Optimal control of nilpotent systems: a sub-Riemannian approach

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Abstract— **We present a general framework for the optimal control of driftless nonlinear systems defined by means of distributions of smooth vector fields that generate nilpotent Lie algebras. A smooth varying inner product on the planes of the distribution, yields the energy functional that allows to approach the optimal control problem as a sub-Riemannian geodesic problem. This class of systems is relevant because provides good models for nonholonomic systems in mechanics and automation as well as in classical particle physics. We discuss two examples of nonholonomic systems within this formalism, the Cartan geometry that corresponds to the problem of rolling without slipping or twisting, and the classical Foucault pendulum that is accepted as indisputable demonstration of the Earth's rotation movement.**

I. INTRODUCTION

Optimal control problems defined by means of non-linear driftless systems provide good models for some problems in mechanics such as car-like robots, trailers, pendula, etc., see for instance (Bullo and Lewis, 2005). The study of optimal trajectories for such systems can be carried out from the perspective of sub-Riemannian geometry, that generally speaking is the geometry of non-holonomic constraints, we refer the reader to the survey (Vershik and Gershkovich, 1991), as well as the book (R. Montgomery, 2002),

The first non-trivial example of sub-Riemannian geometry was discussed in (R. Brockett, 1981), it is defined by the rank 2 distribution in $\mathbb{R}^3$ given by the vector fields $X_1 = \partial x + y\partial z,\ X_2 = \partial y - x\partial z$, and the inner product $\langle X_i, X_j\rangle = \delta_{ij}$. The usual Heisenberg multiplication endows $\mathbb{R}^3$ with the structure of nilpotent Lie group and the vector fields X_1 and X_2 are left invariant. This example is very well known and stands for the archetype of the theory. In this paper we discuss a general setting for optimal control problems defined for nilpotent systems and energy-like cost functions.

We present two applications of this formalism, first we discuss the problem of *rolling without twisting and slipping* that is based on a rank 2 distribution on $\mathbb{R}^5$ that yields the so-called step-3 nilpotent Cartan Lie algebra. Second, we present the well known *Foucault pendulum* based on an experiment that goes back to 1851 when J.B.L. Foucault suspended a 67 meter, 28 kilogram pendulum from the roof of the Pantheon in Paris, and made the observation that the pendulum's oscillation plane rotated slowly clockwise with respect to the Earth. The experiment is recognized as a feasible demonstration of Earth's rotation movement. The underlying Lie algebra for this problem is an extension of the Heisenberg algebra. To the best of our knowledge this is the first presentation of the problem in the sub-Riemannian setting.

Apart from this introduction the paper contains four sections, in section II, we present the general setting for nilpotent sub-Riemannian systems including some general properties of the Lie algebraic structures involved. In section III we tackle the optimal control problem for nilpotent systems and energy functionals. We derive necessary conditions for extremals by means of the Pontryaguin Maximum Principle, the symplectic structure of the cotangent bundle and the associated Hamiltonian formalism. In section IV we present two examples of sub-Riemannian geodesic nilpotent problems that correspond to the nonholonomic problems of rolling without slipping or twisting and the Foucault pendulum. In both cases we give some preliminary results about the geometry of the solutions. Finally we provide in section V, some conclusions and sketch some ideas for future work.

II. NILPOTENT SUB-RIEMANNIAN GEOMETRY

Let G be a smooth simply connected manifold and let $\Delta \subsetneq TG$ be a rank n distribution of vector fields with $n < \dim(G)$, a recursive definition of modules is given by $\Delta_1 = \Delta$ and $\Delta_{j+1} = \Delta_j + [\Delta_j, \Delta]$ for $j = 2, 3, \ldots$, with the resulting flag, $\Delta_1 \subset \Delta_2 \subset \Delta_3 \cdots$ The growth vector of Δ at $g \in G$ is written as follows: $(\dim(\Delta_1(g)), \dim(\Delta_2(g)), \dim(\Delta_3(g)), \ldots)$. The distribution is said to be *bracket generating* if for each $g \in G$ the growth vector reaches the value $\dim(G)$, that is, commutators up to certain order suffice to span the tangent space at each point.

