ON THE FEEDBACK PASSIVITY PROPERTY OF NONLINEAR DISCRETE-TIME SYSTEMS ⋆

E.M. Navarro-López * **E. Fossas-Colet** **

* *Programa de Investigación en Matemáticas Aplicadas y Computación*
Instituto Mexicano del Petróleo
Eje Central Lázaro Cárdenas, 152, ed. 2, planta baja, cub.1,
A.P. 14-805, 07730 Mexico, D.F., Mexico
** *Institut d'Organització i Control*
Universitat Politècnica de Catalunya
Avda. Diagonal 647, planta 11, 08028 Barcelona, Spain

Abstract: The feedback passivity problem in nonlinear discrete-time systems is examined in this paper. The characteristics of the relative degree and zero dynamics of the non-passive system are related to its feedback passivity property. The main contribution is the study of the relative degree properties of single-input single-output (SISO) passive systems in general form, and the use of them in the proposal of sufficient conditions to render this class of systems passive by means of a static state feedback control law. Some notes, based on previous results, referring the feedback passivity problem of multiple-input multiple-output (MIMO) nonlinear systems which are affine in the input are also given.

Keywords: Discrete-time systems, Nonlinear systems, Energy control, Passive elements, Feedback stabilization.

1. INTRODUCTION. MOTIVATIONS

Dissipative and passive systems present highly desirable properties which may simplify the system analysis and control design.

Dissipativity and passivity implications in the continuous-time setting have been broadly studied. Nevertheless, a great variety of problems concerning discrete-time dissipative and passive systems remain unsolved. This is the case of the problem of rendering a nonlinear discrete-time systems dissipative (passive) by means of a static state feedback control law or the study of the relative degree properties of nonlinear discrete-time dissipative (passive) systems.

The action of rendering a system dissipative (passive) by means of a static state feedback is known as feedback dissipativity (feedback passivity or passification). Systems which can be rendered dissipative (passive) are regarded as feedback dissipative (feedback passive) systems. In this paper, the feedback passivity problem is considered for SISO nonlinear discrete-time systems in general form, and for MIMO nonlinear discrete-time systems affine in the control input. Conditions for this class of systems to be feedback passive are proposed by means of the relative degree and the zero dynamics of the original system.

⋆ Corresponding author: Eva María Navarro-López. Tel. +52 55 3003 7235. Fax +52 55 3003 6277. E-mail addresses: enavarro@imp.mx (E.M. Navarro-López), enric.fossas@upc.es (E. Fossas-Colet). This paper has been partially supported by "Proyecto Interno de Incubación de Investigación y Desarrollo Tecnológico", project number IMP-D.21516.02.002, and partially done in the context of the project GeoPlex with reference code IST-2001-34166 (http://www.geoplex.cc).

This paper follows the same approach given in (Byrnes and Lin, 1994) in the sense that the feedback passivity problem is based on the properties of the relative degree and zero dynamics of the non-passive system, with the difference that in (Byrnes and Lin, 1994), the problem of feedback losslessness for affine-in-input systems is treated. Therefore, the results here presented can be considered as an extension to the passivity general case of the ones given in (Byrnes and Lin, 1994). The feedback passivity methodology presented in this paper is an alternative one to the ones proposed in (Navarro-López, 2002a; Navarro-López *et al.*, 2002b). In these works, the approaches followed are different in essence to the one given here; the feedback dissipativity methodologies are based on the basic dissipativity inequality. In (Navarro-López, 2002a), the discrete-time version of the speed-gradient algorithm is also used.

Conditions for a system to be feedback passive can be obtained by means of the properties of the relative degree and the zero dynamics of the system. This idea is inherited from the continuous-time setting where the study of the properties of the relative degree and the zero dynamics of a passive system has played an important role in understanding problems such as feedback passivity or the stabilization of passive systems, see (Byrnes *et al.*, 1991). These properties give valuable information concerning the relation between the input and the output of the system. As the passivity property is an input-output property, the relative degree and zero dynamics of a passive system will present distinctive features. For general discrete-time systems, the implications of dissipativity and passivity in the relative degree and the zero dynamics have not been established yet, they have been studied for the losslessness and passivity nonlinear affine-in-input case (see (Byrnes and Lin, 1994) and (Navarro-López and Fossas-Colet, 2002c)) and for the passivity linear case (Byrnes and Lin, 1994; Monaco and Normand-Cyrot, 1999). In this paper, the characteristics of the relative degree and zero dynamics of passive nonlinear discrete-time systems in general form will be related to the feedback passivity property.

