Digital Redesign of a Sliding Mode Tracking Controller for Chua's Circuit

Juan G. Barajas-Ramírez
Departamento de Electrónica y Telecomunicaciones
CICESE
Ensenada, B. C., México 22680
E-mail: jbarajas@cicese.mx

Abstract—In this paper, a method to design a discrete-time version of a sliding mode tracking controller is presented. In the proposed method, digital redesign techniques are use to obtain a discrete-time equivalent on a state matching sense to a continuous-time sliding mode controller designed such that the chaotic Chua's circuit tracks a periodic orbit corresponding to an unstable limit cycle contain on its dynamics. Numerical simulations are used to illustrate the effectiveness of the proposed discrete-time sliding mode tracking controller**.**

I. Introduction

In recent years interest on the control of nonlinear systems that exhibit chaotic behavior has increased considerably, as a result numerous approaches have been proposed for chaos control, the reader is refer to [1] for a comprehensive review of methodologies and perspectives on chaos control. One of the most attractive engineering feedback approaches to chaos control is sliding mode controller (SMC) design. Sliding mode is the principal operation of variable structure systems. A VSS is a set of subsystems related under an appropriated switching logic. The SMC control action consists on switching the structure of the feedback controller to force the system to move from one subsystem to another depending on some variables of interest. The switching is designed such that the system evolves "sliding" along a chosen sliding surface, where a set of control objectives are satisfy.

Under some conditions the structural change moves the dynamics so strongly that the system is send not to the sliding surface but a little past it. In consequence, the next control action requires another structure change that does the same, resulting on a high frequency switching around the sliding surface. This effect is called chattering and represents one of the main complications in practical applications of SMC. Fortunately, this effect can be reduced or even eliminated utilizing some modern SMC design techniques, like the reaching law approach and high dimensional sliding mode design [2, 3].

Different methods to design SMC for nonlinear systems are available on the literature. Many of them are based on the transformation of the system into canonical forms by coordinate transformation or feedback linearization. On this paper, an alternative representation of Chua's circuit as a set of switching linear systems is used to derive the corresponding sliding mode tracking controller (SMTC) designed such that Chua's circuit tracks an unstable limit cycle contain on the dynamics of the system.

Nowadays, given the accessibility and low cost of digital systems the occurrence of hybrid control of complex nonlinear systems is ever increasing. However, most of the SMC schemes proposed on the literature are designed either on continuous or discrete time. On this paper the case of a hybrid control system is consider, where the SMC is implemented on a digital device and is applied to a continuous-time plant.

On this regard, the commonly used assumption that with a sufficiently small sampling period the design performance of the continuous-time design will be maintain on the digital device does not hold in general. In fact, the effects of sampling on the stability of the system and the limits on the maximum sampling frequency that a real system can endure, result on degraded performance or even instability. This is further complicated on the case of chaotic systems by the extreme sensibility of chaotic dynamics to small changes on parameters and initial conditions. An alternative design technique for a discrete-time implementation of a continuous-time controller that has been successfully applied to chaotic systems is the state-matching digital redesign approach.

Digital redesign is a design technique that consists on deriving a discrete-time controller for a continuous-time plant by first designing a continuous controller to satisfy a set of control specifications, and then converting it to an equivalent digital controller such that the states of the continuous and discrete-time closed-loop systems are matched at least at each sampling instant for the entire process. During the last decade, Shieh et al. have thoroughly investigated this topic and develop several digital redesign methods that allow for (sub)-optimal control performance even for relatively large sampling periods, while at the same time requiring significantly smaller control energy [4-7].

The rest of the paper is organized as follows: In Section II, the sliding mode tracking controller for Chua's circuit is designed. In Section III, digital redesign is applied to obtain a discrete-time equivalent of the SMTC on the state-matching sense. Numerical simulations, which prove the effectiveness of the proposed methodology, are presented in Section IV. Finally in Section V, conclusions are presented.

II. Sliding Mode Tracker Controller Design

The transient dynamics of a SMC system consist of two modes: 1) the reaching mode, and 2) the sliding mode. Therefore, the design procedure has two fundamental steps: (Step I) Determining an appropriated sliding surface $s(\cdot)$, designed such that the system will have the desired dynamics in sliding mode, and (Step II) Design a control law that guaranties that the reaching and sliding conditions are satisfied.