Definition 1: A sub-Riemannian structure for G consists of a pair $(\Delta, \langle\cdot,\cdot\rangle)$ of a bracket generating distribution together with a smooth varying inner product defined on the planes $\Delta(g) \subsetneq T_gG,\ g \in G$.

The derived series of an arbitrary Lie algebra $\mathfrak{g}$ is written as $\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(k)} \supseteq \cdots$, with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$; whereas the lower central series is written as $\mathfrak{g} \supseteq \mathfrak{g}^2 = \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^k \supseteq \cdots$, with $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$. The Lie algebra is said *solvable* if $\mathfrak{g}^{(h)} = 0$ for some h, (the index of solvability), and it is said *nilpotent*, if $\mathfrak{g}^{\ell} = 0$ for some ℓ, (the nilpotency), for details see (Jacobson, 1962).

Definition 2: A sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ is said to be $N-$step nilpotent of type $(\text{rank}(\Delta), \dim(G))$, if the Lie algebra generated by Δ is nilpotent with nilpotency equal to N.

The contact sub-Riemannian geometry in $\mathbb{R}^3$ corresponds to the 2-step nilpotent sub-Riemannian geometry of type $(2,3)$ realized through the aforementioned three dimensional Heisenberg group. The 2-step nilpotent sub-Riemannian geometry of type $(n, n(n+1)/2)$ corresponds to higher dimensional Heisenberg groups, and is discussed in (Liu and Sussmann, 1995) and (Anzaldo and Monroy, 2006), the particular case of type $(3,6)$ is worked out in detail in (Myasnichenko, 2002).

We consider the sub-Riemannian structure on G given by the smooth varying inner product defined on the planes $\Delta(g) \subset T_gG$ by means of $\langle X_i, X_j\rangle_g = \delta_{ij}$.

Definition 3: An absolutely continuous arc-length parameterized curve $t \mapsto g(t)$ is said to be Horizontal or admissible for Δ if satisfies $\dot{g}(t) \in \Delta(g(t))$ a.e.

The Chow-Rashevsky's theorem together with the connectedness of G guarantees the existence of an admissible curve $t \mapsto g(t),\ t \in [0, T_g]$ connecting any two given points $g_i, g_f \in G$, that is $g(0) = g_i$ and $g(T_g) = g_f$, for details see for instance (Agrachev and Sachkov, 2004).

The sub-Riemannian geodesic problem is the variational problem of minimization of the energy functional in the class of admissible curves. We approach this problem as an optimal control problem, that is, as the problem of minimizing the functional

$$\Lambda(\mathbf{u}, g) = \int \langle \dot{g}, \dot{g}\rangle = \int u_1^2 + \cdots + u_n^2, \qquad (1)$$

among the solutions of the system

$$\dot{g}(t) = u_1 X_1(g) + \cdots + u_n X_n(g),\ g \in G, \qquad (2)$$

where the controls $\mathbf{u} = (u_1, \ldots, u_n)$, are assumed as measurable and bounded. The set of admissible controls is denoted as $\mathcal{U}$.

II-A. A model for Nilpotent sub-Riemannian geometries

For distributions of analytic vector fields, the Lie algebra generated by Δ, in general infinite dimensional, can be described by writing the vector fields in local coordinates $g = (x_1, \ldots, x_n, y_1, \ldots, y_\ell) = (x, y) \in \mathbb{R}^n \times \mathbb{R}^\ell$, where ℓ is the co-rank of Δ, as follows

$$X_k = \sum_{i=1}^{n} \varphi^{ki}(g)\frac{\partial}{\partial x_i} + \sum_{j=1}^{\ell} \xi^{kj}(g)\frac{\partial}{\partial y_j},\ k = 1, \ldots, n$$

