Synchronization problem via an asymptotic polynomial observer

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Abstract— **In this paper a new observer is proposed for the synchronization problem, this new observer is an asymptotic polynomial observer for a class of nonlinear oscillators which turns out be robust against output noises. Furthermore this observer is of high order polynomial type. Stability analysis in a Lyapunov sense for the synchronization error is performed. The proposed methodology is applied to synchronization of chaotic systems with success: the performance of this observer is shown by using Rössler system.**

Keywords: Nonlinear systems, state observers, Riccati equation, synchronization.

I. INTRODUCTION

In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields, for instance in (Fradkov, 2007) and (Chen et al., 2005). It is well known that study of the synchronization problem for nonlinear systems has been very important for nonlinear science, in particular the applications to biology, medicine, cryptography, secure data transmission and so on. In general, the synchronization research has been focused onto two areas. The first one relates with the employ of state observers, where the main applications lies on the synchronization of nonlinear oscillators (Hua and Guan, 2005) and (Martínez-Guerra et al., 2006). The second one is the use of control laws, which allows achieve the synchronization with different structure and order between nonlinear oscillators (Femat and Solis-Perales, 2008). A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master- slave configuration (Pecora and Caroll, 1990). Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer.

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general indeed, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances.

The design of observers for nonlinear systems is a challenging problem (even for accurately known systems) that has received a considerable amount of attention. Since the observers developed by Kalman and Luenberger several years ago for linear systems, different state observation techniques have been proposed to handle the systems nonlinearities. A first category of techniques consists in applying linear algorithms to the system linearized around the estimated trajectory. These are known as the extended Kalman and Luenberger observers. Alternatively, the nonlinear dynamics are split into a linear part and a nonlinear one. The observer gains are then chosen large enough so that the linear part dominates the nonlinear one. Such observers are known as high-gain observers (Aguilar et al., 2003), (Martínez-Guerra et al., 2000). In a third approach the nonlinear system is transformed into a linear one by an appropriate change of coordinates (Keller, 1987). The estimate is computed in these new coordinates and the original coordinates are recovered through the inverse transformation. In most approaches, nonlinear coordinate transformations are employed to transform the nonlinear system into a block triangular observer canonical form. Then, high gain (Gauthier et al, 1992), backstepping (Young and Farrel, 2000), or sliding mode observers (Levant, 2001) can be designed.

Many problems in engineering and other applications are globally Lipschitz for instance the sinusoidal terms in robotics. Nonlinearities which are square or cubic in nature are not globally Lipschitz, however, they are locally so, moreover when such functions occur in physical systems, they frequently have a saturation in their growth rate, making them globally Lipschitz functions (Raghavan, 1994). Thus, this class of systems covered by this note is fairly general.

The main contribution of this paper consists in the solution of the synchronization problem via an asymptotic polynomial observer. The obtained state space estimation error is shown to be bounded, and this bound depends on observer's gain and a Lipchitz constant. This communication presents some fundamental insights into polynomial observer design for the class of Lipschitz nonlinear systems, that it means, that any autonomous nonlinear system of the form $\dot{x} = f(x,u)$ can be regarded as Lipschitz continuous system with respect to x, with a Lipchitz constant L.

The intention of choosing an example as the Rössler system is to clarify the proposed methodology. However, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

In what follows, an asymptotic polynomial observer is proposed as well as an easy numerical design is given. Numerical results show its satisfactory performance. Finally we close this paper with some concluding remarks.

II. MAIN RESULT

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \tag{1a}$$

where $x \in \Re^n$ is the vector of the state variables; $f(\circ): \Re^n \times \Re^l \to \Re^n$, $(l \le n)$ is a nonlinear smooth vector function and Lipschitz in x and uniformly bounded in u, $y \in \Re$ is the vector of measured states.

Any nonlinear system of the form (1a) can be expressed in the form (1b) as long as $f(x,u)$ is differentiable with respect to x.

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx \quad , \quad x_0 = x(t_0) \end{aligned} \tag{1b}$$

In system (1b), $\Psi(x,u)$ is a nonlinear vector function which satisfies the Lipschitz condition with a Lipschitz constant L, i.e,

$$\|\Psi(x,u) - \Psi(\hat{x},u)\| \le L\|x - \hat{x}\|$$

In this paper, we always assume that the pair (A,C) is observable.

We have the main result.

