master and slave systems (initial conditions $x_{m1} = 0.1, x_{m2} = 0.1, x_{m3} = 0.1$ and $x_{s1} = 0.4, x_{s2} = 0.4, x_{s3} = 0.4$ respectively) showing synchronization of Sprott Q attractor.

In Fig. 1 we can appreciate the master/slave pair of the original Sprott Q attractor and in Fig. 2 we see semi-logarithmic graph of the state's absolute errors from which we can determine that they have achieved synchronization as there is an effective convergence to zero between the error of all the state variables of both systems.

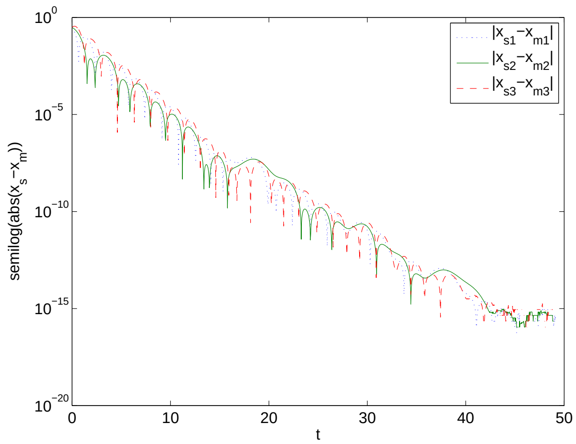

Figure 2. Magnitude of error $|e| = |x_s - x_m|$ between master and slave systems.

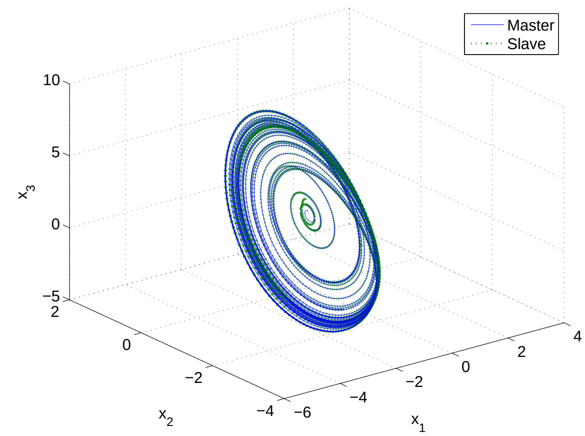

Figure 3. Master and slave systems (initial conditions $x_{m1} = 0.1, x_{m2} = 0.1, x_{m3} = 0.1$ and $x_{s1} = 0.4, x_{s2} = 0.4, x_{s3} = 0.4$ respectively) showing synchronization of modified ($w = 0.8$) Sprott Q attractor.

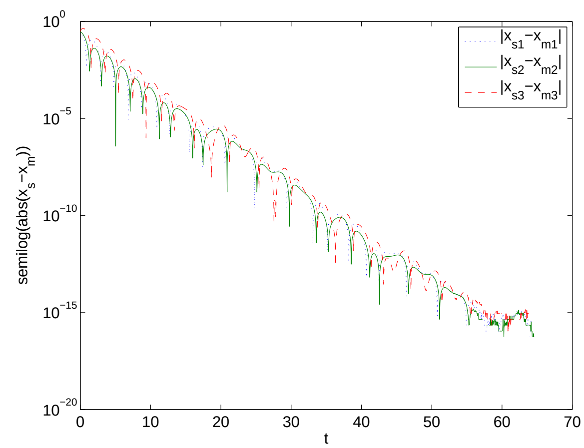

Figure 4. Magnitude of error $|e| = |x_s - x_m|$ between modified master and slave systems.

B. Modified Sprott Q System

The explicit method employed to obtain a system's Jacobian and feedback matrices after their characteristic polynomials have been modified is similar to the one described in (Becker-Bessudo *et al.*, 2009) and hence will not be shown here in detail, we will merely deal with the resulting matrices derived from it. The method is based on the Controllable Canonical Form (Åström y Wittenmark, 1990) which allows us to reconstruct a new matrix (A_w) based on the modified characteristic polynomial.

Consider a modification of the characteristic polynomial's coefficients of the Sprott Q system's Jacobian matrix (A) and state feedback matrix (KC) using $w = 0.8$. The resulting Jacobian ($A_{w=0.8}$) and state feedback ($KC_{w=0.8}$) matrices after modification are

$$A_{w=0.8} = \begin{pmatrix} -0.1734 & -0.0405 & -0.9010 \\ 1.0000 & -1.0000 & 0 \\ 2.9266 & -0.0405 & 0.5990 \end{pmatrix},$$

$$KC_{w=0.8} = \begin{pmatrix} -0.3638 & -0.2122 & 0.1283 \\ 0 & 0 & 0.5012 \\ -0.3638 & -0.2122 & 2.1681 \end{pmatrix}$$

Simulation and errors of this new master/slave system can be seen in the following figures

In Fig. 4 we have the semi-logarithmic absolute error plot of the master/slave system configuration after its characteristic polynomial and feedback matrix have been modified. We can appreciate a longer time span until synchronization occurs but still there is an effective convergence to zero for the error of the master/slave pair for all states.

Looking at the modified attractor in Fig. 3, as far as we can see, the chaotic dynamics are preserved.

C. Importance of Controller Modification

To illustrate the importance of employing the modification methodology on both the system and the control we refer to the simulations in Figs. 5, 6, 7 and 8 where we see two identical systems that have been identically modified. However in Figs. 5 and 6 the applied controller has been modified ($KC_{w=3.6}$), whereas in Figs. 7 and 8 the original controller (KC) was used.

