SYNTHESIS OF POSITIVE CONTROLS FOR THE GLOBAL CLF STABILIZATION OF SYSTEMS*

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Abstract. In this paper, we propose an explicit formula for bounded continuous feedback laws taking values in the hyperbox $U := \left[-r_1^-, r_1^+\right] \times \cdots \times [-r_m^-, r_m^+]$, that renders an affine system globally asymptotically stable. The case of bounded *positive* feedback controls ($r_i^- = 0$, for $i = 1, \ldots, m$) is also included.

Keywords: Global asymptotic stabilization, bounded feedback control, positive control, control Lyapunov function

I. Introduction

Consider the multiple-input affine system

$$\dot{x} = f(x) + G(x)\, u, \tag{1}$$

where the vector function f and the $n \times m$ matrix function G are smooth on $\mathbb{R}^n$, and the control function u is restricted to take values in the hyperbox $U := \left[-r_1^-, r_1^+\right] \times \ldots \times [-r_m^-, r_m^+] \subset \mathbb{R}^m$, with $r_i^- \geq 0$ and $r_i^+ > 0$, for all $1 \leq i \leq m$. If $r_i^- = 0$ for $i = 1, \ldots, m$, the restricted feedback control u is a *bounded positive function*. Without loss of generality, we shall assume that $f(0) = 0$.

In recent years, there has been an increasing interest in the design of feedback controllers by means of *Lyapunov functions* due to the results obtained in [1, 12]. Artstein [1] proved the existence of a continuous stabilizing feedback function defined on any convex control value set, provided a *control Lyapunov function* (**clf**) exists (see definition of a **clf** below). Although the proof in [1] is nonconstructive, the interest on the design of feedback controllers by means of **clf**'s have been increased due to the explicit formula for the feedback controller ("*universal formula*") obtained by Sontag in [12] (defined for the control value set $U = \mathbb{R}^m$). One of the main difficulties in the design of control functions by means of **clf**'s correspond to fact that the proposed "universal formula" depends on the particular control value set U. In [6], Lin and Sontag obtained a "universal formula" for the case when the control value set is the unit ball in the standard Euclidean space. Recently, in [8], Malisoff and Sontag extended that result to the case when control inputs are restricted to "Minkowski open unit balls", $\operatorname{int} \mathcal{B}_1^m(p) := \{u \in \mathbb{R}^m : \|u\|_p < 1\}$, where $\|u\|_p = \sqrt[p]{|u_1|^p + \ldots + |u_m|^p}$ and $1 < p \leq 2$. The feedback functions obtained in [6, 8] are smooth on $\mathbb{R}^n \backslash \{0\}$ and everywhere continuous (*almost smooth* functions). Using a similar approach, in [14], Suárez, Solís and Álvarez proposed a continuous feedback control function with values in $\mathcal{B}_r^m(p)$, for the general case $1 < p < \infty$, and then defined a controller for the case when the control value set is the m-dimensional r-hyperbox, $\mathcal{B}_r^m(\infty) := \left[-r_1^-, r_1^+\right] \times \ldots \times [-r_m^-, r_m^+] \subset \mathbb{R}^m$, with $r_i^\pm > 0$, by increasing p when x approaches the boundary of $\mathcal{B}_r^m(\infty)$. Recently, for a general compact convex control value set, a continuous feedback function was finally obtained in [13].

In all the aforementioned papers, the case when the null control is contained in the boundary of the control value set, *i.e.* $0 \in \partial U$, and in particular the positive con-

