On generalized synchronization of different order affine chaotic systems: a submanifold approach

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***Resumen*— Regulation theory is used to address the synchronization phenomena of chaotic systems. Our results are based on the solution of Francis-Isidori-Byrnes equations to derive the synchronization submanifold. Thus conditions for complete or partial synchronization are depicted. This analysis is not restrictive with respect to the master and the slave systems dimension, therefore it can be applied to strictly different systems with the same order or even different order systems. Finally, workbench examples are presented to illustrate the results. © AMCA.**

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I. Introduction

Synchronization of chaotic systems is an interesting topic that, since early 90's, has caught the attention of the nonlinear science community. Two research directions have been already conformed in synchronizing chaos: (i) analysis and (ii) synthesis. Analysis problem comprises (a) the classification of synchronization phenomena (Femat y Solis-Perales, 1999), (Brown y Kocarev, 2000); (b) the comprehension of the synchronization properties as, for instance, robustness (Kocarev *et al.*, 2000) or geometry (Josic, 2000; Martens *et al.*, 2002); and (c) the construction of a general framework for unifying chaotic synchronization (Brown y Kocarev, 2000; Boccaleti *et al.*, 2001). On the other hand, synthesis of synchronization focus on the problem of finding the control effort such that two chaotic systems share the same time evolution in some sense (see, e.g., among the others, Ott *et al.*, 1990; Cicogna y Fronzoni, 1990; Loskutov, 2001; Chacón, 2006). Both analysis and synthesis directions are active research areas and one of the current challenges is to achieve and explain the synchronization of chaotic system with different models.

In regard to the analysis of strictly-different systems, the reported studies have been focussed on the existence of synchronization manifolds for coupled systems. These studies have shown that such manifolds are strongly dependent on measures from Lyapunov exponents (Josic, 2000; Martens *et al.*, 2002). Synchronization of different models has been addressed in nonidentical space-extended systems (for the case of parameter mismatching) (Boccaletti *et al.*, 1999) and structurally nonequivalent systems including delay (Boccaletti *et al.*, 2000). In (Josic, 2000) chaotic synchronization has been also analysed from invariant manifolds in terms of the existence of a diffeomorphism between the attractor of the coupled systems, which is closely related to generalized synchronization (GS). Josic (2000) had also included synchronization of different systems, and illustrative examples have shown the existence of synchronization manifolds (e.g., between Rössler and Lorenz). This analysis has departed from rigorous definitions and is thorough for the complete synchronization (i.e., the synchronization of all master states with all corresponding states of the slave system, Femat y Solis-Perales, 1999). Unfortunately, such a formalism for other synchronization phenomena (as, for example, the partial-state synchronization, Femat y Solis-Perales, 1999) is still obscure. Additionally, the generalized synchronization problem between different order systems is still open and few works have pointed at this direction; however, all efforts have been focussed on particular systems (see for instance, Ge y Yang, 2008; Rodríguez *et al.*, 2008) and general results must be established.

On the other hand, broadly speaking, one of the solutions to the tracking problem in dynamical systems which are subjected to external disturbances and reference signals (both generated by a dynamic system better known as the exosystem) can be formulated as the problem of finding a solution to the so called regulation problem. Under such a formulation, the problem consists, in finding a feedback scheme such that the equilibrium point of the closed loop system is asymptotically stable and the tracking error approaches zero even under the influence of external signals. The regulation problem has been extensively studied in linear systems (Francis, 1977). These ideas were then extended to the nonlinear case in (Isidori y Byrnes, 1990), where it was demonstrated that the corresponding solution depends on the solution of a pair of matrix partial differential equations, known henceforth as the Francis-Isidori-Byrnes equations. Technically speaking, the regulation problem and the synchronization problem are different from the point of view of physical meaning and their application fields, however, some analogies can still be established between them.

In this paper, borrowing the regulation theory from the control framework, we address the synchronization problem of nonlinear systems with not necessarily the same dimension in order to explain how the complete or partial synchronization phenomena is achieved. The paper is organized as follows: In section II the regulation theory is presented; then, in section III an analogy of this theory is applied to the synchronization to derive conditions for complete or partial state synchronization. The analysis is carried out on SISO affine systems. Workbench examples are analysed in section IV. Finally, this work is closed with some concluding remarks.

