AN EXTENSION TO CHAOS CONTROL VIA LIE DERIVATIVES

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Abstract— **The technique of using Lie derivatives to control chaos is extended in this contribution. Here, by using Lie derivatives in an extended space-state, it is proved that chaos can be practically suppressed via feedback in spite the Lie derivative is ill-posed at the reference. The main idea is to construct a dynamically equivalent system. In this way, the chaotic system can be practically stabilized around any point of singularity x^o. Lorenz equation is used as illustrative example to show the application on the chaos control context.**

I. INTRODUCTION

Application of the Lie derivative in dynamical systems can be a powerful tool for localization of periodic orbits [1] and controlling dynamical systems [2]. Thus the history of the Lie derivative has been held in physical [3] and chemical processes [4]. In addition, Lie derivative has been applied to chaos control [2]. As matter of fact, chaotic synchronization can be achieved [1] and feedback approaches can be performed from Lie derivative toward chaos applications [5],[6]. The idea behind control via Lie derivatives is to design a feedback such that the nonlinear terms in dynamical system can be counteracted because the dynamical system is inverted via Lie derivatives of the measured states (system output). In this sense, designing the feedback control via Lie derivatives departs from considering a nonlinear system that is given by (affine nonlinear system): $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control command, $f(x)$ and $g(x)$ are smooth vector fields. It is often assumed that the measured states are given by a smooth function, i.e. $y = h(x) \in \mathbb{R}^k$. The following condition is commonly imposed for the control design [2]: *The relative degree should be well- posed in all point belonging to space-state.* This implies that a coordinate transformation, which is constructed via Lie derivatives, is invertible at any points belonging to space-state. One should note that main control objective is to lead the trajectories of the nonlinear affine system to any prescribed point (or points set) within space-state. The interesting problem arises if: (a) coordinates transformation is not invertible at any point x^o within the prescribed reference or (b) the trajectory of the controlled system go through the point x^o. Such a problem could be solved, if it is physically possible, when output $h(x)$ and/or vector field $g(x)$ are modified. However, on the one hand, physical restrictions to get measurements of any observable and to select the vector field $g(x)$ can involve that in a given system neither output function $h(x)$ nor vector field $g(x)$ can be changed. On the other hand, from the mathematical viewpoint, an interesting problem is to approach the chaos suppression via Lie derivative when the reference includes the point of singularity. Here, an extension to Lie derivatives in dynamical systems is detailed.

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The extension is focused in chaos control. The main contribution of the paper is that chaotic systems can be controlled by means of Lie derivative even if the resulting feedback yields an unstable closed-loop behavior when the vector field $g(x)$ and output function $h(x)$ are given. The main idea is to construct an extended system which is dynamically equivalent via an augmented state. After that, a state estimator is designed to get a dynamic output feedback. In this way, the estimated value of the augmented state leads the trajectories of the system at any arbitrarily small neighborhood U of the point of singularity x^o.

The paper is organized as follows. Some preliminaries and the problem statement are presented in next section. Third section contains main results. Two illustrative examples are discussed in Sect. 4. Concerning chaos control, Lorenz equation is used to illustrate the approach in the chaos context. The text is closed with some concluding remarks.

II. STATE-FEEDBACK CHAOS CONTROL

Definition 1 *Let $\mu(x) = (\mu_1(x), \mu_2(x), ...\mu_n(x))^T$ be a set of n smooth vector fields where $x \in \mathbb{R}^n$. A distribution is the assignation $\Delta = span\{\mu_1, \mu_2, ..., \mu_n\}$ of a vector space to each point x in $\mathbb{R}^n$ and the value of the distribution at the point $x \in \mathbb{R}^n$ is given by $\Delta(x) = span\{\mu_1(x), \mu_2(x), ..., \mu_n(x)\}$.* •

Lemma 1 *(Isidori [2]) Let $\Delta = span\{\mu_1, \mu_2, ..., \mu_n\}$ be a smooth distribution for a vector fields $\mu(x) = (\mu_1(x), \mu_2(x), ...\mu_n(x))^T$. The distribution Δ is involutive if and only if the Lie product $[\mu_i, \mu_j] \in \Delta$, for $1 \le i, j \le n$.* ∎