Proposition 1: the following nonlinear dynamic system is a full order state observer of the system (1b):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \Psi(\hat{x},u) + K_1C(x - \hat{x}) + K_2[C(x - \hat{x})]^m \\ \hat{x}_0 &= \hat{x}(t_0) \end{aligned} \tag{2}$$

If the following assumptions are considered,

- $m \in Z^+, m \ \ odd, m > 1$ (3)
- K_1 can be chosen such as the following Algebraic Riccati Equation (ARE) has a symmetric positive-definite solution P for some $\varepsilon > 0$

$$(A - K_1C)^T P + P(A - K_1C) + L^2PP + I + \varepsilon I = 0 \tag{4}$$

- $\lambda_{\min}(PK_2C) \ge 0$ (5)

In (2), $\hat{x} \in \Re^n$, $K_1 = [k_{1,1} \quad k_{2,1} \quad \ldots \quad k_{n,1}]^T \in \Re^n$, $K_2 = [k_{1,2} \quad k_{2,2} \quad \ldots \quad k_{n,2}]^T \in \Re^n$.

Proof. Defining the estimation error as $e = x - \hat{x}$, the corresponding dynamic of the estimation error is:

$$\dot{e} = (A - K_1C)e - K_2[Ce]^m + [\Psi(x,u) - \Psi(\hat{x},u)] \tag{6}$$

Consider the Lyapunov function candidate,

$$V = e^T Pe$$

Its derivative is

$$\begin{aligned} \dot{V} &= \dot{e}^T Pe + e^T P\dot{e} \\ &= e^T\left[(A - K_1C)^T P + P(A - K_1C)\right]e - \\ &\quad - 2(Ce)^{m-1} e^T PK_2Ce + 2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \end{aligned} \tag{7}$$

In (Raghavan and Hedrick, 1994) is presented the next inequality as a lemma which is useful for this proof,

$$2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \le L^2 e^T PPe + e^T e$$

From Rayleigh inequality, and taking into account (5), we have,

$$-e^T PK_2Ce \le -\lambda_{\min}(PK_2C)\|e\|^2$$

Equation (7) leads to

$$\begin{aligned}\dot{V} \leq \; & e^T\left[(A-K_1C)^T P + P(A-K_1C)\right]e - \\ & -2(Ce)^{m-1}\lambda_{\min}(PK_2C)\|e\|^2 + L^2 e^T PP e + e^T e \\ = \; & e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2PP + I\right]e - \\ & -2(Ce)^{m-1}\lambda_{\min}(PK_2C)\|e\|^2 \qquad (8)\end{aligned}$$

From assumption (3), the second term in the right hand side of the inequality (8) always will be positive or zero,

$$\dot{V} \leq e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2PP + I\right]e \qquad (9)$$

According with assumption (4), since $\varepsilon > 0$, it is clear that

$$(A-K_1C)^T P + P(A-K_1C) + L^2PP + I < 0$$

Hence, $\dot{V} < 0$ .This implies that system (2) is an observer for system (1b) and the corresponding dynamic of the estimation error (6) is asymptotically stable.

♦

III. APPLICATION TO SINCRONIZATION OF CHAOTIC SYSTEMS

To illustrate our methodology, we give an application to chaotic systems. In fact, this is an application to the so-called Rössler's system (Rössler, 1976) which presents a chaotic behavior and exhibits the simplest possible strange attractor, arose from work in chemical kinetics.

A. The model

We consider the popular nonlinear Rössler's System, which is described by

$$\begin{aligned}\dot{x}_1 &= -(x_2 + x_3) \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \\ y &= x_1\end{aligned} \qquad (10)$$

It is well known that in a large neighborhood of $\{a = b = 0.2,\ c = 5\}$ this system has a chaotic behavior.

Remark 1: it is not difficult to prove that above system is Lipschitz.

B. Observability condition

The system (10) may be written in the form given by (1b), where $x = [x_1 \quad x_2 \quad x_3]^T$,

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

It can easily be shown that with the selection of $y = x_1$, the corresponding pair (A, C) is observable. By choosing the observer's gain K_1 and K_2 appropriately, the observer given by (2) may achieve local synchronization.