II. Fundamental regulation theory

In the literature of control of dynamical systems, regulation problem is often addressed as forcing the output of a dynamical system to reach a predetermined reference signal. Although this is the case for many systems, due to their nature, for others, such as synchronization systems, varying reference signals are imposed to obtain a suitable behavior. In this section, a brief review of results in regard to the regulation problem is presented. Let us consider the following nonlinear time-invariant system

$$\dot{x}_S = F_S(x_S, w, u) \quad (1)$$

$$e = h(x_S, w), \quad (2)$$

where the first equation (1) describes the dynamics of a plant, whose state x_S is defined in a neighborhood U of the origin in $\mathbb{R}^m$, with a control input $u \in \mathbb{R}^p$ and subject to a set of exogenous input variables $w \in \mathbb{R}^n$ which includes disturbances to be rejected and/or references to be tracked. We consider that the first approximation matrices of system (1) are respectively, $A = [\partial F_S/\partial x_S]_{(x_S,w,u)=(0,0,0)}$ and $B = [\partial F_S/\partial u]_{(x_S,w,u)=(0,0,0)}$. Equation (2) defines an error variable $e \in \mathbb{R}^p$ expressed as the function $h : U \times \mathbb{R}^n \to \mathbb{R}^p$. Let us assume that the exogenous inputs w, is a family of all functions of time that are solution of a homogeneous differential equation

$$\dot{w} = F_w(w) \quad (3)$$

with initial condition $w(0)$ ranging in a neighborhood W of the origin in $\mathbb{R}^n$. System (3) is a mathematical generator of all possible exogenous input functions and it is better known as *the exosystem*. Moreover, it is assumed that F_S, h and F_w are smooth functions and, without loss of generality, it is also assumed that $F_S(0,0,0) = 0$, $F_w(0) = 0$, and $h(0,0) = 0$. Thus, for $u = 0$, the composite system (1)-(3) has an equilibrium state $(x_S, w) = (0,0)$ yielding zero error.

The *State Feedback Regulation Problem* for system (1)-(3) is defined for tracking reference signals and rejecting the disturbance signals while maintaining the closed-loop stability property. The regulation problem can be formulated as the problem of determining a certain submanifold of the state space (x_S, w), where the tracking error e is zero, which is rendered attractive and invariant by feedback. Then the *Nonlinear Regulation Problem* (NRP) consists in finding, a function $u = \alpha(x_S, w)$ such that the following conditions hold:

C1 **Stability:** The equilibrium point $x_S = 0$, of the closed-loop system without disturbances is asymptotically stable.

C2 **Regulation:** For each initial condition $(x_S(0), w(0))$ in a neighborhood of origin, the solution of the closed-loop system satisfies the condition $\lim_{t\to\infty} e(t) = 0$.

The next theorem states conditions for the existence of a solution to the NRP.

Theorem 1: (Isidori, 1995) The Nonlinear Regulation Problem is locally solvable if and only if the pair (A, B) is stabilizable and there exist mappings

$$x_S = \pi(w), \quad and \quad u = \gamma(w) = \begin{pmatrix} \gamma_1(w) \\ \vdots \\ \gamma_p(w) \end{pmatrix}, \quad (4)$$

with $\pi(0) = 0$ and $\gamma(0) = 0$, both defined in a neighborhood $W^\circ \subset W$ of the origin, satisfying the conditions

$$\frac{\partial \pi(w)}{\partial w} F_w(w) = F_S(\pi(w), w, \gamma(w)), \quad (5)$$

$$0 = h(\pi(w), w), \quad (6)$$

for all $w \in W^\circ$.■

Conditions (5) and (6) are known as the Francis-Isidori-Byrnes equations (FIB) (Byrnes y Isidori, 2000) used to find the zero tracking error submanifold. The mapping $x_S = \pi(w)$ represents the steady-state zero output submanifold whose time derivative produces (5), while $u = \gamma(w)$ is the steady state input which makes invariant this steady state zero output submanifold.

III. Synchronization analysis

III-A. Problem statement

Let us consider the following master nonlinear dynamical system

$$\dot{x}_M = f_M(x_M), \quad (7)$$

$$y_M = h_M(x_M), \quad (8)$$

where x_M is a state vector defined in a neighborhood W of the origin in $\mathbb{R}^n$ and $F_M(x_M)$ is a smooth vector field. $y_M \in \mathbb{R}$ denotes the output of master system. If system (7)-(8) is chaotic, its trajectories are bounded. Additionally, let us now take a dynamical system

$$\dot{x}_S = f_S(x_S) + g_s(x_S)u, \quad (9)$$

$$y_S = h_S(x_S), \quad (10)$$

where x_S, defined in a neighborhood U of the origin in $\mathbb{R}^m$, denotes the state vector of the slave system, $u \in \mathbb{R}$

is the control command, f_S is a smooth vector field, and $y_S \in \mathbb{R}$ is the output of slave system.