For then, system (2) is written as follows

$$\dot{g} = (\dot{x}, \dot{y}) = (\mathbf{u}\ \Phi, \mathbf{u}\ \Xi), \qquad (3)$$

where $\Phi = (\varphi^{\alpha\beta})$ and $\Xi = (\xi^{\mu\nu})$ are matrices of order $n \times n$ and $n \times \ell$ respectively. The understanding of the problem at this level of generality if far from being complete. We consider here the nilpotent finite dimensional approximation, obtained by considering finite truncations of the power series of functions $\varphi^{\alpha\beta}$ and $\xi^{\mu\nu}$, and restrict them to take values in the variable $x = (x_1, \ldots, x_n)$ only. This approach fits into the hierarchy of non-holonomic constraints proposed in (Brockett and Dai, 1993). We take Φ as $n \times n$ identity matrix and the entries of the matrix Ξ as polynomials in the variable $x = (x_1, \ldots, x_n)$ only, that is

$$X_k = \frac{\partial}{\partial x_k} + \sum_{j=1}^{\ell} \xi^{kj}(x)\frac{\partial}{\partial y_j},\ k = 1, \ldots, n \qquad (4)$$

In this case system (3) reduces to

$$\dot{x} = \mathbf{u}, \quad \text{and} \quad \dot{y} = \mathbf{u}\ \Xi(x). \qquad (5)$$

Proposition 1: The Lie algebra generated by the vector fields (4) is $N-$step nilpotent with $N = \max\{\deg(\xi^{kj}(x))\}$.

Proof:

Let $\xi^{\alpha\beta}_{\gamma}$ denote the partial derivative $\dfrac{\partial \xi^{\alpha\beta}}{\partial x_\gamma}$, a direct calculation shows

$$[X_j, X_k] = \sum_{\nu=1}^{\ell} = (\xi^{k\nu}_j(x) - \xi^{j\nu}_k(x))\frac{\partial}{\partial y_\nu}.$$

An elementary induction argument shows that for any multi-index $(i_1, i_2, \ldots, i_r)$ with $r < N$ one has that $[X_{i_r}, [X_{i_{r-1}}, [\cdots[X_{i_3}, [X_{i_2}, X_{i_1}]]\cdots]]]$ is equal to $\sum_{\nu=1}^{\ell}\left(\xi^{i_1\nu}_{\widehat{i_1} i_2\cdots i_r}(x) - \xi^{i_2\nu}_{i_1\widehat{i_2}\cdots i_r}(x)\right)\frac{\partial}{\partial y_\nu}$. For then, all commutators of length greater than N vanish.

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II-B. The nilpotent Lie-algebraic structure

The Lie algebra generated Δ determines most of the geometry of optimal solutions, it is therefore important to describe bases for the Lie algebra. There is a collecting process of high order brackets that yields a basis in a constructive manner, it was originally presented by Philip Hall in (P. Hall, 1934), and further explained by Bourbaki (Bourbaki, 1989) and M. Hall (M. Hall, 1950), such a basis has been recently utilized in control theory, see (Laferriere and Sussmann, 1993).

Let $\mathfrak{g}$ be the Lie algebra generated by, each vector field is considered of being of degree one, higher order brackets $[X_{i_1}[X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]\ldots]]$, are called Lie monomials of degree k, any element of $\mathfrak{g}$ is written as linear combination of monomials. Linear combinations of monomials of arbitrary degree j are said to be homogeneous of degree j. Since there are only a finite number of monomials of an arbitrary degree, then for each positive integer j there is a finite collection η_j of monomials, which are linearly independent, and have the property that each homogeneous expression of degree j is written as linear combination of elements of η_j, the elements of such a collection are called standard monomials and are formally defined by induction.

Definition 4: The standard monomials of degree one are the $X_i's$. Assume that the standard monomials of degree $n-1$, are defined, and that they are $\prec$ −ordered in such a way that $u \prec v$ provided $\deg(u) < \deg(v)$. If $\deg(x) = i, \deg(v) = j$ and $\deg[x,v] = i+j = n$, then $[x,v]$ is a standard monomial if and only if satisfies:

1. x and v are standard monomials with $x \prec v$.
2. If $v = [y,z]$ then $y \preceq x$ and $y \prec z$.

An element of the Lie algebra $\mathfrak{g}$ is said to be in standard form if it is written as linear combination of standard monomials.The standard monomials form the so-called a Philip Hall basis for the Lie algebra $\mathfrak{g}$.