Comparing both cases it is immediately clear that the original controller was not capable of stabilizing the error dynamics of the modified system while the modified controller successfully synchronized the slave system. We feel it is important to note, given this evidence, that we are not concerned with finding or proving robustness of the controller used in these examples but rather the tools that will allow us to preserve synchronization under the proposed modification.

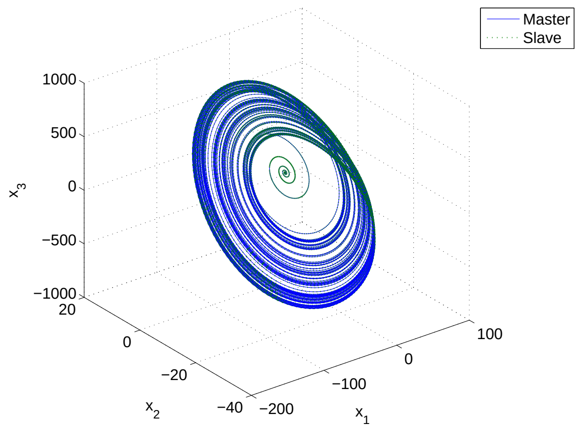

Figure 5. Master and slave systems (initial conditions $x_{m1} = 0.1, x_{m2} = 0.1, x_{m3} = 0.1$ and $x_{s1} = 0.4, x_{s2} = 0.4, x_{s3} = 0.4$ respectively) showing synchronization of modified ($w = 3.6$) Sprott Q attractor.

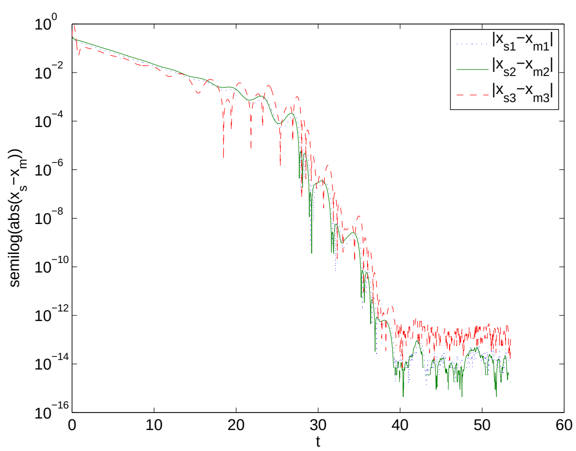

Figure 6. Magnitude of error $|e| = |x_s - x_m|$ between modified master and slave systems.

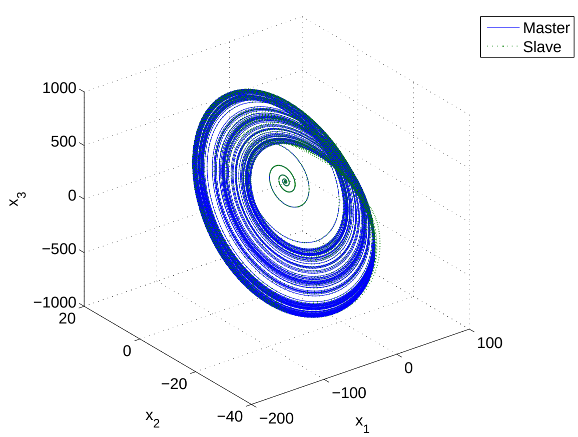

Figure 7. Master and slave systems (initial conditions $x_{m1} = 0.1, x_{m2} = 0.1, x_{m3} = 0.1$ and $x_{s1} = 0.4, x_{s2} = 0.4, x_{s3} = 0.4$ respectively) of modified ($w = 3.6$) Sprott Q attractor.

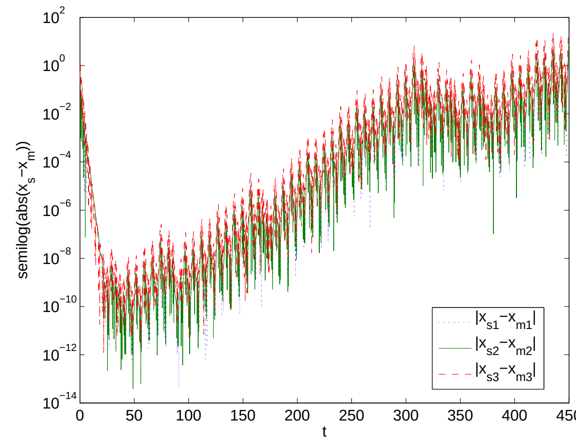

Figure 8. Magnitude of error $|e| = |x_s - x_m|$ between modified master and slave systems.

VI. Conclusions

The preservation of stability (hyperbolic behavior) in chaotic synchronization is based in the preservation of the signature of the linear part of the vector fields in nonlinear dynamical systems. Given this basic premise we set specific constraints for the possible nonlinear modifications and systems they can be performed to. Having established a viable transformation, i.e. power modification of the coefficients of the characteristic polynomial associated with the Jacobian matrix, we designed a control law that would allow us to preserve synchronization under the same transformation. These results present a significant difference to previous *linear* modification methods like the ones seen in (Becker-Bessudo *et al.*, 2008) as well as an important extension on previous results concerning nonlinear modifications, as presented in (Becker-Bessudo *et al.*, 2009). The effectiveness of the proposed method can be ascertained by the results seen in the simulations.

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