Consider an affine nonlinear system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ the input vector $(m \le n)$, $f(x(t))$ and $g(x(t))$ are nonlinear functions of appropriated dimension.

The objective of the reaching mode is to reach the switching surface:

$$s(t) = s(x(t)) = \hat{H}\, x(t) = 0 \tag{2}$$

with $rank(\hat{H}) = m$, $\hat{H} \in \Re^{m\times n}$ and $s(x(t)) \in \Re^m$.

Step I: Design an appropriated m dimensional sliding surface $s(x(t))$.

The sliding surface is assigned such that while in sliding mode the system satisfies a set of design objectives like stability, minimization of a performance index, reduction of chattering, etc. In the case of tracking control the sliding surface is define in terms of tracking error

$$e(t) = x(t) - x_r(t) \tag{3}$$

where $x_r(t)$ is the reference or desired trajectory.

Then, a sliding surface can be chosen to be

$$s(e(t)) = \{e(t) \mid s(e,t) = \hat{H}\, e(t) = 0\} \tag{4}$$

where $s(e(t))$ is the error sliding surface and $\hat{H}$ is the coefficient matrix of the sliding surface. Matrix $\hat{H}$ is chosen either as the result of an optimization process or simply as the best choice available from a trial and error search.

Step II: Design the sliding mode control law.

In a SMC the objective of the control law is to ensure that the state moves toward and reach the sliding surface, this is establish as the reaching condition. The control input is determined from the description of this condition, which can be specified following one of three different approaches:

Chronologically, the first approach used to describe the reaching condition was the direct switching function

$$s_i(t)\dot{s}_i(t) < 0 \text{ with } i = 1,2,\ldots,m \tag{5}$$

this is a global reaching condition but there is no assurance of finite reaching time, also for the multi-input case is very difficult to describe.

Another way to specify the reaching condition is through an appropriately chosen Lyapunov-like function

$$V(x,t) = s(t)^T s(t) \tag{6}$$

then, a global reaching condition that guaranties a finite reaching time is obtain as

$$\dot{V}(x,t) < -\varepsilon \quad \text{when} \quad s \ne 0 \text{ and } \varepsilon > 0 \tag{7}$$

An alternative way to express the reaching condition is as a set of differential equations that specify the dynamics during reaching conditions called the reaching law. This is a simple form to express the condition since any differential equation describing an asymptotically stable switching manifold is itself a reaching condition. A bonus of this approach is that using the parameters of the differential equation the dynamic characteristics of the sliding motion in the reaching mode can be assigned.

A commonly used expression of the reaching law is the constant plus proportional reaching law

$$\dot{s}(x(t)) = -\hat{Q}\,\mathrm{sgn}(s(x(t))) - \hat{K}s(x(t)) \tag{8}$$

where $\hat{Q} = diag[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_m],\ \hat{q}_i > 0,$

$$\hat{K} = diag[\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_m],\ \hat{k}_i > 0,$$

$$\mathrm{sgn}(s(t)) = [\mathrm{sgn}(s_1(t)), \ldots, \mathrm{sgn}(s_m(t))]$$

The control law obtained from the time derivative of the constant plus proportional reaching law (8) along the trajectories of the nonlinear system (1) is

$$\frac{\partial s(t)}{\partial x(t)} f(x(t)) + \frac{\partial s(t)}{\partial x(t)} g(x(t))u(t) = -\hat{Q}\,\mathrm{sgn}(s(t)) - \hat{K}\, s(t) \tag{9}$$

with the term $\frac{\partial s(t)}{\partial x(t)} g(x(t))$ nonsingular, one can solve for the control law:

$$u(t) = -\left[\frac{\partial s(t)}{\partial x(t)} g(x(t))\right]^{-1} \left[\frac{\partial s(t)}{\partial x(t)} f(x(t)) + \hat{Q}\,\mathrm{sgn}(s(t)) + \hat{K}\, s(t)\right] \tag{10}$$

In the SMC design is desirable to have a fast approximation to the vicinity of the sliding surface while at the same time, once the switching surface is close, the control action must be small to avoid the over corrections and high frequency switching that leads to the chattering problem at the moment of reaching it. In this regard, tuning the matrices $\hat{Q}$ and $\hat{K}$ of the constant plus proportional reaching law, allows for fast convergence when the state is far and slows down to a constant speed as the state gets close to the switching surface, this almost completely eliminates the chattering effects while the control law remains relatively simple.