Example 1: For $\Delta = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$, the standard monomials of degree 2 are the following

X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}
	X_{23}	X_{24}	X_{25}	X_{26}	X_{27}
		X_{34}	X_{35}	X_{36}	X_{37}
			X_{45}	X_{46}	X_{47}
				X_{56}	X_{57}
					X_{67}

All the monomials of degree $3, X_{ijk}$ are listed in the following arrangement

X_{112}	X_{212}	X_{312}	X_{412}	X_{512}	X_{612}	X_{712}
X_{113}	X_{213}	X_{313}	X_{413}	X_{513}	X_{613}	X_{713}
X_{1144}	X_{214}	X_{314}	X_{414}	X_{514}	X_{614}	X_{714}
X_{115}	X_{215}	X_{315}	X_{415}	X_{515}	X_{615}	X_{715}
X_{116}	X_{216}	X_{316}	X_{416}	X_{516}	X_{616}	X_{736}
X_{117}	X_{217}	X_{317}	X_{417}	X_{517}	X_{617}	X_{717}
X_{123}	X_{223}	X_{323}	X_{423}	X_{523}	X_{623}	X_{723}
X_{124}	X_{224}	X_{324}	X_{424}	X_{524}	X_{624}	X_{724}
X_{125}	X_{225}	X_{325}	X_{425}	X_{525}	X_{625}	X_{725}
X_{126}	X_{226}	X_{326}	X_{426}	X_{526}	X_{626}	X_{726}
X_{127}	X_{227}	X_{327}	X_{427}	X_{527}	X_{627}	X_{727}
X_{134}	X_{234}	X_{334}	X_{434}	X_{534}	X_{634}	X_{734}
X_{135}	X_{235}	X_{335}	X_{435}	X_{535}	X_{635}	X_{735}
X_{136}	X_{236}	X_{336}	X_{436}	X_{536}	X_{636}	X_{736}
X_{137}	X_{237}	X_{337}	X_{437}	X_{537}	X_{637}	X_{737}
X_{145}	X_{245}	X_{345}	X_{445}	X_{545}	X_{645}	X_{745}
X_{146}	X_{246}	X_{346}	X_{446}	X_{546}	X_{646}	X_{746}
X_{147}	X_{247}	X_{347}	X_{447}	X_{547}	X_{647}	X_{747}
X_{156}	X_{256}	X_{356}	X_{456}	X_{556}	X_{656}	X_{756}
X_{157}	X_{257}	X_{357}	X_{457}	X_{557}	X_{657}	X_{757}
X_{167}	X_{267}	X_{367}	X_{467}	X_{567}	X_{667}	X_{767}

and the standard monomials are located above the horizontal lines.

II-C. *The 3-step nilpotent Lie algebra with solvability index 2*

For the case 3-step nilpotent with solvability 2, the Lie brackets with more than 3 factors as well as the ones of the form $[[X_i, X_j], [X_k, X_l]]$ vanish, then $\mathfrak{g} = \text{span}\{\Delta, [\Delta,\Delta], [[\Delta,\Delta],\Delta]\}$. The elements of $[\Delta,\Delta]$ are linear combinations of first order Lie brackets of the generators $X_{i_1 i_2}$ with $i_1 < i_2$. The elements of $[[\Delta,\Delta],\Delta]$ are linear combinations of second order Lie brackets $X_{i_1 i_2 i_3}$. In this case Jacobi identity reads $X_{i_1 i_2 i_3} + X_{i_2 i_3 i_1} + X_{i_3 i_1 i_2} = 0$, or equivalently

$$X_{i_1 i_2 i_3} = X_{i_3 i_2 i_1} - X_{i_3 i_1 i_2} \qquad (6)$$

Jacobi identity leads to nothing new if two indices are equal, we consider distinct indices, say $i_3 < i_1 < i_2$, for then equation (6) means that a bracket $X_{i_1 i_2 i_3}$ with $i_3 < i_1 < i_2$ can be expressed in terms of brackets X_{ijk} such that $i < j$ and $i < k$. Since the bracket X_{iji}, with $i < j$ is subject to no conditions, we conclude that X_{ijk} with $i < j$ and $i \le k$ are the linearly independent brackets.