C. The observer for the Rössler's System

According with proposition 1, we get the following equations system (slave system) as the observer,

$$\begin{aligned}\dot{\hat{x}}_1 &= -(\hat{x}_2 + \hat{x}_3) + k_{1,1}(x_1 - \hat{x}_1) + k_{1,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_2 &= \hat{x}_1 + a\,\hat{x}_2 + k_{2,1}(x_1 - \hat{x}_1) + k_{2,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_3 &= b + \hat{x}_3(\hat{x}_1 - c) + k_{3,1}(x_1 - \hat{x}_1) + k_{3,2}(x_1 - \hat{x}_1)^m\end{aligned} \qquad (11)$$

where, $K_1 = [k_{1,1} \quad k_{2,1} \quad k_{3,1}]^T$, $K_2 = [k_{1,2} \quad k_{2,2} \quad k_{3,2}]^T$ and $a, b, c > 0$.

D. Numerical simulations

The design of the full-order observer presented in this paper is based on the solution of the Riccati equation which can be obtained by using the Matlab function ARE.

We have chosen the values for the Rössler's system (10) and the observer (11) as $a = b = 0.2$, $c = 5$, $m = 3$, and the observer's gain have been taken as $K_1 = [5 \quad -5 \quad 5]^T$ and $K_2 = [10 \quad 10 \quad 10]^T$. All the simulations results in this paper were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox.

In this work, the performance index of the corresponding synchronization process was calculated as,

$$J(t) = \frac{1}{t + 0.001}\int_0^t \|e(t)\|_{Q_0}^2 \, d\tau \qquad (12)$$

where $e(t)$ denotes the estimation error and $Q_0 = I$.

Figure 1 shows the convergence of the estimated states (slave system) to the real states (master system), without noise in the system output. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = 3$, $\hat{x}_3 = -4$.

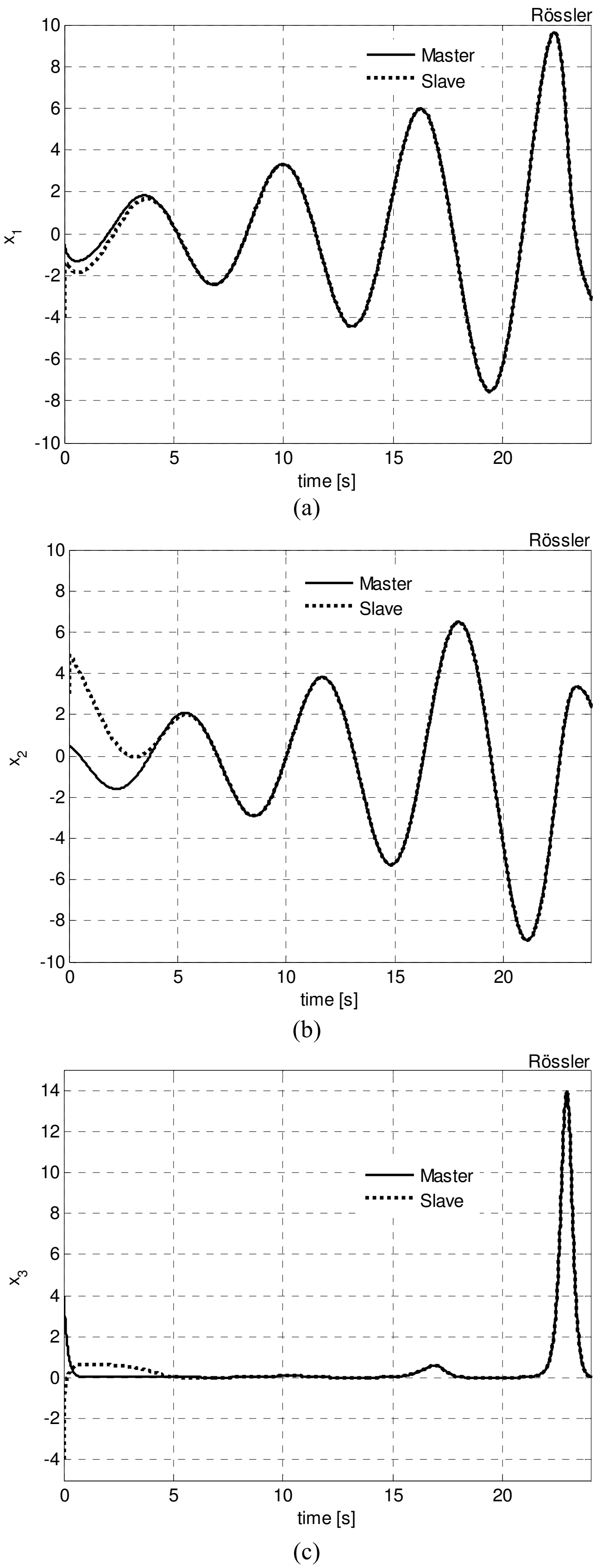


Figure 1. State estimation, without any noise in the system output.