II.A. Design of a SMTC for Chua's Circuit

In its dimensionless form Chua's circuit is given by the equation [1]:

$$\begin{aligned}\dot{x}_1(t) &= \alpha\, x_2(t) - \alpha\, x_1(t) - \alpha\, NL(x_1(t))]\\ \dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t)\\ \dot{x}_3(t) &= -\beta\, x_2(t),\end{aligned} \tag{11}$$

where $NL(x_1(t))$ is a piecewise-linear function of $x_1(t)$ describe by

$$NL(x_1(t)) = m_0 x_1(t) + 0.5(m_1 - m_0)v(t)$$

with $v(t) = (|x_1(t)+1| - |x_1(t)-1|)$ and $m_0, m_1 < 0$. This circuit can also be expressed as a set of switching linear systems:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + D_1 & for \quad x_1(t) > 1.0\\ A_2 x(t) + D_2 & for \quad |x_1(t)| \le 1.0\\ A_3 x(t) + D_3 & for \quad x_1(t) < -1.0\end{cases} \tag{12}$$

with $A_1 = A_3 = \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0\\ 1 & -1 & 1\\ 0 & -\beta & 0\end{bmatrix}$,

$$A_2 = \begin{bmatrix} -\alpha(1+m_1) & \alpha & 0\\ 1 & -1 & 1\\ 0 & -\beta & 0\end{bmatrix}, D_1 = \begin{bmatrix} -\alpha(m_1 - m_0)\\ 0\\ 0\end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}, D_3 = \begin{bmatrix} \alpha(m_1 - m_0)\\ 0\\ 0\end{bmatrix}.$$

The equation on (12) for the parameters $\alpha = 9$, $\beta = 100/7$, $m_0 = -5/7$ and $m_1 = -8/7$, is chaotic and produces the well-known double-scroll attractor.

Let the control objective be to track the unstable limit cycle that encircles the chaotic Chua's circuit attractor, which was reported by Chen and Dong on [8] to by approximated by the equation

$$\begin{aligned}x_{r1}(t) &= a\cos(\xi)\cos(\omega t) - b\sin(\xi)\sin(\omega t)\\ &\quad + c\cos(\xi)\cos(2\omega t) - d\sin(\xi)\sin(2\omega t)\\ x_{r2}(t) &= e\,\left(a\cos(\xi)\cos(\omega t) - b\sin(\xi)\sin(\omega t)\right)\\ &\quad + f\,\left(c\sin(\xi)\cos(2\omega t) - d\cos(\xi)\sin(2\omega t)\right)\\ \dot{x}_{r3}(t) &= -\beta\, x_{r2}\end{aligned} \tag{13}$$

with the parameters $a = 2.6$, $b = 1.2$, $c = d = 0.2$, $e = 0.6$, $f = 0.3$, $\xi = \pi/18$, $\omega = 1.77$ and the initial condition for the third coordinate $x_{r3}(0) = -6$.

From the alternative representation of Chua's circuit (12) a SMCT can be design for each of the linear subsystems as follows:

First, choosing a sliding surface for tracking in the form

$$s(e(t)) = \{e(t) \mid s(e,t) = \hat{H}\left(x(t) - x_r(t)\right) = 0\}$$

Then, the reaching law at each linear subsystem is obtain from (10) as

$$\begin{aligned}u(t) = -\hat{H}^{-1}\Big(&\hat{H}\, A_j\, x(t) + \hat{H}\, D_j - \hat{H}\, \dot{x}_r(t)\\ &+ \hat{Q}\, \mathrm{sgn}(s(e(t))) + \hat{K}\, s(e(t))\Big)\end{aligned} \tag{14}$$

with A_j and D_j for $j = 1, 2, 3$ as defined on (12).

III. Digitally Redesigned SMTC

Consider a controllable linear system:

$$\dot{x}_C(t) = Ax_C(t) + Bu_C(t), \quad x_C(0) = x_0 \tag{15}$$

where A and B are constant matrices of appropriate dimensions, and the state feedback controller is given by

$$u_C(t) = -K_C\, x_C(t) + E_C r(t) \tag{16}$$

where the feedback gain $K_C \in \Re^{m\times n}$ and feedforward gain $E_C \in \Re^{m\times m}$ have been previously obtained to satisfy a set a of control objectives, and $r(t)$ is an $m \times 1$ reference input.