The paper is organized as follows. Section (2) revisits the basic definitions about passive systems for the discrete-time case. Section (3) is devoted to the properties of the relative degree and zero dynamics of passive SISO nonlinear discrete-time systems in general form. Section (4) deals with the feedback passivity problem through the relative degree and zero dynamics properties for the class of systems treated in Section (3). Some notes concerning the feedback passivity property and the characteristics of the relative degree of MIMO nonlinear *Quadratic Storage*-passive systems which are affine in the input are given in Section (5). The conclusions are given in the last section.

2. BASIC DEFINITIONS

This section introduces some basic definitions concerning the notions of passivity in the discrete-time setting. These concepts are an adaptation of those given in the continuous-time case (Willems, 1972).

Let a system of the form,

$$x(k+1) = f(x(k), u(k)) \quad (1a)$$
$$y(k) = h(x(k), u(k)) \quad (1b)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are smooth maps, and $k \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Let (x^*, u^*) be an isolated fixed point of the system. There is no loss of generality in considering $(x^*, u^*) = (0, 0)$, $f(0, 0) = 0$ and $h(0, 0) = 0$. In addition, it is assumed that $h(x, u) = 0 \Longleftrightarrow x = 0$, $u = 0$ in a neighbourhood $\mathscr{X} \times \mathscr{U}$ of $x = 0$, $u = 0$ with $\mathscr{X} \subset \mathbb{R}^n$, $\mathscr{U} \subset \mathbb{R}^m$.

Definition 1. A positive definite $\mathscr{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0 \Longleftrightarrow x = 0$ is addressed as storage function.

Definition 2. (Byrnes and Lin, 1994) System (1) is said to be pasive if there exists a storage function V such that

$$V(f(x, u)) - V(x) \leq h^T(x, u)u, \;\; \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad (2)$$

If inequality (2) becomes a strict inequality, the system is said to be strictly passive.

Definition 3. System (1) is said to be locally passive if there exists a storage function V such that inequality (2) is satisfied $\forall (x, u) \in \mathscr{X} \times \mathscr{U}$, with $\mathscr{X} \times \mathscr{U}$ a neighbourhood of $x = 0$, $u = 0$.

Lemma 4. Let a system of the form (1) be locally passive in a neighbourhood $\mathscr{X} \times \mathscr{U}$ of $x = 0, u = 0$. Then, there exists a storage function $V : \mathscr{X} \to \mathbb{R}$ such that the functions $\phi_1 : \mathscr{X} \times \mathscr{U} \to \mathbb{R}$, $\phi_2 : \mathscr{X} \times \mathscr{U} \to \mathbb{R}$:

$$\begin{aligned} \phi_1(x, u) &= V(x) + h^T(x, u)u - V(f(x, u)) \\ \phi_2(x, u) &= V(x) + h^T(x, u)u \end{aligned} \quad (3)$$

have an isolated minimum at $x = 0, u = 0$.

Let $\alpha : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a smooth function. A nonlinear static state feedback control law is denoted by the expression $u = \alpha(x, v)$. The system $x(k+1) = f(x(k), \alpha(x(k), v(k)))$ is referred to as the *feedback transformed system.*

Definition 5. A feedback control law $u = \alpha(x, v)$ is regular if for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$ it follows that $\partial\alpha/\partial v$ is invertible.

Definition 6. Consider system (1) and assume that there exists a storage function $V(x)$. The system is said to be feedback passive (feedback strictly passive) if there exists a regular static state feedback control law of the form $u = \alpha(x, v)$, with v as the new input, such that the feedback transformed system is passive (strictly passive).