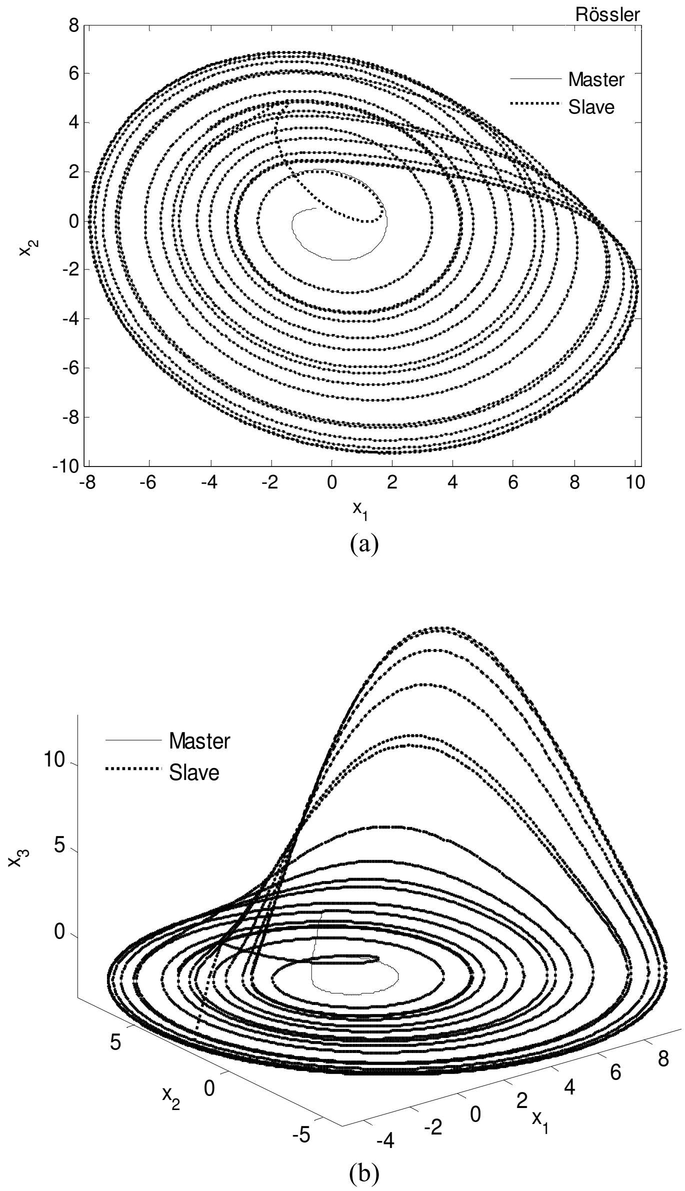
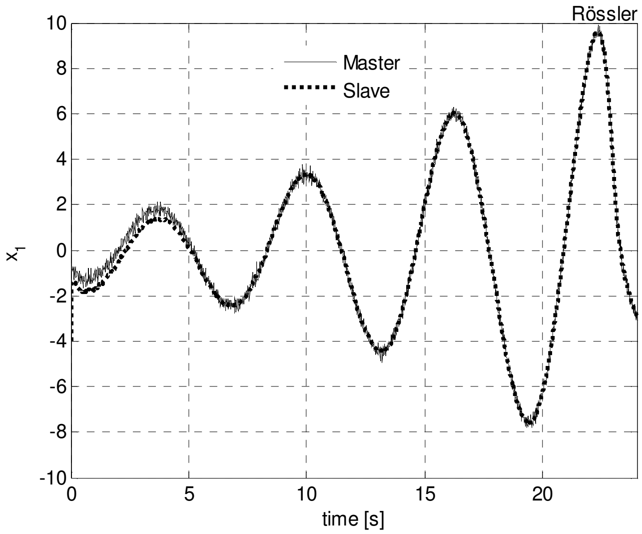