Let us consider the affine nonlinear system $\dot{x} = f(x)+g(x)u$, where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ stands for the control command, $f(x)$ and $g(x)$ are smooth vector fields. Here, we assume that the relative degree $\rho = n$ (i.e., affine system is fully linearizable). If conditions (A.1) and (A.2) hold for any $x \in U^o$ of x^o, there exists a coordinates transformation $z = \Phi(x)$ such that the affine nonlinear system can be rewritten in the following normal form

$$\begin{aligned} \dot{z}_i &= z_{i+1}; i = 1, 2, \ldots, n-1 \\ \dot{z}_n &= \alpha(\Phi(x)) + \gamma(\Phi(x))u \qquad (1) \\ y &= z_1 \end{aligned}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}$ and the nonlinear functions $(\alpha(\Phi(x)) = \mathscr{L}_f^\rho h(x)$ and $\gamma(\Phi(x)) = \mathscr{L}_g \mathscr{L}_f^{\rho-1} h(x)$ are continuos functions. Since relative degree $\rho = n$ is defined for any $x \in U^o$, then there exists a function $u = u(z)$ such that the local asymptotic stabilization problem can be solved. That is, the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is diffeomorphic[1, 2].

Remark 1. System (2) can be constructed because the distribution in involutive at any point $x \in \mathbb{R}^n$. This signifies that the distribution $\Delta(x)$ in integrable. Hence, since the assignation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is diffeomorphic, there exists a map $\Phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $x = \Phi^{-1}(\Phi(x))$.

Corollary 1. Under the nonlinear feedback $u = (\mathscr{L}_f^r h(x) + \sum_i K_i \mathscr{L}_f^i h(x))/\mathscr{L}_g \mathscr{L}_f^{r-1} h(x) = D\Phi(x)^{-1} K(h(x) - r(t))$ for any $K = (K_1, K_2, \ldots, K_n)^T$, the system (2), and as consequence the affine system, can be leaded to any prescribed reference $r(t)$. ∎

III. MAIN RESULTS

Proposition 1. Let $\Delta^*(x)$ be an involutive distribution at any $x \in U \subset U^o$ of x^o, where the subset U is arbitrarily small and $x^o \neq x$. Let $g(x)$ be a vector field such that $g(x) \in \Delta^*(x)$. Then, there exists a vector field $\hat{g}(x)$ (bounded away from zero at x^o) such that $\Delta^*(x) = span\{g(x), \hat{g}(x)\}$ is involutive at any $x \in U^o$ of x^o.

Proof. The proof is straightforward if we consider any vector fields $g, \hat{g} \in \Delta$ at $x \in U$ such that the Lie bracket, $[g, \hat{g}](x)$, belongs to the distribution $\Delta(x)$ for any $x \in U \subset U^o$ of x^o, *i.e.* $\Delta^*(x) = span\{[g(x), \hat{g}(x), [g, \hat{g}](x)\}$. ∎

Remark 2. If Proposition 1 holds for any $x \in U \subset U^o$ of x^o, where x is arbitrarily close to x^o, then this property implies that an estimated value of the vector field $g(x)$ can be represented by vector field $\hat{g}(x) \neq g(x)$.

Now, let us assume the following: *Assumption S.1)* $f(x), g(x)$ and $h(x)$ are such that $\mathscr{L}_g \mathscr{L}_f^\rho h(x) \neq 0$ for any positive integer ρ and $x \neq x^o$ contained in U^o of x^o but $\mathscr{L}_g \mathscr{L}_f^\rho h(x) = 0$ at $x^o \in U^o$ for any x^o within reference. *Assumption S.2)* $\hat{g}(x)$ is known and bounded away from zero at $x^o \in U^o$. *Assumption S.3)* $\hat{g}(x)$ is such that (i) $\mathscr{L}_{\hat{g}} \mathscr{L}_f h(x) \neq 0$ at $x \in U^o$ of x^o and (ii) $sign(\hat{g}(x^o)) \equiv sign(g(x^o))$.