In synchronization, under master-slave interconnection, system (7) describes the goal dynamics while system (9) represents the experimental system to be controlled. Thus, the *Synchronization Problem* can be stated as follows: *Given the transmitted signal y_M, to design a signal $u(t)$ which synchronizes the output of the slave system (10) with the output of the master system (8).* That is, given the synchronization error

$$e = h_M(x_M) - h_S(x_S), \tag{11}$$

find a function $u(x_S, x_M)$ such that $\lim_{t\to\infty} e(t) = 0$, for all t and any $x_S(0) \in U$, $x_M(0) \in W$.

III-B. Synchronization submanifold

Several kinds of synchronization have been defined (Boccaletti *et al.*, 1999; Femat y Solis-Perales, 1999): i) Complete exact synchronization (CES) (where $\|x_S(t) - x_M(t)\| \equiv 0$ for all $t \geq 0$), ii) Complete inexact synchronization (where $\|x_S(t) - x_M(t)\| \approx 0$ for all $t \geq 0$), iii) Partial synchronization (where at least for one state $x_i(t)$, for any $i \leq n$, $\|x_S(t) - x_M(t)\| \neq 0$) and iv) Almost synchronization (where only the phase of the driving system is similar to the response system with a different amplitude).

Actually, the synchronization problem can be addressed as a regulation problem for the above definitions. Since the master dynamical system (7) is similar to system (3), the slave system (9) is a particular case of system (1) and the synchronization error (11) has similarities with the regulation error (2). Hence, Theorem 1 can be adapted to solve the synchronization problem. The next assumption is instrumental to the following analysis.

Assumption 1: Let us consider that the relative degree of (9) is well defined and equal to ρ.

Since the relative degree is well defined, it is possible to find a diffeomorphism which transforms system (9) into a normal form. Moreover, for the synchronization analysis, one may split the inner dynamics in order to consider the asymptotic stable and Poisson stable modes; hence one may assume that there exists a diffeomorphism

$$\underline{x}_S = \Phi_S(x_S) = \begin{pmatrix} \zeta \\ \eta \\ \underline{x}_{S2} \end{pmatrix} \tag{12}$$

where the states variables $\zeta \in \mathbb{R}^{\rho}$, $\eta \in \mathbb{R}^{m_1-\rho}$ and $\underline{x}_{S2} \in U_2 \subset \mathbb{R}^{m_2}$ with $m = m_1 + m_2$, which in particular are defined as $\zeta_i = L_{fS}^{i-1} h_S(x_S)$, $i = 1, 2, \ldots, \rho$, with $L_{fS}^{0} h_S(x_S) = h_S(x_S)$, $\eta_j = \phi_{S,j}(x_S)$, $j = 1, 2, \ldots, m_1 - \rho$, with $\phi_{S,j}(x_S)$ such that $L_{gS}\phi_{S,j}(x_S) = 0$, and $\underline{x}_{S2} = \psi_{S,j}(x_S)$, $j = 1, 2, \ldots, m_2$. This diffeomorphism transforms system (9) into the normal form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, 2, \ldots, \rho - 1, && \text{(13a)} \\ \dot{\zeta}_\rho &= a(\zeta, \eta, \underline{x}_{S2}) + b(\zeta, \eta, \underline{x}_{S2})\, u, && \text{(13b)} \\ \dot{\eta} &= q(\zeta, \eta, \underline{x}_{S2}), && \text{(13c)} \\ \dot{\underline{x}}_{S2} &= \underline{F}_{S2}(\underline{x}_{S2}), && \text{(13d)} \end{aligned}$$

where $a(\zeta, \eta, \underline{x}_{S2}) = \left[L_{f_s}^{\rho} h_S(x_S)\right]_{x_S = \Phi_S^{-1}(\underline{x}_S)}$, $b(\zeta, \eta, \underline{x}_{S2}) = \left[L_{gS} L_{f_s}^{\rho-1} h_S(x_S)\right]_{x_S = \Phi_S^{-1}(\underline{x}_S)}$, $q_j(\zeta, \eta, \underline{x}_{S2}) = \left[L_{f_s}\phi_{S,j}(x_S)\right]_{x_S = \Phi_S^{-1}(\underline{x}_S)}$ for $j = 1, 2, \ldots, m_1 - \rho$ and $\underline{F}_{S2}(\underline{x}_{S2}) = \left[L_{f_s}\psi_{S,j}(x_S)\right]_{x_S = \Phi_S^{-1}(\underline{x}_S)}$ for $j = 1, 2, \ldots, m - m_1$. Here, we are assuming that equation (13d) is Poisson stable. That is, system (13d) yields trajectories which will return at future time arbitrarily close to any initial condition $\underline{x}_{S2}(0) \in U_2 \subset \mathbb{R}^{m_2}$ persistently. This fact arises because of oscillatory nature of chaos. In addition we are assuming that $\dot{\eta} = q(0, \eta, 0)$ is asymptotically stable. Then, the states ζ and η are stabilizables via feedback control.