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trol case, was not addressed. In fact, the set of systems globally asymptotically stabilizable by continuous control functions taking values in the control value set $U = [0, r]$ is a small part of the set of systems *globally asymptotically stabilizable* (**GAS**) by controllers restricted to the set $U = [-r, r]$. For instance, it is well known that the *two-integrator* is **GAS** by a continuous controller with values in $U = [-r, r]$, but it is not stabilizable by a positive control function (see [3] where Brammer characterized linear systems locally controllable by positive controllers (the case $0 \in \partial U$)). For system (1) there are few results in this direction: The local stabilizability problem has been addressed by Smirnov in [10], characterizing the nonlinear systems that are locally stabilizable by positive controllers. The methodology consists on determining the controllability of the linearized system in order to infer the local stabilizability of the nonlinear system. In [9], Saperstone and Yorke outlined the following mechanical problem: Can the pendulum be taken to the stable equilibrium point by means of the application of a finite continuous force in a single direction? They show that this system is *null-controllable.* In [11], Smirnov also presented a non-Lipschitzian stabilizer for that mechanical problem, but to achieve it requires *full-controllability* and knowing a Lyapunov function. In [5], Korobov establishes necessary and sufficient conditions to determine the local controllability of the family of systems $dx/dt = Ax + g(u)$, including the case $0 \in \partial U$. The global stabilization problems has been considered in the following particular cases: In [2], Bastin and Praly, presented a design of a positive control law for the feedback stabilization of a class of positive *mass-balance* system, where the state-solution of feedback system converges to the *iso-mass* (which is a portion of a hyperplane in the space-state). Finally, Lin and Sontag, [7], have considered the single input control design problem when the control value set is given by the open sets $U = (0, 1)$ or $U = (0, \infty)$. The obtained control functions are not necessarily continuous at $x = 0$.

In this paper, we propose an explicit formula for a one-parameterized family of continuous controllers $u = u_s(x)$, with parameter $s \in (0, \infty)$ taking values in the hyperbox $U := \left[-r_1^-, r_1^+\right] \times \cdots \times [-r_m^-, r_m^+]$, that renders system (1) **GAS**. The case of bounded positive feedback controls ($r_i^- = 0$, for $i = 1, \ldots, m$) is also included. Additionally, the proposed bounded feedback laws approximate (for large values of the parameter s) the controller which optimizes the robust stability margin.

II. The Feedback Control Function

We will say that the $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a *control Lyapunov function* (**clf**) with respect to (1) if $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, $\lim_{\|x\| \to \infty} V(x) = \infty$ and the following inequality is satisfied

$$\inf_{u \in U} \{\nabla V(x) \cdot (f(x) + G(x)u)\} < 0, \quad \forall x \neq 0. \tag{2}$$

In order to obtain control functions continuous at the origin the next definition was introduced in [1].

Definition 1 *The control Lyapunov function $V(x)$ has the small control property* ***(scp)*** *respect to (1), if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \|x\| < \delta$, then a control $u \in U$ exists with $|u| < \varepsilon$ so that*

$$\nabla V(x) \cdot (f(x) + G(x)u) < 0, \quad \forall x \neq 0.$$

Denote by

$$a(x) := L_f V = \nabla V(x) \cdot f(x) \quad \text{and} \quad B^T(x) := (b_1, ..., b_m),$$

where $b_i := L_{g_i} V = \nabla V(x) \cdot g_i(x)$, and define the non-negative function

$$\|B(x)\|_{1,r} := \sum_{i=1}^{m} |b_i|\, r_i, \text{ where } r_i = \begin{cases} r_i^+, & \text{if } b_i \leq 0, \\ r_i^-, & \text{if } b_i > 0. \end{cases}$$

Given the system (1), suppose that a **clf** V exists satisfying the **scp.** Define the set of *stabilizing optimal control* feedbacks $u^* = (u_1^*, u_2^*, ..., u_m^*)^T \in U$ as the set of functions from $\mathbb{R}^n$ to U that satisfy:

$$\nabla V \cdot (f + Gu^*) = \inf_{u \in U} \{\nabla V(x) \cdot (f(x) + G(x)u)\}, \tag{3}$$

for all $x \neq 0$. It is not difficult to see that there is a unique stabilizing optimal control $u^*(x)$ defined by

$$u_i^*(x) = -r_i \operatorname{sign}(b_i(x)), \quad \text{for } i = 1, \ldots, m, \tag{4}$$

and all x such that $b_i(x) \neq 0$, but it is discontinuous whenever $b_i(x) = 0$.