The closed-loop system becomes

$$\dot{x}_C(t) = (A - BK_C)x_C(t) + BE_C r(t) \tag{17}$$

Now, consider a discrete piecewise-constant control law

$$u_d(t) = u_d(kT) = -K_d \quad x_d(kT) + E_d r(kT) \tag{18}$$

for $kT \le t < (k+1)T$ where k is an integer, $T > 0$ the sample-hold period, $K_d \in \Re^{m\times n}$ and $E_d \in \Re^{m\times m}$ are the feedback and feedforward digital gains, respectably. With this controller the closed-loop digitally controlled system becomes

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(kT), \tag{19}$$

The objective of the digital redesign is to find the digital gains (K_d, E_d) in (18) from the continuous-time controller gains (K_C, E_C) in (16), such that the closed-loop continuous-time controlled states $x_C(t)$ in (17) closely match the closed-loop digitally controlled state $x_d(t)$ in (19), at every sampling instant, throughout the entire process.

The corresponding discrete-time representation of (17) for a piecewise-constant reference, $r(t)=r(kT)$, $kT \le t < (k+1)T$, is given by

$$x_C(kT+T) = G_C x_C(kT) + H_C E_C r(kT) \qquad (20)$$

where $G_C = e^{A_C T}$ and $H_C = (G_C - I_n)A_C^{-1}B$ with $A_C = A - BK_C$.

Applying via a zero-order hold device the control law (18) to (15), the closed-loop hybrid system becomes

$$\begin{aligned} \dot{x}_d(t) &= Ax_d(t) - BK_d x_d(kT) + BE_d r(kT) \\ x_d(0) &= x_0 \end{aligned} \qquad (21)$$

The corresponding discrete-time model of the hybrid system is

$$\begin{aligned} x_d(kT+T) = (G - HK_d)x_d(kT) \\ + HE_d r(kT) \end{aligned} \qquad (22)$$

where $G = e^{AT}$ and $H = (G - I_n)A^{-1}B.$

The digital gains (K_d, E_d) can be determine taking into account the intersampled values of the continuous-time system using the approximation method call general Chebyshev quadrature formula, which is define as follows:

$$\int_a^b w(\lambda) f(\lambda) d\lambda \cong W \sum_{i=0}^{N} f(\lambda_i) \qquad (23)$$

where $w(\lambda)$ is a constant sign weighting function in $[a,b]$, W is the weighting factor determined by $W = \frac{1}{(N+1)} \int_a^b w(\lambda) d\lambda,$ and $f(\lambda_i)$ are the values of the function $f(\lambda)$ evaluated at $\lambda = \frac{a + i(b-a)}{N}$, for $i = 0,1,2,..., N$.

Rewriting the discrete-time model (20) as

$$x_C(kT+T) = Gx_C(kT) - \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)} BK_C x(\tau) d\tau + HE_C r(kT) \qquad (24)$$

The matrix convolution integral on (24) can be solved by way of the Chebyshev quadrature formula has:

$$\int_{kT}^{(k+1)T} e^{A(kT+T-\tau)} BK_C x(\tau) d\tau = \frac{1}{N+1} HK_C x_C(kT + iTN)$$

The intersampled state $x_C(kT + iT_N)$, with $T_N \equiv \frac{T}{N}$ is given by

$$x_C(kT + iT_N) = G_C^{(i)} x_C(kT) + H_C^{(i)} E_C r(kT) \qquad (25)$$

where $G_C^{(i)} = e^{A_C T_N}$ and $H_C^{(i)} = (G_C^{(i)} - I_n)A_C^{-1}B.$

Using these results the discrete-time model (20) is found to be

$$\begin{aligned} x_C(kT+T) &= Gx_C(kT) \\ &- H\left[\frac{1}{N+1} K_C \sum_{i=0}^{N} G_c^{(i)}\right] x_C(kT) \\ &+ H\left[I_m - \frac{1}{N+1} K_C \sum_{i=0}^{N} H_c^{(i)}\right] E_C r(kT) \end{aligned} \qquad (26)$$