Notice that when synchronization error (11) is equal to zero, the states of the slave system related to y_S become a function of the master system states related to y_M, while the states $\underline{x}_{S2}$ evolve independently in a region of U_2 depending on the initial conditions $\underline{x}_{S2}(0)$. In general, when $e = 0$, ζ and η are function of x_M and $\underline{x}_{S2}$, i.e. there exists a synchronization submanifold $\mathrm{col}(\zeta, \eta) = \pi(x_M, \underline{x}_{S2})$ and this submanifold becomes invariant under the input $u = \gamma(x_M, \underline{x}_{S2})$. This idea is formalized in the next Proposition.

Proposition 2: The Synchronization Problem is locally solvable if and only if the pair (A, B) is stabilizable and there exist mappings

$$\begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \pi(w), \quad \textit{and} \quad u = \gamma(w), \tag{13n}$$

with $\pi(0) = 0$ and $\gamma(0) = 0$, both defined in a neighborhood $W^{\circ} \times U_2^{\circ} \subset W \times U_2$ of the origin, satisfying the conditions

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} F_w(w) &= \underline{F}_{S1}(\pi(w), w) && \text{(13ña)} \\ &\quad + \underline{G}_{S1}(\pi(w), w)\,\gamma(w), \\ 0 &= h_M(x_M) - \underline{h}_S(\pi(w)), && \text{(13ñb)} \end{aligned}$$

where

$$\begin{aligned} w &= \begin{pmatrix} x_M \\ \underline{x}_{S2} \end{pmatrix}, \quad F_w(w) = \begin{pmatrix} F_M(x_M) \\ \underline{F}_{S2}(\underline{x}_{S2}) \end{pmatrix}, \\ \underline{F}_{S1}(\pi(w), w) &= \mathrm{col}(\zeta_2, \ldots, \zeta_\rho, a, q) \quad \textit{and} \\ \underline{G}_{S1}(\pi(w), w) &= \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix}_{\rho\times 1} \\ (0)_{(m_1-\rho)\times 1} \end{pmatrix} \end{aligned}$$

for all $w \in W^{\circ} \times U_2^{\circ}$.■

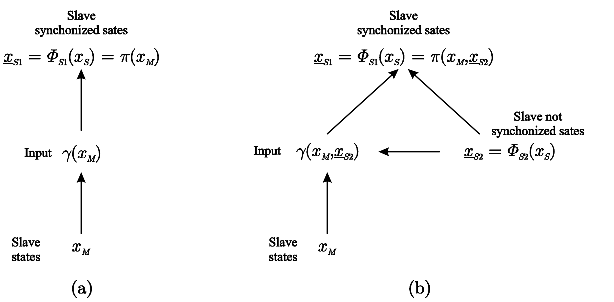

Figura 1. Interconnections diagram for generalized synchronization. (a) Total synchronization, when $\dim(\underline{x}_{S2}) = 0$. (b) Partial synchronization, when $\dim(\underline{x}_{S2}) \neq 0$.

Demostración: The proof is straightforward from the analogy with the regulation problem (Isidori, 1995). ■

Also notice that if $\dim(\underline{x}_{S2}) = 0$, in the framework of generalized synchronization, the slave system is totally synchronized; i.e., there exists the map $x_S = \Phi_S^{-1}(\pi(x_M))$ which allows to calculate x_S from x_M, despite a different dimension between the master and the slave, by means of the mapping $\pi(x_M)$ which is a contraction if $m_1 < n$ or an immersion if $m_1 > n$. This relation is schematically presented in the interconnections diagram of Figure 1a. Also notice that, if in addition $m_1 = n$, because of the map $\Phi_S^{-1}(\pi(\cdot))$, not necessarily $x_S(t) = x_M(t)$, unless both the master and the slave system were identical, since in this case, map $\Phi_S^{-1}(\pi(\cdot))$ would be the identity and the traditional complete exact synchronization would be reached. On the other hand, if $\dim(\underline{x}_{S2}) \neq 0$, only the first m_1 states of $\underline{x}_S$ are synchronized with the combined variables x_M and $\underline{x}_{S2}$, i.e., when the synchronization is achieved (and therefore $e = 0$), the m_1 states of vector $\underline{x}_{S1} = \mathrm{col}(\zeta, \eta)$ reside in the invariant submanifold $\pi(w)$ which depends in general on $m_2 + n$ states. Hence, in some sense, m_1 states are synchronized with the master system; however they may also depend on the remaining m_2 states of the slave system, $\underline{x}_{S2}$, therefore, one gets the map $\Phi_{S1}(x_S) = \pi(x_M, \Phi_{S2}(x_S))$ (see Figure 1b). In this case only partial synchronization will be reached.