From the definition of $\|B(x)\|_{1,r}$, we get

$$\min_{u \in U} \left\{a + B^T u\right\} = a + B^T u^* = a - \|B\|_{1,r}. \tag{5}$$

From (5) we have that the **clf** property (2) is equivalent to the following inequality

$$a(x) < \|B(x)\|_{1,r}, \quad \forall x \neq 0. \tag{6}$$

Consider the one-parameter multiple-input continuous feedback $u_s(x) = (u_{1s}(x), \ldots, u_{ms}(x))^T$ given by

$$u_{is}(x) := \rho_{is}(x)\, u_i^*(x), \tag{7}$$

where $u_i^*(x)$ is given in (4),

$$\rho_{is}(x) = \begin{cases} 1 - (1 - \frac{(|a|+a)\,\sigma_i}{2\|B\|_{1,r}}) \exp(\tau_{is}\sigma_i), & \text{if } \sigma_i > 0, \\ 0, & \text{if } \sigma_i = 0, \end{cases} \tag{8}$$

where $\sigma_i(x) = |b_i| r_i / \|B\|_{1,r}$ and $\tau_{is}(x)$ is a non-positive function defined as

$$\tau_{is}(x) = \begin{cases} m\dfrac{\ln(\lambda(x))}{\lambda(x)} - s|b_i| r_i, & \text{if } \|B(x)\|_{1,r} > 0, \\ 0, & \text{if } \|B(x)\|_{1,r} = 0, \end{cases} \tag{9}$$

for $i = 1, \ldots, m$, where $\lambda(x) = 1 - \frac{1}{2}(a(x) + |a(x)|) / \|B(x)\|_{1,r}$ and $s > 0$ is a tuning parameter. It can be seen that when $s \to \infty$, the control function $u_i(x)$ converges to the stabilizing optimal control $u_i^* = -r_i \operatorname{sign}(b_i)$, and as a consequence of $0 < \exp(\tau_{is}\dfrac{|b_i| r_i}{\|B\|_{1,r}}) \leq 1$ for each $s > 0$, we get that the control constraint condition $-r_i^- \leq u_{is}(x) \leq r_i^+$ for $i = 1, \ldots, m$, is satisfied for all $x \in \mathbb{R}^n$.

III. Main Results

In this section, we will prove that the control (7) is a continuous function that globally asymptotically stabilizes the origin.

In the next remark we introduce a function, and its properties, essential for showing the stability of the control feedback function.

Remark 1 *Consider the non negative function $h_m : \mathbb{Y}_m \times (0,1) \to \mathbb{R}$ defined by*

$$h_m(\mathbb{Y}, z) = h(y_1, \ldots, y_{m-1}, z) = \sum_{i=1}^{m} \frac{y_i - z y_i^2}{1 - z + z m y_i} \tag{10}$$

with $\mathbb{Y}_m = \left\{ (y_1, \ldots, y_{m-1}) \subset [0,1]^{m-1} : \sum\limits_{i=1}^{m-1} y_i < 1 \right\}$, *where* $y_m = 1 - \sum\limits_{i=1}^{m-1} y_i$. *A short calculation yields that the function $h_m(\mathbb{Y}, z)$ satisfies the properties at the point* $Y_0 = \left(\frac{1}{m}, \ldots, \frac{1}{m} \right) \in \operatorname{int} \mathbb{Y}_m$, *for each $z \in (0,1)$.*