Definition 7. A system of the form (1) is said to be locally feedback passive (locally feedback strictly passive) if it is feedback passive (feedback strictly passive) in a neighbourhood $\mathscr{X}\times\mathscr{U}$ of $x=0, v=0$, with v as the new input to the system.

Now, a class of nonlinear discrete-time passive systems which are affine in the control input is introduced. Let a system of the form,

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{4a}$$
$$y(k) = h(x(k)) + J(x(k))u(k) \tag{4b}$$

where $x\in\mathbb{R}^n$, $u, y\in\mathbb{R}^m$, f, g, h, J are smooth maps and $f(x)\in\mathbb{R}^n$, $g(x)\in\mathbb{R}^{n\times m}$, $h(x)\in\mathbb{R}^m$, $J(x)\in\mathbb{R}^{m\times m}$. Consider $f(0)=0$, $h(0)=0$.

Definition 8. A system of the form (4) is said to be (strictly) *QS* (Quadratic Storage)-passive if it is (strictly) passive with a storage function V such that $V(f(x)+g(x)u)$ is quadratic in u $\forall f$, $\forall g$.

Remark 9. Storage functions V such that $V(f(x)+g(x)u)$ is quadratic in u $\forall f$, $\forall g$ can be proposed as a quadratic form such that $V = x^T Px$, with P a constant positive definite matrix.

3. RELATIVE DEGREE AND ZERO DYNAMICS OF PASSIVE GENERAL NONLINEAR DISCRETE-TIME SYSTEMS

In this section, the relative degree and zero dynamics of nonlinear passive discrete-time systems are analyzed. Systems of the form (1) are examined and the SISO case is treated. These properties can be considered as an extension of those given in (Navarro-López and Fossas-Colet, 2002c) where the passivity case is treated for affine-in-input nonlinear discrete-time systems.

The relative degree and zero dynamics for nonlinear discrete-time systems have been studied for systems with outputs independent of the inputs, see (Monaco and Normand-Cyrot, 1987; Monaco and Normand-Cyrot, 1988). This paper focuses on systems with outputs dependent of the inputs. The definition for relative degree is considered as that given in (Byrnes and Lin, 1994).

Definition 10. A system of the form (1) is said to have local relative degree zero for all the outputs at $x=0, u=0$ if $\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}}$ is nonsingular. The system has uniform relative degree zero for all the outputs if $\frac{\partial h(x,u)}{\partial u}$ is nonsingular $\forall x\in\mathbb{R}^n$, $\forall u\in\mathbb{R}^m$.

Remark 11. For system (4), the local relative degree zero and the uniform relative degree zero is defined in terms of the invertibility of $J(0)$ and $J(x)$, $\forall x\in\mathbb{R}^n$, respectively.

Definition 12. Consider system (1). Let $\mathscr{X}_1$ and $\mathscr{U}_1$ be neighbourhoods of $x=0$ and $u=0$, respectively. Consider $u^*: \mathscr{X}_1\to\mathscr{U}_1$ the control which makes $h(x,u^*)=0$, $\forall x\in\mathscr{X}_1$. The zero dynamics of the system is defined by $f^*(x) = f(x,u^*)\in\mathscr{X}_1$ where $(x,u^*)\in\mathscr{Z}^* = \{(x,u):\ x\in\mathscr{X}_1, h(x,u)=0\}$.

Definition 13. A system of the form (1) has locally passive zero dynamics if there exists a storage function V locally defined on a neighbourhood $\mathscr{X}_1$ of $x=0$ in $\mathbb{R}^n$, such that,

$$V(f(x,u^*)) \le V(x),\ \ \forall x\in\mathscr{X}_1 \tag{5}$$

with u^* as given in Definition (12).