Proposition 2: If $\mathfrak{g}$ is a 3-step nilpotent Lie algebra with solvability index 2, generated by $\Delta = \{X_1, \ldots, X_n\}$, then $\dim(\mathfrak{g}) \le \frac{1}{3}n(n+1)(n+\frac{1}{2})$.

Proof:

All the brackets $X_{i_1 i_2}$ with $i_1 < i_2$ are linearly independent, and for each $i_1 = 1, 2, \ldots, n-1$ there are $(n-i_1)(n-i_1+1)$ elements linearly independent of the form $X_{i_1 i_2 i_3}$, by writing $\kappa = n - i_1$ we have that

$$\begin{aligned} D &= \dim([[\Delta,\Delta],\Delta]) = \sum_{\kappa=1}^{n-1} \kappa(\kappa+1) \\ &= \sum_{\kappa=1}^{n-1} \kappa^2 + \sum_{\kappa=1}^{n-1} \kappa = \frac{(n-1)n(n+1)}{3} \end{aligned}$$

in consequence $\dim(\mathfrak{g}) \le n + \frac{(n-1)n}{2} + D$.

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III. The optimal control problem

The Hamiltonian functions associated to the vector fields X_i, X_{ij} and X_{ijk} are denoted as H_i, H_{ij} and H_{ijk}, respectively. The control dependent Hamiltonian writes as follows

$$\mathcal{H}_{\mathbf{u}}^{\lambda_0} = -\frac{\lambda_0}{2}\left[\, u_1^2 + \cdots + u_n^2\right] + u_1 H_1 + \cdots + u_n H_n,$$

we can write then the necessary conditions for optimal trajectories.

Maximum Principle. If trajectory $t \mapsto (g, \mathbf{u})$ is Λ−optimal then it is the projection of an extremal curve $t \mapsto \xi = (g, p)$, satisfying

i. $\mathcal{H}_{\mathbf{u}}^{\lambda_0}(\xi) \ge \mathcal{H}_{\mathbf{v}}^{\lambda_0}(\xi)$, for all admissible controls $\mathbf{v} \in \mathcal{U}$

ii. $\mathcal{H}_{\mathbf{u}}^{0}(\xi)$ is not identically zero.

We assume that $\lambda_0 = 1$, maximality condition implies that the optimal control along extremals is given as $\mathbf{u} = (H_1, \ldots, H_n)$, for then, the system Hamiltonian becomes quadratic

$$\mathcal{H} = \frac{1}{2}(H_1^2 + \cdots + H_n^2) \qquad (7)$$

The covector is written as follows

$$((H_1,\ldots,H_n),(H_{j<k}),H)\in\mathbb{R}^n\times\mathfrak{so}_n\times\mathbb{R}^D,$$

with D given in proposition 2. The adjoint system of Hamiltonian equations write then as follows

$$\dot{H}_j = \{H_j,\mathcal{H}\}=\sum_{k\neq j} H_k H_{jk}, \tag{8}$$

$$\dot{H}_{jk} = \{H_{jk},\mathcal{H}\}=-\sum_{i=1}^{n} H_i H_{ijk}, \tag{9}$$

$$\dot{H}_{ijk} = \{H_{ijk}.\mathcal{H}\}=0. \tag{10}$$

Last equation imply that all the length three Poisson brackets are constant along extremals. In equations (9) we set $H_i=\dot{x}_i$ to obtain $\frac{d}{dt}\left(H_{ij}+\sum_{k=1}^{n}\ x_k\,H_{kij}\right)=0.$ If we introduce the skew-symmetric constant matrix with entries $c_{ij}=H_{ij}+\sum_{k=1}^{n}\ x_k\,H_{kij}\ i<j$, we obtain, from equation (8),

$$\ddot{x}_i-\sum_{j=1}^{n}c_{ij}\dot{x}_j=-\sum_{j,k=1}^{n}x_k\,H_{kij}\,\dot{x}_j.$$

These equations are given only in terms of the x_i, and together with (2), after plugging the optimal controls (7), determine completely the geodesics.