1. $\dfrac{\partial h(Y_0, z)}{\partial y_j} = 0$ *for $j = 1, \ldots, m-1$; with Y_0 as unique equilibrium point.*
2. $\dfrac{\partial^2 h(Y_0, z)}{\partial y_i \partial y_j} = \begin{cases} 2p(z), & \text{if } i = j \\ p(z), & \text{if } i \neq j \end{cases}$ *for $i, j = 1, \ldots, m-1$; where $p(z) = -2z(z - m - 1)(z - 1)$ it is such that $p(z) < 0$ for $z \in (0,1)$. These are the components of the Hessian matrix Hh of the function h_m evaluated at the point (Y_0, z).*
3. $h_m(y_1, \ldots, y_{m-1}, z) \leq h(Y_0, z) = 1 - \dfrac{z}{m}$ *for all $(y_1, \ldots, y_{m-1}, z) \in \mathbb{Y}_m \times (0,1)$. That is to say, Y_0 it is a global maximum for h_m.*

The last issue of the remark is a consequence of the following facts. The Hessian matrix evaluated at Y_0 is given by

$$Hh_m(Y_0, z) = p(z) M_m,$$

where M_k is a $k-1 \times k-1$ matrix, given by

$$M_k = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}$$

The $(k-1) \times (k-1)$ matrix M_k has determinant $\det(M_k) = k$. This implies that matrix M_m has all its principal minors positive, from where we get that M_m is a positive definite matrix. Thus, the Hessian matrix $Hh_m(Y_0)$ is negative definite for any $z \in (0,1)$.

One way to find the *global maximum* of h_m is to determine all of the *local maxima* and then we choose the one that gives us the biggest value for h_m, which we also compare with the values that it reaches the function in the frontier of the domain.

By means of direct calculations, it is easy to obtain the following equivalences for this function

$$\begin{aligned} &\frac{\partial h_m(y_1, \ldots, y_{m-1}, z)}{\partial y_i} = 0 \Leftrightarrow \\ &1 - \sum_{j=1}^{m-1} y_j = y_i \quad \text{for } i = 1, \ldots, m-1 \Leftrightarrow \\ &y_i = \tfrac{1}{m} \quad \text{for each } i = 1, 2, \ldots, m-1. \end{aligned}$$

Therefore, the point $Y_0 = \left(\frac{1}{m}, \ldots, \frac{1}{m} \right)$ is the unique equilibrium point of the function $h_m(y_1, \ldots, y_{m-1}, z)$. Also, it is completed that

$$\max_{\partial \mathbb{Y}_m} h_m \leq \max_{\mathbb{Y}_m} h_m = 1 - \frac{z}{m},$$

for all $z \in (0,1)$ and $m = 2, 3, \ldots$

Then, we conclude that

$$h_m(y_1, \ldots, y_{m-1}, z) \leq h(Y_0, z) = 1 - \frac{z}{m},$$

for all $(y_1, \ldots, y_m) \in [0,1]^m$.

Proposition 2 *If $a(x) < \|B(x)\|_{1,r}$ for all $x \neq 0$, then the closed-loop system (1)-(7) is **GAS**.*

Proof. We prove first that the feedback system (1)-(7) satisfies $\dot{V} < 0$ for $x \neq 0$. For x ($x \neq 0$) such that $\|B(x)\|_{1,r} = 0$, we have that $a(x) < 0$ and $u(x) = 0$. This implies that

$$\dot{V} = \nabla V(x) \cdot (f(x) + g(x)u) = a + Bu = a < 0.$$

Suppose that $\|B(x)\|_{1,r} > 0$, therefore

$$\begin{aligned}\dot{V} &= a + Bu_s = a + (b_1, \ldots, b_m)\begin{pmatrix} u_{1s} \\ \vdots \\ u_{ms}\end{pmatrix} \\ &= a - \|B\|_{1,r} + \sum_{i=1}^{m} |b_i| r_i \exp\left(\tau_{is}\frac{|b_i| r_i}{\|B\|_{1,r}}\right) \\ &- \sum_{i=1}^{m}\left(\frac{|b_i| r_i}{\|B\|_{1,r}}\right)^2\left(\frac{|a|+a}{2}\right)\exp\left(\tau_{is}\frac{|b_i| r_i}{\|B\|_{1,r}}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\dot{V}}{\|B\|_{1,r}} &= \frac{a}{\|B\|_{1,r}} - 1 + \sum_{i=1}^{m}\frac{|b_i| r_i}{\|B\|_{1,r}}\exp\left(\tau_{is}\frac{|b_i| r_i}{\|B\|_{1,r}}\right) \\ &- \sum_{i=1}^{m}\left(\frac{|b_i| r_i}{\|B\|_{1,r}}\right)^2\left(\frac{|a|+a}{2\|B\|_{1,r}}\right)\exp\left(\tau_{is}\frac{|b_i| r_i}{\|B\|_{1,r}}\right).\end{aligned} \tag{11}$$