The relative degree and zero dynamics of locally passive systems of the form (1) with $m=1$ are established as follows:

Proposition 14. Let system (1) with $m=1$ be locally passive in a neighbourhood $\mathscr{X}\times\mathscr{U}$ of $x=0$ and $u=0$. Then,

i) if for some i, $\left.\frac{\partial h(x,u)}{\partial x_i}\right|_{\substack{x=0\\u=0}} \neq 0$, then the system has local relative degree zero at $x=0$, $u=0$.

ii) If the system has local relative degree zero at $x=0$, $u=0$ then the zero dynamics of system (1) locally exists at $x=0$ and is locally passive.

Proof

i) Since the system is locally passive, there exists a storage function $V(x)>0$ such that $V(x)=0\Leftrightarrow x=0$ and

$$V(f(x,u)) - V(x) - h(x,u)u \le 0,\ \ \forall (x,u)\in\mathscr{X}\times\mathscr{U} \tag{6}$$

From Lemma (4), function ϕ_2 has an isolated local minimum at $x=0$, $u=0$. Consequently, the Hessian matrix of ϕ_2 at $x=0$, $u=0$ must be positive definite. This condition will now be checked. The Hessian matrix of ϕ_2 at $x=0$, $u=0$ takes the following form:

$$Hess(\phi_2)|_{\substack{x=0\\u=0}} = \begin{pmatrix} \left.\dfrac{\partial^2 V}{\partial x^2}\right|_{\substack{x=0\\u=0}} & \left.\left(\dfrac{\partial h(x,u)}{\partial x}\right)^T\right|_{\substack{x=0\\u=0}} \\ \left.\dfrac{\partial h(x,u)}{\partial x}\right|_{\substack{x=0\\u=0}} & 2\left.\dfrac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}} \end{pmatrix} \tag{7}$$

By a linear change of coordinates, the Hessian matrix of V evaluated at $x=0$ can be transformed into a diagonal matrix, and then (7) is rewritten as,

$$Hess(\phi_2(\tilde{x},u))|_{\substack{\tilde{x}=0\\u=0}} = \begin{pmatrix} \lambda_1 & \dots & \dots & h_1^T \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \dots & \lambda_n & h_n^T \\ h_1 & \dots & h_n & h_u \end{pmatrix} \tag{8}$$

with $(\tilde{x},u)=(\tilde{x}_1,\cdots,\tilde{x}_n,u)$ the coordinate system diagonalizing $\left.\frac{\partial^2 V}{\partial x^2}\right|_{\substack{x=0\\u=0}}$, λ_i, $i=1,\ldots,n$ the eigenvalues of $\left.\frac{\partial^2 V}{\partial x^2}\right|_{\substack{x=0\\u=0}}$ and $h_u=2\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}}$. Note that the terms h_i appearing in matrix (8) do not necessarily satisfy $h_i(\tilde{x},u)=\left.\frac{\partial h(x,u)}{\partial x_i}\right|_{\substack{x=0\\u=0}}$. Nevertheless, it can be shown that

$$h_i(\tilde{x},u)\neq 0 \Longleftrightarrow \left.\frac{\partial h(x,u)}{\partial x_i}\right|_{\substack{x=0\\u=0}}\neq 0,\ \forall i=1,\ldots,n$$

The condition for the local minimum at $x=0$, $u=0$ yields to the following relations:

(1) $\lambda_i>0, \forall i=1,\ldots,n$, due to the fact that $V(x)$ has a strict local minimum at $x=0$;

(2) $\det(Hess(\phi_2(\tilde{x},u)))|_{\substack{\tilde{x}=0\\u=0}}>0$, that is,

$$\lambda_1\cdots\lambda_n h_u-\sum_{i=1}^{n}h_i^2\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_n>0 \qquad (9)$$

where the hat symbol denotes that the term is not present. Considering that for some i, h_i is not zero, the function h_u must be different from zero; otherwise, (9) would be negative which would be a contradiction with the hypothesis of $x=0$, $u=0$ being an isolated local minimum of $\phi_2(x,u)$.

ii) If the system has local relative degree zero at $x=0$, $u=0$, it follows that,

$$\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}}\neq 0 \qquad (10)$$

and $h(0,0)=0$, then by the implicit function theorem, there exists $u^*:\mathscr{X}\to\mathscr{U}$ with $\mathscr{X}$, $\mathscr{U}$ neighbourhoods of $x=0$ and $u=0$, respectively, such that $h(x,u^*(x))=0$, $\forall x\in\mathscr{X}$ and the set $\mathscr{Z}^*$ is not empty. Consequently, the zero dynamics of system (1) locally exists in a neighbourhood of $x=0$ in $\mathbb{R}^n$.