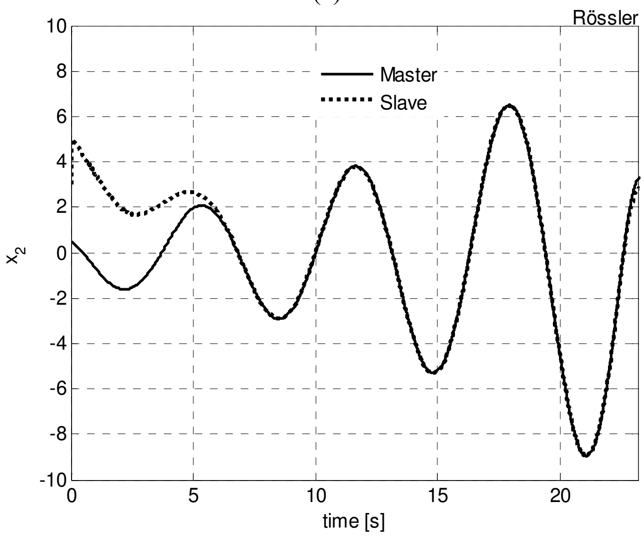
Figure 2. Rössler's system trajectories

Figure 2 shows the chaotic behaviour to the Rössler's System (master system) and its observer (slave system), and also shows the convergence of the estimated states to the real states, without noise in the system output.

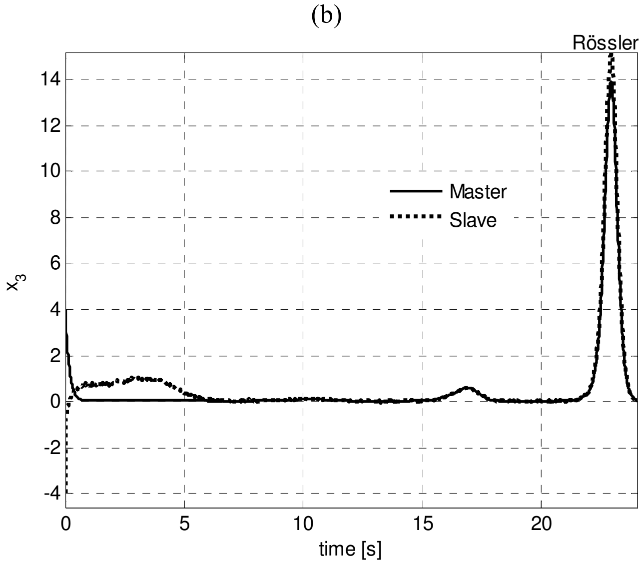
Now, we analyze the effect of noise in the measurements. In figure 3 are presented the numerical results when a noise is added in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). We can see that synchronization is possible, i. e, the estimated states tend to the real states.



(a)



(b)



(c)

Figure 3. State estimation, with white noise in the system output.

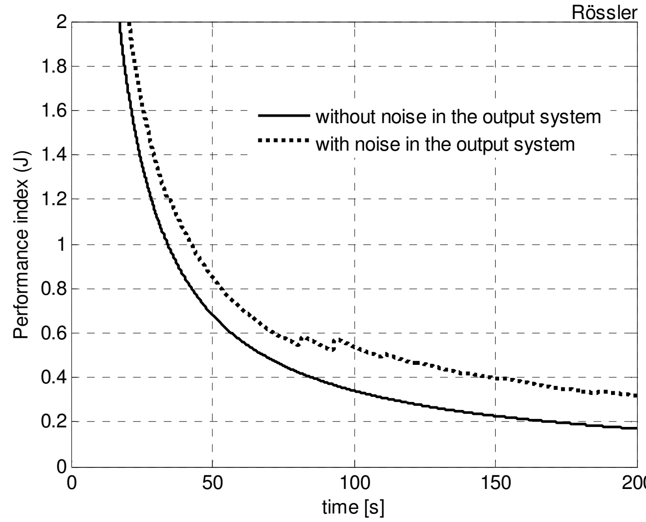


Figure 4. Quadratic estimation error.

In figure 4 is illustrated the performance index given by (12) for the corresponding synchronization process, without any noise system output and with noise in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). It should be noted that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease. Clearly, we can see that the proposed observer is robust against noisy measurements.

IV. CONCLUSION

In this paper, we have designed a new asymptotic polynomial observer (high order polynomial type) for a class of nonlinear oscillators to attack the synchronization problem. Also, we have proven the asymptotic stability of the resulting state estimation error and by means of simple algebraic manipulations we construct the observer (slave system). Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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