To satisfy the requirement that the closed-loop states $x_C(kT+T)$ and $x_C(kT)$ in (26), considering the intersampled states $x_C(kT + iT_N)$ $i = 0,1,2,..., N$, matched the closed-loop states $x_d(kT+T)$ and $x_d(kT)$ in (22), one must have

$$\begin{aligned} G - HK_d &= G - H\left[\frac{1}{N+1} K_C \sum_{i=0}^{N} G_c^{(i)}\right] \\ HE_d &= H\left[I_m - \frac{1}{N+1} K_C \sum_{i=0}^{N} H_c^{(i)}\right] E_d \end{aligned} \qquad (27)$$

By letting $N \to \infty$ and solving from (27) the digital gains are found to be

$$\begin{aligned} K_d &= K_C (A_C T)^{-1} (G_C - I_n) \\ E_d &= [I_m + (K_C - K_d) A_C^{-1} B] E_C \end{aligned} \qquad (28)$$

To apply this digital redesign technique to the continuous-time SMTC designed for Chua's circuit is necessary to express (14) in the structure of a feedback controller similar to (16). This can be done defining the new variables

$$K_{jC} = \hat{H}^{-1} \hat{K} \hat{H} + A_j \qquad (29)$$

$$E_{jC} = \hat{H}^{-1} \qquad (30)$$

and a new reference

$$\begin{aligned} ref_C(t) &= \hat{K}\,\hat{H}\,x_r(t) + \hat{H}\,\dot{x}_r(t) \\ &- \hat{H}\,D_j - \hat{Q}\,\mathrm{sgn}(s(e(t))) \end{aligned} \qquad (31)$$

The matrix gain K_{jC} and E_{jC} represent the feedback and feedforward gains of the SMTC and the discontinuous reference input $ref_C(t)$ corresponds to the original reference input $x_r(t)$ and the discontinuous parts of the controller. Then, the SMTC (14) can be express as

$$u_C(t) = -K_{jC}\, x_C(t) + E_{jC} ref_C(t) \qquad (32)$$

Applying the Chebyshev based digital redesign method (28), a discrete-time equivalent to (32) can be found as:

$$u_d(kT) = -K_{jd}\, x_d(kT) + E_{jd} ref_d(kT) \qquad (33)$$

where $x_d(kT)$ and $ref_d(kT)$ are obtain through a sample-hold device, such that they are piecewise-constant.

Is important to note that during sliding mode the SMTC imposes the sliding dynamics on the system. Therefore, the controlled system is extremely robust to parameter uncertainties. In fact, for ideal switching the controller is invariant to changes on the parameters of the system. Unfortunately, invariance is lost when an approximated realizable switching scheme is used. In this case, the dynamics evolve along a layer around the sliding surface the closeness between the sliding surface and the layer depends on how well the scheme approximates ideal switching.

Using the discrete-time control law (33) implies approximations to ideal switching for both the alternative model of Chua's circuit and the discontinuous controller, here the switching will not occur at a continuous-time instant as required by the original design, but at the discrete-time moment when the conditions for switching are satisfy. In other words, the discrete-time version of the SMTC uses an approximation to the switching laws of the model and the sign function. Therefore, a layer around the sliding surface will by generated, on which the hybrid system will evolve. On this regard, given that the discrete-time SMTC is derived to match the continuous-time design is assume that the resulting layer is also attracting for the hybrid control system, this assumption is confirm by the numerical results presented on the next section.

IV. Numerical Simulations

The sampled-data hybrid SMTC for Chua's circuit was implemented using SimuLink of MatLab with a fixed integration step $\tau = 0.005$ fourth order Runge-Kutta integration algorithm and a sample-hold period of $T = 0.1$ At this point is important to note two things about the implementation of the SMTC for Chua's circuit:

(1) The coefficient matrix is chosen after a trial and error search to be

$$\hat{H} = \begin{bmatrix} -8.9851 & -7.3033 & 7.2837 \\ -7.2973 & -7.2944 & -6.7039 \\ 5.6752 & 1.5060 & -9.4379 \end{bmatrix}$$