Lemma 2. Let us define $\delta(z) = \gamma(z) - \hat{\gamma}, \Theta(z, u) = \alpha(z) + \delta(z)u$, where $\alpha(z) = \mathscr{L}_f^\rho h(x)$

and $h(x)$ is the system output. In addition, let $\eta(t) = \Theta(z(t), u(z(t))$ be a state variable such that the system (3) can be computed. Then, there exists a time-invariant manifold $\Psi(z, \eta, u) = 0$ such that the solution of the system (1) is a projection of the solution of the following dynamical system

$$\begin{aligned} \dot{z}_i &= z_{i+1} \\ \dot{z}_\rho &= \eta + \hat{\gamma} u \\ \dot{\eta} &= \Gamma(z, \eta, \mathcal{U}) \\ y &= z_1 \end{aligned} \tag{2}$$

where $i = 1, 2, \ldots, \rho$ and $\Gamma(z, \eta, \mathcal{U}) = \mathscr{L}_\Phi \alpha(z) + u \mathscr{L}_\Phi \delta(z) + \delta(z)\dot{u}$ is given by the Lie derivative along the vector field $\Phi(z) = (z_2, z_3, \ldots, z_\rho, \eta)^T$ and $\mathcal{U} = (u, \dot{u}, \ddot{u}, \ldots, u^m)^T$ is a vector whose components are the input function and its time-derivative; *i.e.*, the system (3) is dynamically equivalent to the system (1).

Proof. The manifold $\Psi(z, \eta, u) = \eta - \Theta(z, u) = 0$ is, by definition, time-invariant. In fact, it is straightforward to prove that the set $\Psi = \{\Psi(z, \eta, u) = \eta - \Theta(z, u) : \dot{\eta} = \sum_{j=1}^{r-1} z_{j+1} \partial_j \Theta(z, u) + (\eta + \hat{\gamma} u) \partial_r \Theta(z, \nu, u) + \delta(z, \nu)\dot{u}\}$ satisfies $d\Psi dt \equiv 0$ for all $t \geq 0$. Now, from the equality $\Psi(z, \eta, u) = 0$ and condition $d\Psi/dt \equiv 0$, one can take the first-integral of the system (3) to get $\eta(t) = \Theta(z(t), u(z(t))$. When the first-integral is back-substituted into the system (3), we obtain the solution of the system (1). Hence, the solution of the system (1) is a projection of the system (3) via the module $\pi \cdot (z, \eta) = z$. This is, the system (3) is dynamically equivalent to the system (1) if initial conditions, $(z(0), \eta(0))$, are contained in $\Psi(z, \eta,)$. ∎

Remark 3. According to Proposition 1, around the point x^o, there exists an estimated value $\hat{\gamma}$ of the nonlinear function $\gamma(z)$ such that $sign(\gamma(z)) = sign(\hat{\gamma})$. This is, since the vector field $\hat{g} \in span\{g(x), \hat{g}(x)\}$ at the point $x \in U \subset U^o$ of $x^o \neq x$, for any x arbitrarily close to x^o, the Lie derivative of the system output along $\hat{g}(x)$, namely $\mathscr{L}_{\hat{g}} \mathscr{L}_f h(x)$, can be used as an approach of the Lie derivative sof output along $g(x)$; i.e., the estimated value can be computed for the real-valuated function, $\hat{\gamma} = \mathscr{L}_{\hat{g}} \mathscr{L}_f h(x)$, at the point x^o.

Theorem 1. Let us consider the state feedback $u = -(\eta + K^T(z - z^*))/\hat{\gamma}$, where $z^* = [x^o, \dot{x}^o, \ddot{x}^o, \ldots, x^{(r-1)o}]^T$ and $K > 0$. Under above controller, the trajectories of the system (3) is asymptotically practically stable at a neighborhood arbitrarily small U of the point of singularity x^o.