III-C. Details on designing synchronization force

Since synchronization equations (13ñ) hold for any dimension for the slave and master systems, depending on the master system dimension and the relative degree of system (9), two cases are considered:

Case 1: If $\rho < n$, the diffeomorphism $\xi = \Phi_M(x_M)$, where $\xi_i = L_{f_M}^{i-1} h_M(x_M)$, $i = 1, 2, \ldots, \rho$, while $\xi_j = \phi_{M,j}(x_M)$, $j = \rho + 1, \ldots, n$, transforms (7) into

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1, 2, \ldots, \rho - 1 \\ \dot{\xi}_\rho &= r_1(\xi) \qquad\qquad (16a) \\ \dot{\xi}_j &= r_{j-\rho+1}(\xi), \qquad j = \rho + 1, \ldots, n \end{aligned}$$

where $r_j(\xi) = \left[L_{f_M} \phi_{M,j}(x_M)\right]_{x_M = \Phi_M^{-1}(\xi)}$ for $j = 1, \ldots, n - \rho + 1$.

Case 2: If $\rho \geq n$, the diffeomorphism $\xi = \Phi_M(x_M) = \mathrm{col}\left\{L_{f_M}^{i-1} h_M(x_M), i = 1, 2, \ldots, n\right\}$, transforms (7) into

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1, 2, \ldots, n - 1 \qquad (17) \\ \dot{\xi}_n &= r(\xi), \end{aligned}$$

where $r(\xi) = \left[L_{f_M}^n h_M(x_M)\right]_{x_M = \Phi_M^{-1}(w)}$.

Notice that for case 2 the master system dimension is necessarily smaller than the slave system dimension, while for case 1 such dimension may smaller, greater or equal.

Using the normal form for both the master and slave systems [equations (17) or (16a) and (13a)], the synchronization error can be written as

$$e = \xi_1 - \zeta_1. \qquad (18)$$

Now let us analyse each case.

III-C.1. Case 1: $\rho < n$: Considering (16a) and (13) for the master and slave systems, respectively, and (18) as the synchronization error, from equations (13ñ) we deduce the following:

The states ζ are completely synchronized with the first ρ states of ξ , since

$$\zeta_i = \pi_i(\xi) = \xi_i, \qquad i = 1, 2, \ldots, \rho, \qquad (19)$$

which in original coordinates is equivalent to

$$L_{f_S}^{i-1} h_S(x_S) = L_{f_M}^{i-1} h_M(x_M), \qquad i = 1, 2, \ldots, \rho. \quad (20)$$

From (20) we deduce that the hyperplane defined by $L_{f_S}^{i-1} h_S(x_S) - L_{f_M}^{i-1} h_M(x_M) = 0$ contains the evolution of the remaining states of $\underline{x}_S$, i.e. η and $\underline{x}_{S2}$. The synchronization input necessary to obtain (20) is given by the mapping

$$\gamma = \frac{r_1(\xi) - a(\xi, \eta_{ss}, \underline{x}_{S2})}{b(\xi, \eta_{ss}, \underline{x}_{S2})} \qquad (21)$$

where η_{ss} is the solution of

$$\dot{\eta}_{ss} = q(\xi, \eta_{ss}, \underline{x}_{S2}), \qquad (22)$$

for a given initial condition $\eta_{ss}(0) = \eta_0$. However, since $\dot{\eta} = q(0, \eta, 0)$ is asymptotically stable, there exists class $\mathcal{K}$ function α_w and a class $\mathcal{KL}$ function β such that

$$\|\eta_{ss}(t)\| \leq \beta(\eta_0, t) + \alpha_w(w) \qquad (23)$$

where $w = \begin{pmatrix} \xi^T & \underline{x}_{S2}^T \end{pmatrix}^T$. Notice that w contains the master and slave variables. Given a large enough time $t \geq T$, $\beta(\eta_0, t) \to 0$, hence for $t \geq T$, η_{ss} no longer depends on initial condition. Then, η_{ss} together with