The reaching mode dynamics are set by the choice of the parameters of the reaching law (8) to have a fast convergence and small chattering the following values were used:

$$\hat{Q} = diag([0.7, 0.7, 0.7]) \qquad \hat{K} = diag([17, 17, 17])$$

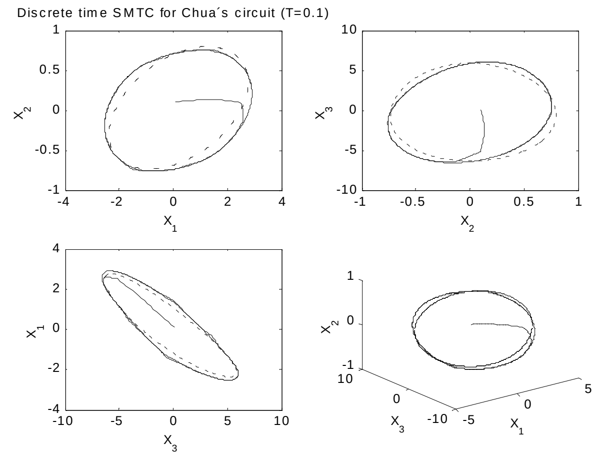

Figure 1. Projections of the digitally redesigned SMTC for Chua's circuit with $T = 0.1$

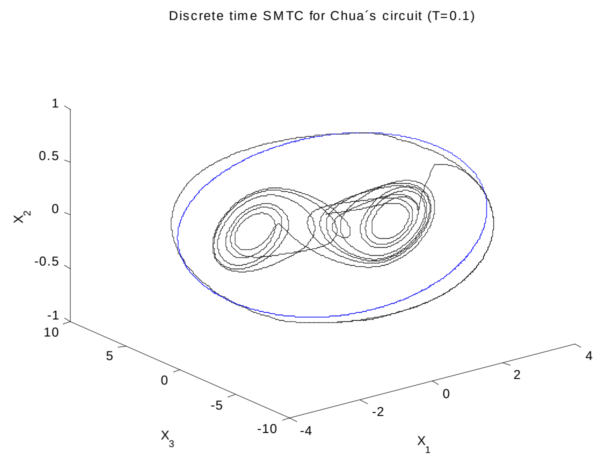

Figure 2. Performance of the digitally redesigned SMTC for Chua's circuit with $T = 0.1$

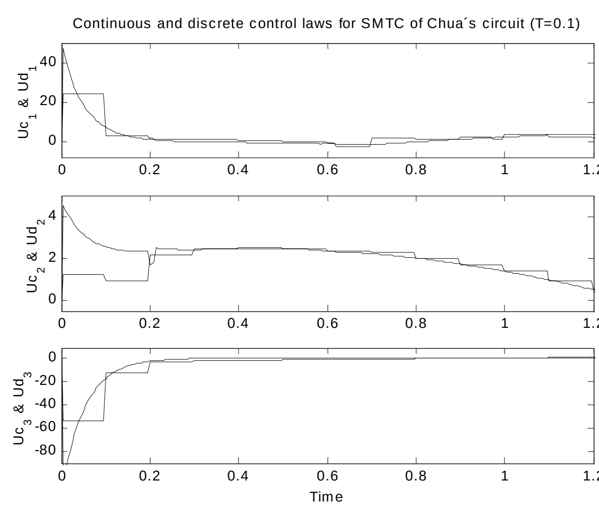

Figure 3. Comparison between the continuous-time and digitally redesigned control laws for SMTC of Chua's circuit with $T = 0.1$

(2) The SMTC control law requires the used of the derivate of the reference trajectory $\dot{x}_r(t)$, which is obtained numerically using the formula

$$\dot{x}_r(t) = \frac{x_r(k\tau) - x_r((k+1)\tau)}{\tau} \quad (34)$$

where k is the simulation index.

The performance of the digitally redesign SMTC for Chua's circuit are presented on Figures 1 and 2. As a comparison, in Figure 3 the continuous and discrete-time SMCT control laws are presented.

V. Conclusions

In this paper, a method to obtain a discrete-time version of a sliding mode tracking controller for the chaotic Chua's circuit is presented. The proposed methodology is based on an alternative representation of the system as a set of switching linear system for which a sliding mode controller is designed. Then, the general Chebyshev quadrature approximation is used to obtain an equivalent discrete-time version of the SMTC in a state matching sense.

As shown by the numerical results the used of the digital redesign technique presented here is a viable alternative for the realization of a SMC on digital devices. Further investigations on stability, invariant robustness and the extension of the proposed method for smooth systems will be reported on future communications.

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