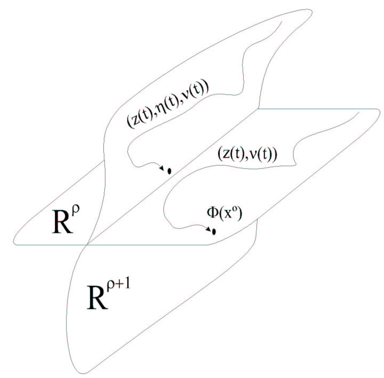

Figure 1. *Schematic notion of the Lemma 2.*

Proof. Without lost of generality, let us choose the prescribed and no regular point at $x^o = 0$. Now, by definition, $\alpha(z)$ and $\gamma(z)$ are smooth, hence $\Gamma(z, \eta, \mathcal{U}) = \mathscr{L}_\Phi \alpha(z) + u \mathscr{L}_\Phi \delta(z) + \delta(z)\dot{u}$ is a smooth function. As a consequence, the dynamics of the state feedback controller, $\dot{u} = -(\Gamma(z, \eta, \mathcal{U}) + \sum_i K_i z_{i+1} + K_r(\eta + \hat{\gamma} u))/\hat{\gamma}$, is also smooth. In addition, from the fact that $\eta = \alpha(z) + (\gamma(z) - \hat{\gamma})u$ and under the state feedback, we have that $\eta = (\hat{\gamma}\alpha(z) + \delta(z) K^T z)/\gamma(z)$; from where if z is bounded then η is bounded, which implies that the state feedback is bounded.

On the other hand, $\Psi(z, \eta, u) = \eta - \Theta(z, u)$ is the first integral of the system (3). Therefore, convergence to zero follows from the fact that closed-loop system is in cascade form. This is, since $K^T > 0$ are coefficient of the Hurwitz polynomial $P_\rho(s) = s^\rho + K_1 s^{\rho-1} + \ldots + K_{\rho-1} s + K_\rho = 0$, then $z(t) \to 0$, which implies that $(z, \eta) \to 0$. ∎

Corollary 2. Consider the point x^o where $\mathscr{L}_g \mathscr{L}_f^{\rho-1} h(x) = 0$ (*i.e.*, it is point of singularity). If there exists a vector field $\hat{g}(x) \neq g(x)$ such that: (i) $sign(\hat{\gamma}) = sign(\gamma)$ at any point $x \in U \subset U^o$ of x^o; where x is close to x^o (*i.e.*, $\|x - x^o\| < \delta$, for any $\delta > 0$ arbitrarily small) and (ii) $\hat{\gamma} = \mathscr{L}_{\hat{g}} \mathscr{L}_f h(x) \neq 0$. Therefore the system (3) can be stabilized around the point x^o by the state feedback, $u = u(z, \eta)$. That is, the state feedback $u = u(z, \eta)$ leads the trajectories of the system (1) to a neighborhood arbitrarily small of the point of singularity x^o. ■

Theorem 2. There is a set of initial conditions, $(z(0), \eta(0)) \in M \subseteq \mathbb{R}^{\rho+1}$, such that the states trajectories of the system (3) can be reconstructed, for every admissible input function, $u = u(z, \eta)$, by means of the following observer

$$\begin{aligned} \dot{\hat{z}}_i &= \hat{z}_{i+1} + L^i \kappa_i(\hat{z}_1 - z_1); i = 1, 2, \ldots, \rho - 1 \\ \dot{\hat{z}}_\rho &= \hat{\eta} + \hat{\gamma} u + L^\rho \kappa_\rho(\hat{z}_1 - z_1) \\ \dot{\hat{\eta}} &= L^{\rho+1}\kappa_{\rho+1}(\hat{z}_1 - z_1) \end{aligned} \qquad (3)$$

where $(\hat{z}, \hat{\eta})$ are estimated values of (z, η), respectively.