$$\dot{w} = R(w), \qquad (24)$$

where $R(w) = \mathrm{col}\{\xi_2, \ldots, \xi_\rho, r_1(\xi), \ldots, r_{n-\rho+1}(\xi), \underline{F}_{S2}(\underline{x}_{S2})\}$, generate a central manifold and for this reason for large enough time $\eta_{i,ss} = \pi_{i+\rho}(w)$, $i = 1, 2, \ldots, m_1 - \rho$ and γ eventually depend only on w. From the previous discussion, one conclude that when synchronization is achieved, ζ is totally synchronized, while η is, in some sense, synchronized with the master system; however, it also depends on the Poisson stable states of the slave system, $\underline{x}_{S2}$. Notice that if $\dim(\underline{x}_{S2}) = 0$, the slave system is totally synchronized, i.e., there exists the map $x_S = \Phi_S^{-1}(\pi(\Phi_M(x_M)))$. Here map π allows that $n \neq m$.

Another interesting case is when the master and the slave systems are identical, then it can be defined the same diffeomorphism for both systems, i.e., $\Phi_S(x_S) = \Phi_M(x_M)$. However, if the initial conditions for the Poisson stable subsystems are not the same, its evolution through time may differ and only partial synchronization may be achieved. On the other hand, if the initial conditions of the Poisson stable subsystems are identical, total synchronization can be achieved.

III-C.2. Case 2: $\rho \geq n$: Considering (17) and (13) for the master and slave systems, respectively, and (18) as the synchronization error, from equations (13ñ) we deduce the following:

The first n states of ζ are directly synchronized with ξ, since

$$\zeta_i = \pi_i(\xi) = \xi_i, \qquad i = 1, 2, \ldots, n, \tag{25}$$

while the $\rho - n$ following states of ζ are in the tangent space of ξ_n since

$$\zeta_i = \pi_i(\xi) = L_R^{i-n-1} r(\xi), \qquad i = n+1, n+2, \ldots, \rho, \tag{26}$$

where $R(\xi) = \mathrm{col}\{\xi_2, \ldots, \xi_n, r(\xi)\}$. In original coordinates

$$L_{f_S}^{i-1} h_S(x_S) = L_{f_M}^{i-1} h_M(x_M), \tag{27}$$

$$L_{f_S}^{j-1} h_S(x_S) = L_R^{j-n-1} L_{f_M}^{n} h_M(x_M), \tag{28}$$

where $i = 1, 2, \ldots, n$ and $j = n+1, n+2, \ldots, \rho$, with the elements of R as $R_i(x_M) = L_{f_M}^{i} h_M(x_M)$, $i = 1, 2, \ldots, n$. Therefore complete synchronization is achieved for the first m_1 states of $\underline{x}_S$. On the other hand, The synchronization input is given by the mapping

$$\gamma = \frac{r(\xi) - a(\xi, \eta_{ss}, \underline{x}_{S2})}{b(\xi, \eta_{ss}, \underline{x}_{S2})} \tag{29}$$

where η_{ss} is the solution of

$$\dot{\eta}_{ss} = q(\xi, \eta_{ss}, \underline{x}_{S2}), \tag{30}$$

which, for the same reasons described in the case 1, forms a central manifold with ξ and $\underline{x}_{S2}$ and therefore the synchronization input γ depends only on ξ and $\underline{x}_{S2}$.

Finally, for both cases, if $\dim(\underline{x}_{S2}) > 0$ but $\dim(\eta) = 0$, only ζ is synchronized with x_M and $\underline{x}_{S2}$ only affects the synchronization input γ.

IV. Examples

In this section we present three workbench examples.

Example 1: **Identical master and slave systems:** Consider a Duffing system

$$y"(t) - y(t) + y^3(t) + \delta\dot{y}(t) = \tau(t) \tag{31}$$

where $\tau(t) = \alpha\sin(ct + \theta)$ represents an oscillatory driving signal, which can be described by the dynamical equation $\tau"(t) = -c^2\tau(t)$. This system is use as a master system to synchronize the system

$$z"(t) - z(t) + z^3(t) + \delta\dot{z}(t) = \widehat{\tau}(t) + u(t) \tag{32}$$

where $u(t)$ is the input used for the synchronization while $\widehat{\tau}(t)$ is an oscillatory driven input. We consider that the synchronization error is $e = z - y$. The state space representation of the master and the slave systems are

Master: $\dot{x}_{M,1} = x_{M,2}$, $\dot{x}_{M,2} = F(x_M)$, $\dot{x}_{M,3} = cx_{M,4}$ and $\dot{x}_{M,4} = -cx_{M,3}$.