IV. EXAMPLES FROM NON-HOLONOMIC MECHANICS

IV-A. Cartan geometry

Rolling bodies without slipping or twisting is a classical non-holonomic problem that has important applications in control and automation, a nice presentation can be founded for instance in (Sharpe, 2000). A more intrinsec approach has been taken in the recent book (Kobayashi and Oliva, 2008) whereas a control theory perspective is presented in (A.Agrachev and Y.Sachkov, 1999).

From the geometric viewpoint exposed in this paper it consists of a 3-step nilpotent sub-Riemannian geometry given by five dimensional nilpotent Lie group G_5 together with a rank 2 distribution of left invariant vector fields $\Delta=\{X_1,X_2\}$, for which the only non-zero Lie brackets are the following

$$[X_1,X_2]=:X_3,\quad [X_1,X_3]=:X_4,\quad [X_2,X_3]=:X_5.$$

The Lie algebra $\mathfrak{n}_5=\text{span}\{X_1,X_2,X_3,X_4,X_5\}$, is known as the Cartan Lie algebra or *diamond* Lie algebra and is customarily represented by the graph

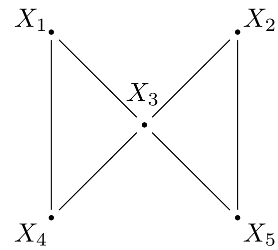

FIGURE 1. Cartan Lie algebra

If H_i denotes the Hamiltonian associated with the vector field X_i, then we have the following non-trivial Poisson brackets

$$\{H_1,H_2\}=H_3,\quad \{H_1,H_3\}=H_4,\quad \{H_2,H_3\}=H_5.$$

The corresponding system Hamiltonian writes as follows

$$\mathcal{H}=\frac{1}{2}(u_1^2+u_2^2),+u_1H_1+u_2H_2 \tag{11}$$

the maximality condition of the Maximum Principle, readily yields $u_1=H_1$ and $u_2=H_2$, therefore the systems Hamiltonian is quadratic $\mathcal{H}=H_1^2+H_2^2$,
and the adjoint system can be directly written as follows

$$\dot{H}_1 = \frac{1}{2}\{H_1,\mathcal{H}\}=H_2H_3 \tag{12}$$

$$\dot{H}_2 = \frac{1}{2}\{H_2,\mathcal{H}\}=-H_1H_3 \tag{13}$$

$$\dot{H}_3 = \frac{1}{2}\{H_3,\mathcal{H}\}=-H_1H_4-H_2H_5 \tag{14}$$

$$\dot{H}_4 = \frac{1}{2}\{H_4,\mathcal{H}\}=0 \tag{15}$$

$$\dot{H}_5 = \frac{1}{2}\{H_5,\mathcal{H}\}=0 \tag{16}$$

Observe that H_4 and H_5 are central elements. Multiplying third equation by H_3, we get

$$H_3\dot{H}_3 = \left[\frac{1}{2}\frac{d}{dt}\left(H_3^2\right)\right] = H_3(-H_1H_4-H_2H_5)$$
$$= \dot{H}_2H_4-\dot{H}_1H_5=\frac{d}{dt}[H_2H_4-H_1H_5]$$

therefore we obtain the constant of integration

$$c_2:=\frac{1}{2}H_3^2-H_2H_4+H_1H_5. \tag{17}$$

Further derivation of (14) yields

$$\ddot{H}_3=-\dot{H}_1H_4-\dot{H}_2H_5=c_2H_3-\frac{1}{2}H_3^3,$$

in consequence

$$\dot{H}_3\ddot{H}_3 = \left[\frac{1}{2}\frac{d}{dt}(\dot{H}_3)^2\right] = c_2 H_3\dot{H}_3 - \frac{1}{2}H_3^3\dot{H}_3$$
$$= c_2\left[\frac{1}{2}\frac{d}{dt}(H_3)^2\right] - \frac{1}{2}\left[\frac{1}{4}\frac{d}{dt}(H_3)^4\right]$$

we obtain then another constant of integration

$$c_4 := \frac{1}{4}H_3^4 - c_2 H_3^2 + (H_1H_4 + H_2H_5)^2$$

Lemma 1: The elements of set $\{\mathcal{H}, H_4, H_5, c_2\}$ are independent first integrals in involution, whereas $\mathcal{K} := H_4^2 + H_5^2$ and c_4 are neither independent nor in involution.