Two cases can be distinguished:

(i) If $a \leq 0$, it follows that

$$\frac{\dot{V}}{\|B\|_{1,r}} < \frac{a}{\|B\|_{1,r}} - 1 + \sum_{i=1}^{m}\frac{|b_i| r_i}{\|B\|_{1,r}}\exp\left(-\frac{(|b_i| r_i)^2}{\|B\|_{1,r}}\right) < 0$$

since, $\tau_{is}(x) = -s|b_i| r_i$, $\dfrac{a}{\|B\|_{1,r}} < 0$ and the following inequality is satisfied

$$\sum_{i=1}^{m}\frac{|b_i| r_i}{\|B\|_{1,r}}\exp\left(-\frac{(|b_i| r_i)^2}{\|B\|_{1,r}}\right) < \sum_{i=1}^{m}\frac{|b_i| r_i}{\|B\|_{1,r}} = 1.$$

(ii) On the other hand, suppose that $a > 0$. Denote by $y_i = |b_i| r_i / \|B\|_{1,r}$, for $i = 1, \ldots, m$, and $z = a/\|B\|_{1,r}$, then we get

$$\begin{aligned}\frac{\dot{V}}{\|B\|_{1,r}} &= z - 1 + \sum_{i=1}^{m}\left(y_i - zy_i^2\right)\exp\left(\tau_{is} y_i\right) \\ &= z - 1 + \sum_{i=1}^{m}\left(y_i - zy_i^2\right)(1-z)^{\left(\frac{my_i}{1-z}\right)}\exp\left(-sy_i^2\|B\|_{1,r}\right) \\ &< z - 1 + \sum_{i=1}^{m}\left(y_i - zy_i^2\right)(1-z)^{\left(\frac{my_i}{1-z}\right)}\end{aligned} \tag{12}$$

We will use the following couple of well-known inequalities (see [4]):

$$\begin{array}{lll}\ln(r) < r - 1 & (r > 0,\ r \neq 1) & (a) \\ \exp(r) < \frac{1}{1-r} & (r < 1,\ r \neq 0) & (b)\end{array}$$

From the first inequality we have that

$$\ln(1-z) < -z, \quad \forall z \in (0,1).$$

Applying these inequalities to the exponential term

$$(1-z)^{my_i(1-z)^{-1}}$$

we obtain that

$$\begin{aligned}(1-z)^{my_i(1-z)^{-1}} &= \exp(\left(\frac{my_i}{1-z}\right)\ln(1-z)) \\ &< \exp(-z\left(\frac{my_i}{1-z}\right)) \quad \text{for the inequality } (a) \\ &< \left(1 + z\left(\frac{my_i}{1-z}\right)\right)^{-1} \quad \text{for the inequality } (b) \\ &= \tfrac{1-z}{1-z+zmy_i},\end{aligned}$$

so that

$$\begin{aligned}\frac{\dot{V}}{\|B\|_{1,r}} &< z - 1 + \sum_{i=1}^{m}\left(y_i - zy_i^2\right)\left(\tfrac{1-z}{1-z+zmy_i}\right) \\ &= (1-z)\left[-1 + \sum_{i=1}^{m}\tfrac{y_i - zy_i^2}{1-z+zmy_i}\right].\end{aligned}$$

From the aforementioned remark, we have that

$$\sum_{i=1}^{m}\frac{y_i - zy_i^2}{1-z+zmy_i} \leq 1 - \frac{z}{m},$$

for all $(y_1, \ldots, y_m, z) \in [0,1]^m \times (0,1)$, with $m \in \mathbb{N}$.