As system (1) is locally passive, relation (2) is met $\forall(x,u)\in\mathscr{X}\times\mathscr{U}$. Setting $u=u^*$ such that $y=h(x,u^*)=0$, one yields to $f^*(x)$. Since the zero dynamics is restricted to $\mathscr{Z}^*$, relation (2) is converted into (5).

Remark 15. In Proposition (14), it is necessary to consider that for some i, $\left.\frac{\partial h(x,u)}{\partial x_i}\right|_{\substack{x=0\\u=0}}\neq 0$. There are systems of the form (1) with $m=1$ which are locally passive in a neighbourhood of $x=0$, $u=0$ and $\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}}=0$. For instance,

$$\begin{aligned} x(k+1)&=ax(k)+bu^2(k),\\ y(k)&=h(x(k),u(k))=u^3(k)\end{aligned}$$

with $a,b\in\mathbb{R}$ and $a^2+b^2<1$. Consider $V(x)=x^2$, then, the system is locally passive at $x=0$, $u=0$ due to the fact that the function,

$$\begin{aligned}\phi(x,u)&=V(f(x,u))-V(x)-h(x,u)u\\ &=(a^2-1)x^2+2abxu^2+(b^2-1)u^4\\ &=-\left(\sqrt{1-a^2}x-\frac{ab}{\sqrt{1-a^2}}u^2\right)^2+\\ &\quad+\left(\frac{a^2b^2}{1-a^2}-(1-b^2)\right)u^4\\ &=-\left(\sqrt{1-a^2}x-\frac{ab}{\sqrt{1-a^2}}u^2\right)^2-\frac{1-a^2-b^2}{1-a^2}u^4\end{aligned}$$

has a strict local maximum at $x=0$, $u=0$ for some neighbourhood of $x=0$, $u=0$. Nevertheless, $\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}}=0$.

4. A SOLUTION TO THE FEEDBACK PASSIVITY PROBLEM

This section is devoted to rendering a system of the form (1), with $m=1$, passive by means of a static state feedback control law. The results given in the previous section concerning the properties of the relative degree and the zero dynamics of passive systems are used.

Proposition 16. Let a system of the form (1) with $m=1$. If the system has locally passive zero dynamics in a neighbourhood of $x=0$ and local relative degree zero at $x=0, u=0$ then the system is locally feedback passive.

Proof If the system has local relative degree zero, by the implicit function theorem, there exists $u=\alpha(x,v)$ defined in a neighbourhood of $x=0$, $v=0$ such that $h(x,\alpha(x,v))=v$. Thus, system (1) can be rewritten as,

$$\begin{aligned}x(k+1)&=\tilde{f}(x(k),v(k))=f(x(k),\alpha(x(k),v(k)))\\ y(k)&=v(k)\end{aligned} \qquad (11)$$

The goal is to prove that system (11) is locally passive at $x=0$, $v=0$ with respect to the input v. By hypothesis, there exist some storage function $V(x)$ in neighbourhood of $x=0$ for which $V(\tilde{f}(x,0))-V(x)\leq 0$. Let $\tilde{V}(x)=kV(x)$, with $k>0$ a constant. It will be shown that, for an appropriate k, $\tilde{V}(\tilde{f}(x,v))-\tilde{V}(x)-v^2\leq 0$ in a neighbourhood of $x=0$, $v=0$. This is equivalent to proving that the function $\phi(x,v)=v^2+\tilde{V}(x)-\tilde{V}(\tilde{f}(x,v))$ has an isolated local minimum at $x=0$, $v=0$. Indeed, $x=0$, $v=0$ is a critical point of $\phi(x,v)$ and the derivatives of $\phi(x,v)$ with respect to x and v at $x=0$, $v=0$ are zero, that is,