Thus the trajectories in cotangent bundle are given by the intersection of the cylinder $H_1^2 + H_2^2 = 1$ with the parabolic cylinder $\frac{1}{2}H_3 - H_2H_4 + H_1H_5 = c_2$, and they can be visualized as curves on the sphere $(H_1 + H_5)^2 + (H_2 - H_5)^2 + H_3^2 = \mathcal{H} + 2\,c_2 + H_4^2 + H_5^2$. Furthermore, circular coordinates can be used for parameterizing surface (11)

$$H_1 = \mathcal{H}\sin\theta$$
$$H_2 = \mathcal{H}\cos\theta$$

Equations (12) and (13) readily imply $\dot{\theta} = H_3$. The constants of motion H_4 and H_5 determine another quadratic surface say $H_4^2 + H_5^2 = \ell^2$ which can be parameterized similarly as

$$H_4 = \ell\,\cos\theta_0$$
$$H_5 = \ell\,\sin\theta_0$$

Thus, using c_2 we can write (17)

$$\dot{\theta}^2 = 2(c_2 + \mathcal{H}\ell - 2\mathcal{H}\ell\sin^2([\theta + \theta_0]/2).$$

From where a Kinetic analog can be established with a simple pendulum (Whittaker, 1999).

IV-B. The Foucault pendulum

It consists of simple pendulum of length ℓ and point mass m oscillating taking into account Earth's rotation. Let $\{X, Y, Z\}$ an inertial coordinate system such that Earth's rotation coincides with the Z direction, and let $\vec{\omega}$ be the angular velocity of rotational motion. If $\{x, y, z\}$ is the position of the mass measured from a fixed coordinate system with origin located at latitude α on Earth's surface measured from equator, the x direction on a meridian great circle in north-south sense, the y direction on a latitude circle in west-east sense and the z direction perpendicular to the tangent plane at the intersection of both circles. If the origin is at the suspension point of the pendulum, the direction cosines are given as $\cos\phi_x = x/\ell, \cos\phi_y = y/\ell$ and $\cos\phi_z = -z/\ell$.

For a mass m in a gravitational force field $\vec{F}_G = -\vec{\nabla}V_G$, the trajectories are determined by minimizing the total energy functional, moreover if we take into account the holonomic constraint

$$x^2 + y^2 + z^2 - \ell^2 = 0, \qquad (18)$$

we have that the complete functional is written as follows

$$S = \int \Big((\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2gz) + \lambda_0 r^2$$
$$+ \lambda_1(\dot{\xi} + y\dot{z} - z\dot{y}) + \lambda_2(\dot{\eta} + z\dot{x} - x\dot{z})$$
$$+ \lambda_3(\dot{\zeta} + x\dot{y} - y\dot{x}) \Big)\, dt.$$

Set $\lambda_1 = -m\omega\cos(\alpha)$, $\lambda_2 = 0$ and $\lambda_3 = m\omega\sin(\alpha)$ and the differentials $\dot{\xi}dt$, $\dot{\eta}dt$ and $\dot{\zeta}dt$. The last differentials do not alter the Euler-Lagrange equations for $\vec{r}$, but are essential for the sub-Riemannian approach. The action can be reinterpreted as follows: the parameters λ_i are Lagrange parameters associated with the nonholonomic constraints

$$d\xi + ydz - zdy = 0,$$
$$d\eta + zdx - xdz = 0,$$
$$d\zeta + xdy - ydx = 0.$$

This assumption is equivalent to take ω_i as constants, since ξ, η, and ζ are cyclic variables, which is the case of the problems under study. Instead of the last constraints, consider the equivalent 1-forms

$$w_1 = d\xi + ydz - zdy,$$
$$w_2 = d\eta + zdx - xdz,$$
$$w_3 = d\zeta + xdy - ydx,$$

in such a way that that $\ker\{w_1, w_2, w_3\}$ leads to the constraints. Introducing the vector $q = (x, y, z, \xi, \eta, \zeta)$ in a six dimensional smooth manifold $\mathcal{M}$, it follows that

$$\dot{q} = \dot{x}X_1(q) + \dot{y}X_2(q) + \dot{z}X_3(q), \qquad (19)$$

where the vector fields in $T\mathcal{M}$ are given as

$$X_1 = \partial_x + y\partial_\zeta - z\partial_\eta,$$
$$X_2 = \partial_y + z\partial_\xi - x\partial_\zeta,$$
$$X_3 = \partial_z + x\partial_\eta - y\partial_\xi.$$