Therefore, since the following inequalities are true

$$\frac{\dot{V}}{\|B\|_{1,r}} < (1-z)\left(-\frac{z}{m}\right) < 0, \quad \forall z \in (0,1).$$

We have proven that $\dot{V} < 0$, for $x \neq 0$ and all $s \geq 0$. ■

Lemma 3 *If the* ***clf*** V *satisfies the* ***scp****, then the following limits hold:*

(a) $\displaystyle\lim_{x\to 0}\frac{a(x)+|a(x)|}{2\|B(x)\|_{1,r}} = 0$;

(b) $\displaystyle\lim_{\|B(x)\|\to 0}\frac{a(x)+|a(x)|}{2\|B(x)\|_{1,r}} = 0.$

Proof. It is sufficient to prove that $\lim_{x\to 0} a(x)/\|B(x)\|_{1,r} = 0$ for $a(x) > 0$. Suppose first that $u \equiv 0$ is an interior point of U, *i.e.* $r_i^- > 0$ and $r_i^+ > 0$ for all $i = 1, \ldots, m$. Denote by $\overline{r}_i := \min\left\{r_i^-, r_i^+\right\}$ and define

$$\|B\|_{1,\overline{r}} := |b_1|\,\overline{r}_1 + \ldots + |b_m|\,\overline{r}_m$$

By the **scp** and Hölder's inequality we have

$$\begin{aligned}0 < a(x) &< -B^T(x)u \leq |B(x)\cdot u| \\ &\leq \|B(x)\|_{1,\overline{r}}\|u\|_\infty < \|B(x)\|_{1,\overline{r}}\,\varepsilon,\end{aligned}$$

for $0 < \|x\|_2 < \delta$.

The previous inequality is valid for arbitrarily small $r_i^- > 0$, using the continuity of the function $\|B\|_{1,\overline{r}}$ and regarding r_i^- we obtain

$$a(x) \leq \overline{\|B(x)\|}_{1,\overline{r}}\, \varepsilon,$$

where $\overline{\|B(x)\|}_{1,\overline{r}} = \lim\limits_{r_i^- \to 0} \|B(x)\|_{1,\overline{r}}$ for some i. This means that

$$\frac{a}{\overline{\|B\|}_{1,\overline{r}}} \leq \varepsilon, \quad \text{for } 0 < \|x\|_2 < \delta$$

or equivalently

$$\lim_{x \to 0} \frac{a}{\overline{\|B(x)\|}_{1,\overline{r}}} = 0.$$

The proof follows from an induction argument in the number of r_i^-'s equal to zero.

Proof of (b). Let $x_0 \in \left\{x : \|B(x)\|_{1,r} = 0\right\}$ be an arbitrary point. If $x_0 = 0$, then the limit **(b)** coincides with the limit **(a).** Next, we will suppose that $x_0 \neq 0$.

We will have (**b**) if we prove that

$$\lim_{x \to x_0} \frac{a(x) + |a(x)|}{2\,\|B(x)\|_{1,r}} = 0.$$

If $x_0 \in \{x : \|B(x)\|_{1,r} = 0\}$, due to the inequality $a(x_0) < \|B(x_0)\|_{1,r}$, we have $a(x_0) < 0$. Consequently, let $\Omega_\varepsilon(x_0)$ be a neighborhood of x_0 such that

$$a(x) < 0,\ \forall x \in \Omega_\varepsilon(x_0).$$

Let $\{x_n\}$ be sequence such that $\{x_n\} \to x_0$ and $\|B(x_n)\|_{1,r} > 0$ for each n. Then, there exists a positive integer N such that $\{x_n\} \in \Omega_\varepsilon(x_0)$ for every $n > N$, so that $a(x_n) < 0$ if $n > N$. Then

$$\frac{a(x_n) + |a(x_n)|}{2\,\|B(x_n)\|_{1,r}} = 0, \text{ if } n > N.$$

Therefore

$$\lim_{x_n \to x_0} \frac{a(x_n) + |a(x_n)|}{2\,\|B(x_n)\|_{1,r}} = 0.$$

This completes the proof. ∎

Proposition 4 *Consider the affine system (1) such that $V(x)$ is a* ***clf*** *satisfying the condition* ***scp****. Then the feedback (7) is a continuous function.*