$$k\left[\frac{\partial V}{\partial x_i}-\sum_{h=1}^{n}\left.\frac{\partial V}{\partial z_h}\right|_{z_h=\tilde{f}_h(x,v)}\frac{\partial\tilde{f}_h(x,v)}{\partial x_i}\right]_{\substack{x=0\\v=0}}=0$$

$$\left[2v-k\sum_{h=1}^{n}\left.\frac{\partial V}{\partial z_h}\right|_{z_h=\tilde{f}_h(x,v)}\frac{\partial\tilde{f}_h(x,v)}{\partial v}\right]_{\substack{x=0\\v=0}}=0$$

with $i = 1, \ldots, n$. In order to obtain the Hessian matrix of $\phi(x,v)$ at $x = 0$, $v = 0$, the following terms are computed:

$$\left.\frac{\partial^2\phi(x,v)}{\partial x_i \partial x_j}\right|_{\substack{x=0\\v=0}} = k\left[\frac{\partial^2 V}{\partial x_i \partial x_j} - \sum_{h,l}\frac{\partial^2 V}{\partial z_h \partial z_l}\frac{\partial \tilde{f}_h}{\partial x_j}\frac{\partial \tilde{f}_l}{\partial x_i}\right]_{\substack{x=0\\v=0}}$$
$$\left.\frac{\partial^2\phi(x,v)}{\partial x_i \partial v}\right|_{\substack{x=0\\v=0}} = -k\left[\sum_{h,l}\frac{\partial^2 V}{\partial z_h \partial z_l}\frac{\partial \tilde{f}_h}{\partial v}\frac{\partial \tilde{f}_l}{\partial x_i}\right]_{\substack{x=0\\v=0}}$$
$$\left.\frac{\partial^2\phi(x,v)}{\partial v^2}\right|_{\substack{x=0\\v=0}} = 2 - k\left[\sum_{h,l}\frac{\partial^2 V}{\partial z_h \partial z_l}\frac{\partial \tilde{f}_h}{\partial v}\frac{\partial \tilde{f}_l}{\partial v}\right]_{\substack{x=0\\v=0}} \tag{12}$$

with $z_h = \tilde{f}_h(x,v)$, $z_l = \tilde{f}_l(x,v)$, $h = 1,\ldots,n$, $l = 1,\ldots,n$, $i = 1,\ldots,n$, $j = 1,\ldots,n$. Note that,

$$\left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{x=0} - \sum_{h,l}\frac{\partial^2 V}{\partial z_h \partial z_l}\frac{\partial \tilde{f}_h}{\partial x_j}\left.\frac{\partial \tilde{f}_l}{\partial x_i}\right|_{\substack{x=0\\v=0}} =$$
$$= \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{x=0} - \sum_{h,l}\frac{\partial^2 V}{\partial z_h \partial z_l}\frac{\partial \tilde{f}_h(x,0)}{\partial x_j}\left.\frac{\partial \tilde{f}_l(x,0)}{\partial x_i}\right|_{x=0} \tag{13}$$

Taking into account that the zero dynamics of the system is locally passive then matrix (13) is positive definite. Now, it is needed to be checked that the matrix $Hess(\phi(x,v))$ evaluated at $x = 0$, $v = 0$ is positive definite by means of Sylvester's test. Since matrix (13) is symmetric, it can be diagonalized by an appropriate linear change of coordinates. In the new coordinates, the determinant of the Hessian matrix of $\phi(x,v)$ evaluated at $x = 0$, $v = 0$ is

$$\begin{vmatrix} k\lambda_1 & \cdots & \cdots & kb_1 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & k\lambda_n & kb_n \\ kb_1 & \cdots & kb_n & 2-kc \end{vmatrix} =$$
$$= k^n\left[(2-kc)\lambda_1\ldots\lambda_n - k\sum_{i=1}^{n} b_i^2\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_n\right] \tag{14}$$