Proof. In addition, let $e \in \mathbb{R}^{\rho+1}$ an error vector whose components are defined by $e_i = L^{\rho+1-i}(z_i - \hat{z}_i), i = 1, 2, \ldots, n$ and $e_{\rho+1} = \eta - \hat{\eta}$, for any $L > 0$. Then, under a bounded and smooth input function, the dynamics of the estimation error is given by $\dot{e} = LA(\kappa)e + \Psi(z, \eta, N(L)e, u, \dot{u})$, where the matrix $N(L) = diag(L^{-\rho}, L^{-(\rho+1),\ldots,L^{-1},1})$ and $A(\kappa)$ becomes

$$A(\kappa) = \begin{bmatrix} -\kappa_1 & 1 & 0 & \cdots & 0 \\ -\kappa_2 & 0 & 1 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ -\kappa_\rho & \vdots & \vdots & \cdots & 1 \\ -c\kappa_{\rho+1} & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (4)$$

where $c = \gamma(z)/\hat{\gamma}$. The equality $sign(\gamma(z)) = sign(\hat{\gamma})$ implies that $c > 0$, therefore, the estimation constants can be chosen such that the matrix $A(\kappa)$ has all its eigenvalues at the open left-hand of the complex plane. Now, according to Theorem 1, the input function is bounded, therefore $\Psi(z, \eta, N(L), e, u, \dot{u}) = [0, 0, \ldots, \Gamma(z, \eta, u, \dot{u})]^T$ is also bounded. Hence, the integration of the estimation error system yields $e(t) = e(0)exp(A(\kappa)t) + exp(A(\kappa)t) \int exp(-A(\kappa)\sigma)\Psi(z, \eta, N(L)e, u, \dot{u})d\sigma$. Now, using the Triangle and Schwartz inequalities, one has that $\| e(t) \| \leq \| e(0)exp(LA(\kappa)t) \| + exp(LA(\kappa)t) \int \| exp(-LA(\kappa)\sigma)\Psi(z, \eta, N(L)e, u, \dot{u})d\sigma \|$. Since $\Psi(z, \eta, N(L)e, u, \dot{u})$ is bounded, $\| e(t) \| \leq \| e(0)exp(LA(\kappa)t) \| + \beta_1 exp(A(\kappa)t) \| e(0) \| + \beta_2$ for any constants β_1 and β_2 such that $\| \Psi(z, \eta, N(L)e, u, \dot{u}) \| \leq \beta_1 exp(A(\kappa)t) \| e(0) \| + \beta_2$. In other words, $e(t) \to \mathcal{B}(R(L^{-1}))$, where $\mathcal{B}(R(L^{-1}))$ is a ball with radius on the order of L^{-1}. ■

Corollary 3. Let us assume that an estimated value, $(\hat{z}, \hat{\eta})$, of the state (z, η) is provided by the observer (6). Then, the controller $u = -(\hat{\eta} + K^T(\hat{z} - z^*))$, where $K = (K_1, K_2, \ldots, K_\rho)$ and $z^* = [x^o, \dot{x}^o, \ddot{x}^o, \ldots, x^{(r-1)o}]^T$, is a practical stabilizer of the system (3) and consequently of the system (1), at the point x^o. ■

Remark 4. The dynamic feedback output is comprised by the controller $u = -(\hat{\eta} + K^T(\hat{z} - z^*))$ and the observer (4). In this sense, the proposed output feedback does not includes the separation principle, i.e., the dynamical estimator (5) and the above feedback should be simultaneously designed. In addition, since the dynamic estimator is a high-gain observer, peaking phenomenon can be induced. To diminish overshoot, one can use the saturated version of the dynamic output feedback $u_s = Sat\{-\hat{\eta} - K^T(\hat{z} - z^*)\}$, where $Sat : \mathbb{R}^n \to S$ and $S \subset \mathbb{R}^n$ is a bounded set.

Corollary 4. Suppose that the reference signal is a time function, $y^* = y^*(t)$, and assume that it contains no regular points ($i.e.$, $x^o \in X^o \cap Y^*$). Then the system output $y = h(x)$ practically tracks the reference, r, if and only if exists a smooth vector $\hat{g}(x)$ such that $\Delta(x) = span\{g(x), \hat{g}(x)\}$ is involutive at any $x \in U \subset U^o$ of x^o with $x \neq x^o$ and $sign(\hat{\gamma}) = sign(\gamma)$ at x^o. ■