Proof. From the limits (**a**) and (**b**) of the above lemma, the continuity of the term $|b_i|\, r_i$, and because the composition of continuous functions is continuous, we obtain the following limits for each function $u_{is}(x)$:

$$(i) \quad \lim_{x \to 0} u_{is}(x) = 0,$$

$$(ii) \quad \lim_{|b_i| r_i \to 0} u_{is}(x) = 0.$$

Since if $\lim\limits_{|b_i| r_i \to 0} \|B(x)\|_{1,r} > 0$, then (ii) is immediate. We conclude that $u_{is}(x)$ is continuous. ∎

IV. Example

The following example illustrates the stabilizing positive controller design method proposed in this paper.

Example 5 *We consider the affine system*

$$\begin{aligned} \dot{x} &= -x + y^2 + yz^2 \\ \dot{y} &= -y + |x|\, y - 2xy\, u_1 \\ \dot{z} &= -z + |xy|\, z - 2xyz\, u_2 \end{aligned} \tag{13}$$

where $(u_1, u_2)^T \in [0,1]^2$. We propose $V(x,y,z) = \frac{1}{2}\left(x^2 + y^2 + z^2\right)$ as a Lyapunov function candidate. Consequently we denote

$$a = -x^2 + xy^2 + xyz^2 - (1 - |x|)y^2 - (1 - |xy|)z^2,$$
$$b_1 = -2xy^2, \qquad \text{and} \quad b_2 = -2xyz^2.$$

We also define the non-negative functions:

$$r_i = \begin{cases} 1, & \text{if } b_i < 0, \\ 0, & \text{if } b_i \geq 0, \end{cases} \qquad \text{for } i = 1, 2.$$

We have that the ***clf*** *property is satisfied since*

$$\min_{u \in U} \dot{V} = a - \|B\|_{1,r} = -\left(x^2 + y^2 + z^2\right) +$$
$$y^2\left[(x + |x| - 2\,|x|\, r_1) + z^2\,(xy + |xy| - 2\,|xy|\, r_2)\right] < 0$$

for all $(x,y,z) \neq (0,0,0)$. It is not difficult to show that if $\sqrt{x^2 + y^2 + z^2} < \frac{1}{4}$, then $a(x,y,z) < 0$. This implies that

$$\lim_{(x,y,z) \to (0,0,0)} \frac{a + |a|}{2\,\|B\|_{1,r}} = 0$$

meaning that the ***clf*** *V satisfies the* ***scp*** *property. The control $u = (u_1, u_2)^T$ is:*

$$u_{is} = \begin{cases} 1 - \left(1 - \dfrac{(a + |a|)\, \sigma_i}{2\,\|B\|_{1,r}}\right) \exp\left(\tau_{is} \sigma_i\right), & \text{if } b_i < 0, \\ 0, & \text{if } b_i \geq 0. \end{cases}$$

where $\sigma_i(x) = \dfrac{|b_i|\, r_i}{\|B\|_{1,r}}$, $\tau_{is}(x) = m \dfrac{\ln\left(\lambda(x)\right)}{\lambda(x)} - s\,|b_i|\, r_i$ for $i = 1, 2$ with $\lambda(x) = 1 - \frac{1}{2}(a(x) + |a(x)|) / \|B(x)\|_{1,r}$ and $s \geq 0$ is a tuning parameter. Observe also that if $s \to \infty$ then $u_i(x) \to u_i^* = -r_i \operatorname{sign}(b_i)$ -the optimal control. □

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