Slave: $\dot{x}_{S,1} = x_{S,2}$, $\dot{x}_{S,2} = F(x_S) + u(t)$, $\dot{x}_{S,3} = cx_{S,4}$ and $\dot{x}_{S,4} = -cx_{S,3}$,

where $F(x) = x_1 - x_1^3 - \delta x_2 + x_3$, while the synchronization error is $e = x_{S,1} - x_{M,1}$. These systems can be written as (16a) and (13) if we define $\xi = x_M$, $\zeta = \begin{pmatrix} x_{S,1} & x_{S,2} \end{pmatrix}^T$, $\underline{x}_{S2} = \begin{pmatrix} x_{S,3} & x_{S,4} \end{pmatrix}^T$, $a(\zeta, \underline{x}_{S2}) = \zeta_1 - \zeta_1^3 - \delta\zeta_2 + \underline{x}_{S2,1}$ and $b(\zeta, \underline{x}_{S2}) = 1$. Notice that $\dim(\eta) = 0$, since both $\underline{x}_{S2}$ and the last two states of ξ are Poisson stable, when synchronization is achieved $x_{S,1} = x_{M,1}$ and $x_{S,2} = x_{M,2}$ however, if the initial conditions of the last two states of the master and slave systems are different, then $x_{S,3} \neq x_{M,3}$ and $x_{S,4} \neq x_{M,4}$. In this case partial synchronization is achieved. On the other hand, if the initial conditions of the last two states of the master and slave systems are identical, then $x_S = x_M$ and complete synchronization is obtained.

Example 2: **Synchronization of different systems:** Lets consider now the Rössler model as a master system and the Lorenz model as its slave system, where mappings f_M, h_M, f_S, g_S and h_S are respectively:

Rössler's:

$$f_M(x_M) = \begin{pmatrix} -x_{M,2} - x_{M,3} \\ x_{M,1} + ax_{M,2} \\ a + x_{M,3}(x_{M,1} - b) \end{pmatrix}, \quad h_M(x_M) = x_{M,2}, \tag{33}$$

Lorenz's:

$$f_S(x_S) = \begin{pmatrix} \sigma(x_{S,2} - x_{S,1}) \\ \rho x_{S,1} - x_{S,2} - x_{S,1}x_{S,3} \\ -\beta x_{S,3} + x_{S,1}x_{S,2} \end{pmatrix}, \quad g_S(x_S) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{34}$$

and $h_S(x_S) = x_{S,1}$, where a, b, σ, ρ and β are positive constants. The relative degree of the slave system is 2. Therefore, by defining $\zeta = \begin{pmatrix} x_{S,1} & \sigma(x_{S,2} - x_{S,1}) \end{pmatrix}^T$ and $\eta = x_{S,3}$, the normal form of (34) is similar to (13a)-(13d) where $a(\zeta, \eta) = \sigma(\rho - 1)\zeta_1 - (1 + \sigma)\zeta_2 - \sigma\zeta_1\eta$, $b(\zeta, \eta) = \sigma$ and $q(\zeta, \eta) = -\beta\eta + \zeta_1^2 + \sigma^{-1}\zeta_1\zeta_2$. Notice that $\dim(\underline{x}_{S2}) = 0$ and that $q(0, \eta) = -\beta\eta$ is asymptotically stable. On the other hand, if one define $\xi = \begin{pmatrix} x_{M,2} & x_{M,1} + ax_{M,2} & x_{M,3} - a/b \end{pmatrix}^T$, then the Rössler model is similar to (16a) with $r_1(\xi) = -\xi_1 + a(\xi_2 + 1/b) - \xi_3$ and $r_2(\xi) = -b\xi_3 + (\xi_3 + a/b)(\xi_2 - a\xi_1)$. Notice that when $\xi = 0$, $r_2(\xi) = -b\xi_3$, which is asymptotically stable.

The input γ given by (21) is

$$\gamma(\xi) = \left(1-\rho-\frac{1}{\sigma}\right)\xi_1 + \left(1+\frac{1+a}{\sigma}\right)\xi_2 - \frac{1}{\sigma}\xi_3 + \xi_1\eta_{ss}, \tag{35}$$

where η_{ss} is the solution of $\dot{\eta}_{ss} = -\beta\eta_{ss} + \zeta_1^2 + \sigma^{-1}\zeta_1\zeta_2$ given by

$$\eta_{ss}(t) = \eta_0 e^{-\beta t} + \int_0^t \xi_1(\tau)\left[\xi_1(\tau) + \frac{1}{\sigma}\xi_2(\tau)\right]e^{\beta(t-\tau)}d\tau. \tag{36}$$

For large enough time, the first term of (36) disappears and η_{ss} depends only on ξ_1 and ξ_2. Therefore, when synchronization is achieved, $x_{S,1} = x_{M,2}$, and $x_{S,2} = x_{M,1}/\sigma + (1+a/\sigma)x_{M,2}$, while $x_{S,3} = \int_0^t x_{M,2}(\tau)\,x_{S,2}(\tau)\,e^{\beta(t-\tau)}d\tau$ and complete synchronization is achieved.