with $\lambda_i > 0, \forall i = 1,\ldots,n$ the eigenvalues of matrix (13). The hat symbol denotes that the symbol is not present. Note that the relations $b_i = \frac{1}{k}\left.\frac{\partial^2\phi(x,v)}{\partial v \partial x_i}\right|_{\substack{x=0\\v=0}}$, $\forall i = 1,\ldots,n$; $2-kc = \left.\frac{\partial^2\phi(x,v)}{\partial v^2}\right|_{\substack{x=0\\v=0}}$ are not necessarily satisfied. However, the terms b_i, c are directly related to (12). Determinant (14) is strictly positive for k small enough, indeed, for

$$k < \frac{2\lambda_1\lambda_2\ldots\lambda_n}{c\lambda_1\lambda_2\ldots\lambda_n + \sum_{i=1}^{n} b_i^2\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_n}$$

To conclude with, the existence of a new storage function for the feedback transformed system to be locally passive at $x = 0, v = 0$ has been shown.

Remark 17. If there exists a storage function V such that the zero dynamics of a system is locally passive, the system does not have to be locally passive with the same storage function. On the contrary, if there exists a storage function V such that the system is locally passive then the zero dynamics of the system is locally passive for the same storage function V (see Proposition (14)).

5. NOTES ON THE AFFINE-IN-INPUT MIMO CASE

The local and uniform relative degree zero of *QS*-passive systems of the form (4) and the properties of its zero dynamics are studied in this section. These properties are used to treat the feedback passivity problem for systems of the form (4) following the approach given in (Navarro-López and Fossas-Colet, 2002c) and can be considered as an extension to the passivity case of the ones given in (Byrnes and Lin, 1994) where the feedback losslessness problem is treated. The basis of the analysis will be the passivity characterization given in (Byrnes and Lin, 1993) (Theorem (22) in the Appendix) and the characteristics of $g(x)$.

Proposition 18. Let system (4) be *QS*-passive with a storage function V. If $rank\{g(0)\} = m$ then,

i) The system has local relative degree zero at $x = 0$.
ii) If the system has local relative degree zero at $x = 0$ then the zero dynamics of the system (4) locally exists at $x = 0$ and is locally *QS*-passive with V as storage function.

Proof

i) Evaluating condition (A.1c) at $x = 0$, and considering that the Hessian matrix of V at $x = 0$ is positive definite, it is concluded that if $rank\{g(0)\} = m$ then $J^T(0) + J(0)$ must be positive definite, consequently, $J(0)$ is nonsingular and the system has local relative degree zero at $x = 0$.
ii) If system (4) has relative degree zero at $x = 0$ then there is an open neighbourhood $\mathscr{X}$ of $x = 0$ such that $J^{-1}(x)$ is well defined $\forall x \in \mathscr{X}$, therefore, the zero dynamics locally exists in $\mathscr{X}$. The zero dynamics of (4) is defined by $x(k+1) = f^*(x(k)) = f(x(k)) + g(x(k))u^*(k)$, $\forall x(k) \in \mathscr{X}$, with $u^*(k) = -J^{-1}(x(k))h(x(k))$, $\forall x(k) \in \mathscr{X}$ the control which makes the output equal to zero. The result directly follows from relation (2).

The global version of Proposition (18) is proposed in the following way.

Proposition 19. Let system (4) be *QS*-passive with a storage function V which is stricly convex. If $rank\{g(x)\} = m, \forall x \in \mathbb{R}^n$ then,

i) The system has uniform relative degree zero.

ii) If the system has uniform relative degree zero, the zero dynamics of system (4) globally exists and is *QS*-passive with V as storage function.

Proof The proof follows the same lines of the arguments in Proposition (18).

Now, the necessary and sufficient conditions for a system of the form (4) to be rendered locally *QS*-passive proposed in (Navarro-López and Fossas-Colet, 2002c) are written for the global case. The previous conclusions referring the properties of the relative degree and the zero dynamics of such class of systems are used.