IV. ILLUSTRATIVE EXAMPLE

Here the contribution is illustrated in chaos control context. Lorenz equation has been chosen as example to implement the proposed extension of Lie derivative in chaos suppression. Then, let us consider the following affine system

$$\dot{x} = \begin{pmatrix} \pi_1(x_2 - x_1) \\ \pi_2 x_1 - x_2 - x_1 x_3 \\ -\pi_3 x_3 + x_1 x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \qquad (5)$$

where $\pi \in \mathbb{R}^3$ is a parameter vector and system output is given by $y = h(x) = x_1$. In order to obtain the coordinates transformation $z = (z_1, z_2 z_3)^T = \Phi(x) = (h(x), \mathscr{L}_f h(x), \mathscr{L}_f^2 h(x))^T$, one should compute the Lie derivatives of system output $h(x)$ along $f(x)$, thus one has that

$$\Phi(x) = \begin{pmatrix} x_1 \\ \pi_1(x_2 - x_1) \\ \pi_1(\pi_2 x_1 - x_2 - x_1 x_3 - \pi_1(x_2 - x_1)) \end{pmatrix} \quad (6)$$

from where the Jacobian matrix

$$D\Phi(x) = \begin{bmatrix} 1 & 0 & 0 \\ -\pi_1 & \pi_1 & 0 \\ \pi_1(\pi_2 + \pi_1 - x_3) & -\pi_1(1 + \pi_1) & -\pi_1 x_1 \end{bmatrix} \quad (7)$$

is obviously singular only at any point $x^o \in U^o = \{x^o \in \mathbb{R}^3 : x^o = (0, x_2, x_3)\}$. Note that the system (6) cannot be stabilized around origin. Indeed the relative degree $\rho = 3$ for any point $x \notin U^o$ but it is not defined at subset U^o. As matter of fact, since $\mathscr{L}_g \mathscr{L}_f^2 h(x) = -\pi_1 x_1$, the nonlinear state feedback control $u = (\mathscr{L}_f^\rho h(x) + \sum_i K_i \mathscr{L}_f^i h(x))/\mathscr{L}_g \mathscr{L}_f^{\rho-1} h(x)$ is not defined at subset U^o. One should note that the relative degree ρ is defined for any point $x \neq x^o$ belonging to the neighborhood U of x^o. Hence the system (10), at any $x \in U$ of x^o, can be transformed into the normal form to get

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \alpha(\Phi(x)) + \gamma(\Phi(x))u \end{aligned} \quad (8)$$

where $\alpha(\Phi(x)) = \mathscr{L}_f^3 h(x) = \pi_1^2(\pi_1 + \pi_2 - x_3)(x_2 - x_1) + \pi_1(\pi_3 + \pi_1\pi_2 - \pi_2)x_1 + \pi_1(1 + \pi_2)(x_2 + x_1 x_3) - \pi_1 x_1^2 x_2$ and $\gamma(\Phi(x)) = \mathscr{L}_g \mathscr{L}_f^2 h(x) = -\pi_1 x_1$ or, equivalently in coordinates $z = \Phi(x), \alpha(z) = \pi_1\pi_2 z_2 + p_1 z_1 + p_2(z_1 + z_2 + (2z_2 + z_3)/\pi_1) - \pi_1 z_2(z_3 + (1 + \pi_1)/z_1)$, where the parameters becomes $p_1 = \pi_1(\pi_3 + \pi_2(\pi_1 - 1)), p_2 = \pi_1(1 + \pi_1)$ and $\gamma(z) = -\pi_1 z_1$.

Now, taking the vector field $\hat{g}(x) = (0, x^e/(1 - \pi_1), 0)^T$ for any $x^e \neq x^o$ belonging U of x^o, one has that $\mathscr{L}_{\hat{g}} \mathscr{L}_f^2 h(x) \neq 0$ at x^o. In this manner the estimated value $\hat{\gamma}(\Phi(x)) = -\pi_1 x^e$ and, according to Lemma 2, the extended system can be written as follows