Example 3: **Synchronization of systems with different order:**

Now we consider the synchronization of Duffing equation similar to the one considered in example 1,

$$y"(t) - y(t) + y^3(t) + \delta\dot{y}(t) = \tau(t) + u(t) \tag{37}$$

where $\tau(t) = \alpha\sin(ct+\theta)$ is a driving signal, with the Chua system as the master system. Chua system is an electronic circuit with one nonlinear resistive element. The circuit equations can be written as a third order system which is given by the following dimensionless form

$$\dot{x}_{M,1} = \gamma_1\left(x_{M,2} - x_{M,1} - f(x_{M,1})\right) \tag{38}$$

$$\dot{x}_{M,2} = x_{M,1} - x_{M,2} + x_{M,3} \tag{39}$$

$$\dot{x}_{M,3} = -\gamma_2 x_{M,2} \tag{40}$$

where $f(x_{M,1}) = \gamma_3 x_{M,1} + 0{,}5(\gamma_4 - \gamma_3)\cdot[|x_1+1| - |x_1-1|]$, while the slave system is $\dot{x}_{S,1} = x_{S,2}$, $\dot{x}_{S,2} = F(x_S) + u(t)$, $\dot{x}_{S,3} = cx_{S,4}$ and $\dot{x}_{S,4} = -cx_{S,3}$, with $F(x_S) = x_{S,1} - x_{S,1}^3 - \delta x_{S,2} + x_{S,3}$. Notice that $x_S \in \mathbb{R}^4$ and $x_M \in \mathbb{R}^3$, hence the slave system has higher dimension than the master system. Defining $\zeta = \begin{pmatrix} x_{S,1} & x_{S,2}\end{pmatrix}^T$, $\underline{x}_{S2} = \begin{pmatrix} x_{S,3} & x_{S,4}\end{pmatrix}^T$, $a(\zeta, \underline{x}_{S2}) = \zeta_1 - \zeta_1^3 - \delta\zeta_2 + \underline{x}_{S2,1}$ and $b(\zeta, \underline{x}_{S2}) = 1$, Duffing system can be written as (13). On the other hand, Chua system can be written as (16a) if we define $\xi = \begin{pmatrix} x_{M,3} & -\gamma_2 x_{M,2} & -\gamma_2 x_{M,1}\end{pmatrix}^T$. $r_1(\xi) = -\xi_2 - \gamma_2\xi_1 + \xi_3$, and $r_2(\xi) = \gamma_1\left[\xi_2 - \xi_3 - \widehat{f}(\xi_3)\right]$, where $\widehat{f}(\xi_3) = \gamma_3\xi_3 + 0{,}5(\gamma_4 - \gamma_3)(|\xi_3 + \gamma_2| - |\xi_3 - \gamma_2|)$. When synchronization is achieved $x_{S,1} = x_{M,3}$ and $x_{S,2} = -\gamma_2 x_{M,2}$, this explain the chiral behavior (since $\operatorname{sign}(x_{S,2}) = -\operatorname{sign}(x_{M,2})$) (Femat y Solis-Perales, 2008) and why $x_{S,2}$ and $x_{M,2}$ have the same oscillatory frequencies but with a different amplitude (it relation is given by $-\gamma_2$). $x_{S,3}$ and $x_{S,4}$ remain independents and the synchronization input (21) must be $\gamma(\xi, \underline{x}_{S2}) = -\xi_2 - \gamma_2\xi_1 + \xi_3 - \zeta_1 + \zeta_1^3 + \delta\zeta_2 - \underline{x}_{S2,1}$, which depends on ξ and $\underline{x}_{S2}$.

V. Conclusions

A synchronization analysis of chaotic systems has been developed. The proposed analysis approach is based on the extension of the regulation theory. As a result, the synchronization manifold and the driving force necessarily to obtain the synchronization can established by solving a set of partial differential equations. Using this approach it is been possible to explain the behaviors observed in the synchronization practice, for instance, the complete and partial-state synchronization as well as the phase synchronization. This methodology can be systematically applied in order to predict when total or partial synchronization will be achieved and, though the synchronization manifold, to elucidate which is the relation between master and slave synchronized states. A main advantage of this approach lies in the possibility to analyse the synchronization phenomena for strictly different systems as well as for different order systems. Finally some workbench examples has been presented to illustrate the results.

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