The vector fields are dual to the 1-forms w_i and generate a six dimensional step-2 nilpotent Lie algebra $\mathfrak{g}$ with non-zero brackets

$$[X_i, X_j] = X_{ij} = -X_{ji}, \quad i < j,$$

and with

$$X_{12} = -2\partial_\zeta, \quad X_{13} = 2\partial_\eta, \quad X_{23} = -2\partial_\xi.$$

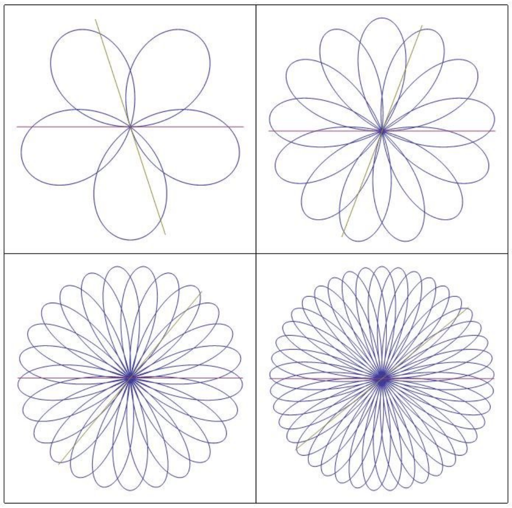

Fig. 1. Cross sections of sub-Riemannian trajectories

Since $T_q\mathcal{M}$ is six dimensional for all $q \in$ M, the distribution $\Delta = \{X_1, X_2, X_3\}$ is bracket generating. Let G be the Lie group associated to $\mathfrak{g}$. The center of $\mathfrak{g}$ is just $[\Delta, \Delta]$, let us denote by G_0 the corresponding center of G. The vector fields in Δ will be called *horizontal*, and the fields in $[\Delta, \Delta]$ will be called *vertical*.

A sub-Riemannian structure for this problem is given by the pair formed by the distribution Δ and the Euclidean metric given by the kinetic energy. The six dimensional space with coordinates $q = (x, y, z, \xi, \eta, \zeta)$ will be called *total space* M, the three dimensional subspace with coordinates (x, y, z) the *base space* B. Define the projection map π : M $\to B$, as $\pi(x, y, z, \xi, \eta, \zeta) = (x, y, z)$, then the *fiber* F_p at $p \in B$ is the set $F_p = \pi^{-1}(p)$ and the spaces $\pi^{-1}(p)$ for all $p \in B$ are homeomorphic to a space F called the typical fiber. Since the group G_0 acts on F by automorphisms, (M$, B, \pi, G_0)$ define a fiber bundle with typical fiber F. Here, it is a principal fiber bundle since F and G_0 coincide as vector spaces.

The trajectories for the Foucault pendulum can be explicitly calculated following standard integration techniques, a detailed presentation of such process together with an exhaustive geometric analysis shall be exposed elsewhere. The Fig. 1, shows some cross sections of the trajectories obtained numerically with a CAS, for some specific values of the frequencies.

V. Conclusions and Perspectives

We present in this paper a general setting for sub-Riemannian nilpotent systems and the associated optimal control problem. The Pontryaguin Maximum Principle provides necessary conditions for optimal trajectories, and yield explicit expressions for the system Hamiltonian in terms of the optimal controls. However, provides no help for discerning the integrability of the system. In some cases the underlying Lie algebraic structure together with some ad hoc integration procedures can be of some use, nevertheless, a more general theory needs to be developed for explaining generic obstructions for integrability.

Non-holonomic mechanics as well as robotics and automation provided a great source of problems that can be formulated within this formalism. The application of the geometric techniques can eventually unveil some properties for solutions of such problems, including control design and stabilization.

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