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \eta + \hat{\gamma} u \\ \dot{\eta} &= \Gamma(z, \eta, u, \dot{u}) \end{aligned} \quad (9)$$

where, by definition, $\eta = \Theta(z, u), \Theta(z, u) = \alpha(z) + \delta(z)u$ and $\delta(z) = -\pi_1(z_1 - x^e)$. The extended dynamics is given by the Lie derivative along the vector field $\Phi(z) = (z_2, z_3, \eta)^T$, i.e., $\Gamma(z, \eta, u, \dot{u}) = \mathscr{L}_\Phi \alpha(z) + u\mathscr{L}_\Phi \delta(z) + \delta(z)\dot{u}$. In this manner the estimator (4) takes the form

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + L\kappa_1(z_1 - \hat{z}_1) \\ \dot{\hat{z}}_2 &= \hat{z}_3 + L^2\kappa_2(z_1 - \hat{z}_1) \\ \dot{\hat{z}}_3 &= \hat{\eta} + \hat{\gamma}u + L^3\kappa_3(z_1 - \hat{z}_1) \\ \dot{\hat{\eta}} &= L^4\kappa_4(z_1 - \hat{z}_1) \end{aligned} \quad (10)$$

where $L > 0$ is the estimation parameter and κ_j's, $j = 1, 2, 3, 4$, are such that the matrix

$$A(\kappa) = \begin{bmatrix} -\kappa_1 & 1 & 0 & 0 \\ -\kappa_2 & 0 & 1 & 0 \\ -\kappa_3 & 0 & 0 & 1 \\ -\kappa_4 & 0 & 0 & 0 \end{bmatrix}$$

has all its eigenvalues at the open left-hand complex plane, that is, the polynomial coefficients of the $\lambda^4 + \kappa_1\lambda^3 + \kappa_2\lambda^2 + \kappa_3\lambda + \kappa_4 = 0$ are positive. Thus, the feedback function $u = (-\hat{\eta} + K_3\hat{z}_3 + K_2\hat{z}_2 + K_1\hat{z}_1)/\hat{\gamma}$ asymptotically steers the trajectories of system (6) around origin. Figure 2 shows the performance of the above controller. The estimation constants were arbitrarily chosen such that the matrix (16) has all its eigenvalues located at -1, i.e., $\kappa = (4.0, 6.0, 4.0, 1.0)^T$ and the control parameters also were chosen such that the polynomial $P_3(s) = s^3 + K_3 s^2 + K_2 s + K_1 = 0$ has all its roots located at -80. Figure 2.a shows the stabilization of Lorenz equation around origin for $L = 20$. Note that system (10) is stabilized around origin via Lie derivative in face of $D\Phi(x)$ is singular. The feedback controller was arbitrarily turned on for time $t = 27.0$. The stabilization of the system (10) is can be carried out because the system (15) provides an estimated value of the output function $y = h(x) = x_1$ (see Figure 2b). The estimated value can be computed via the Lie derivative of the nonlinear function $\Theta(z, u)$. Finally, Figure 2.c shows a projection of the Lorenz attractor on (x_2, x_3)-plane. Note that trajectory is leaded around origin.

V. CONCLUSION

A dynamic output feedback was presented in this contribution. The goal is to show that Lie-based

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feedback control allows the stabilization of chaotic system around points of singularity. That is, even if the coordinates transformation $z = \Phi(x)$ is singular for any point of the reference signal, the practical control of nonlinear systems can be achieved around such a point. The main idea is to construct an extended system based on Lie derivative of a nonlinear function $\Theta(z, u)$. The extended system is, as consequence, which is dynamically equivalent to the system in normal form. After that, a high-gain observer is designed for the extended system. In addition, a solution to the tracking problem can be obtained from this approach even if reference signal contains points of singularity. Thus the practical stabilization around no regular points is proved.

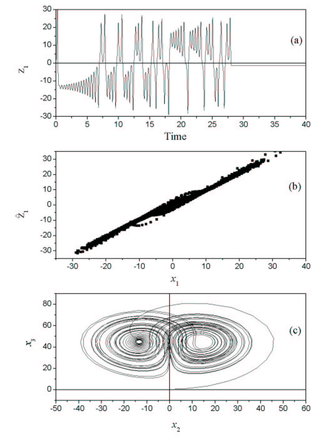

Figure 2. *Stabilization of the Lorenz equation around the singularity point (0,0,0)*

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