Let $\alpha(x)$ and $\beta(x)$ be smooth functions, with $\alpha(0) = 0$. Consider a static state feedback control law of the form,

$$u = \alpha(x) + \beta(x)v \tag{15}$$

Definition 20. A feedback control law of the form (15) is regular if for all $x \in \mathbb{R}^n$ it follows that $\beta(x)$ is invertible.

Theorem 21. Let a system of the form (4). Suppose there exists a strictly convex storage function V such that $V(f(x)+g(x)u)$ is quadratic in u, $\forall f$, $\forall g$. Assume that $rank\{g(x)\} = m$, $\forall x \in \mathbb{R}^n$. Then, system (4) is globally feedback equivalent to a *QS*-passive system with V as storage function by means of a regular feedback control law of the form (15) if and only if the system has uniform relative degree zero and its zero dynamics is globally *QS*-passive.

Proof The proof follows the same lines of the arguments in Theorem (11) given in (Navarro-López and Fossas-Colet, 2002c).

6. CONCLUSIONS

The main contribution of this work is the study of the properties of the relative degree and zero dynamics of passive SISO discrete-time systems in general form. These characteristics have been used to give sufficient conditions to render systems of this class passive by means of a static state feedback control law.

Some comments referring the relative degree, zero dynamics and the feedback passivity properties of MIMO nonlinear systems which are affine in the control input have been also given.

REFERENCES

Byrnes, C.I., A. Isidori and J.C. Willems (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control* **36**, 1228–1240.

Byrnes, C.I. and W. Lin (1993). Discrete-time lossless systems, feedback equivalence, and passivity. *Proceedings of the IEEE 32nd Conference on Decision and Control* pp. 1775–1781.

Byrnes, C.I. and W. Lin (1994). Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems. *IEEE Transactions on Automatic and Control* **39**(1), 83–98.

Monaco, S. and D. Normand-Cyrot (1987). Minimum-phase nonlinear discrete-time systems and feedback stabilization. *Proceedings of the IEEE 26th Conference on Decision and Control* pp. 979–986.

Monaco, S. and D. Normand-Cyrot (1988). Zero dynamics of sampled nonlinear systems. *Systems and Control Letters* **11**, 229–234.

Monaco, S. and D. Normand-Cyrot (1999). Nonlinear representations and passivity conditions in discrete time. *In Robustness in Identification and Control. Lecture Notes in Control and Information Sciences. Springer-Verlag* pp. 422–433.

Navarro-López, E.M. (2002a). *Dissipativity and passivity-related properties in nonlinear discrete-time systems*. Ph.D. thesis. Universitat Politècnica de Catalunya, Barcelona, Spain. ISBN: 84-688-1941-7.

Navarro-López, E.M. and E. Fossas-Colet (2002c). Dissipativity, passivity and feedback passivity in the nonlinear discrete-time setting. *Proceedings of the 15th IFAC World Congress*.

Navarro-López, E.M., H. Sira-Ramírez and E. Fossas-Colet (2002b). Dissipativity and feedback dissipativity properties of general nonlinear discrete-time systems. *European Journal of Control. Special Issue: Dissipativity of Dynamical Systems. Application in Control* **8**(3), 265–274.

Willems, J.C. (1972). Dissipative dynamical systems, part i: general theory; part ii: linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis* **45**(5), 321–351, 352–393.

Appendix A. PASSIVITY CHARACTERIZATION

Theorem 22. (Byrnes and Lin, 1993) Suppose there exists a storage function V such that $V(f(x)+g(x)u)$ is quadratic in u $\forall f$, $\forall g$. Then, a system of the form (4) is *QS*-passive with V if and only if there exist real functions $l(x)$, $k(x)$, $m(x)$ all of appropriate dimensions such that

$$V(f(x)) - V(x) = -l^T(x)l(x) - m^T(x)m(x) \tag{A.1a}$$

$$\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f(x)} g(x) + 2l^T(x)k(x) = h^T(x) \tag{A.1b}$$

$$g^T(x)\left.\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\right|_{\alpha=f(x)} g(x) + 2k^T(x)k(x) = = J^T(x) + J(x) \tag{A.1c}$$