Vibration Control of a Flexible Structure using Feedback Linearization

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Abstract

In this paper, we describe the design of a feedback linearization controller for a single-axis, two-mode model of a flexible structure using a DC motor as its source of actuation. Our main goal is to track the slewing movement to a particular desired end-position assuming no damping and the same control input is delivered to each mode. The beam modes are characterized by a nonlinear saturation function given as atan(x).

1 System Model

A typical single axis linear model of a flexible structure consists of (a) one *rigid mode* describing the motion of the body as if it were rigid, (b) *n flexible modes* describing the vibrational motion caused by the distributed elasticity.

Using Hamilton's Principle, the linear equations of motion of a flexible structure are found to be [1]:

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} + \rho x\ddot{\theta} = 0 \tag{1}$$

$$(J_b + J_m)\ddot{\theta} + \rho\int_0^L x\frac{\partial^2 y}{\partial t^2}dx = \tau_\iota \tag{2}$$

where J_b is the link's mass moment of inertia about the hub, J_m is the mass moment of inertia of the hub, L is the length of the beam, E is Young's modulus, I is the area moment of inertia about the bending axis, and ρ is the mass per unit length of the link.

In this study we consider only small bending motions in a horizontal plane. Figure 1 illustrates the horizontal displacement present in the movement of a flexible beam.

An N-mode expansion for the displacement $y(x,t)$ [2]

$$y(x,t) = \sum_{i=1}^{N}\psi_i(x)q_i(t), \quad x \in [0,L], \quad t \geq 0 \tag{3}$$

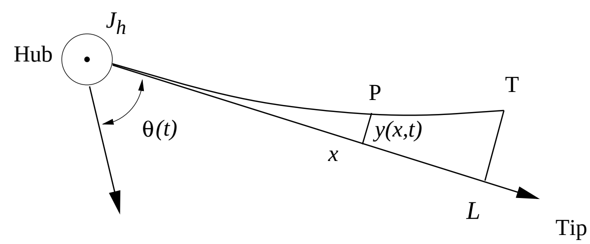

Figure 1: Deflection in a flexible beam

where the spatial-dependent functions $\psi_i(x)$ are the mode shapes, and the time-dependent functions $q_i(t)$ represent the modes, is replaced into the linear equations of motion 1-2.

This substitution leads to the equation of a finite-dimensional model of a flexible link with torque actuation at its hub, expressed as:

$$\ddot{q}_i + w_i^2 h(q_i) + c_i(\dot{q}_i) = b_i u \tag{4}$$

where w are the structural frequencies representing the rigid and flexible modes, $u = \tau$ is the torque delivered by the motor, $h(q_i)$ is the saturation term, and $c_i(\dot{q}_i)$ is the damping term.

Assuming no damping, *i.e.* $c_i(\dot{q}_i) = 0$, we obtain the simplified model from 4 to be:

$$\ddot{q}_i + w_i^2 h(q_i) = b_i u \tag{5}$$

where the nonlinear saturation term $h(q_i)$ is:

$$h(q_i) = \text{atan}(q_i) \tag{6}$$

Specifically, we consider the slewing problem for a two-mode model flexible beam that will steer 5 from any initial position to a final position given at the system's equilibrium point. Each mode equation in our model is:

$$\ddot{q}_0 + w_0^2 h(q_0) = b_0 u \tag{7}$$

$$\ddot{q}_1 + w_1^2 h(q_1) = b_1 u \tag{8}$$

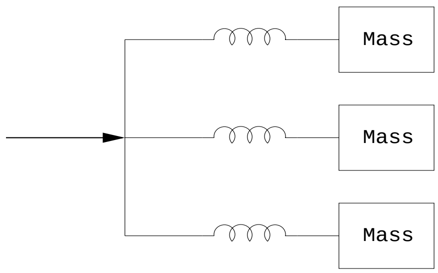

Figure 2: Equivalent representation of the flexible structure

reaching its equilibrium at

$$q_i(x,t) \to q_i^*(x,t), \quad q \in \mathbb{R}^2 \tag{9}$$

Therefore, we find the equilibrium of 5 as

$$\ddot{q}_i = -w_i^2 h(q_i^*) + b_i u^* = 0 \tag{10}$$

thus,

$$q_i^* = h^{-1}\left(\frac{b_i u^*}{w_i^2}\right), \quad h = \text{atan} \tag{11}$$

An equivalent representation of the flexible structure we consider in this paper is shown in Figure 2 as a system of nonlinear springs and masses.

Next section describes the proposed control design to stabilize the induced nonlinear vibrations on the flexible structure.

2 Control Approach

Assuming that our beam consists only of two modes, its structural frequencies are given as $w_0 = 0$ and $w_1 = 1$, and the constant parameters in each mode $b_0 = b_1 = 1$, [3] thus the system is reduced to:

$$\ddot{q}_0 = u \tag{12}$$
$$\ddot{q}_1 = -h(q_1) + u \tag{13}$$

Considering this, the state space model for 12-13 can also be expressed in the form:

$$\dot{x} = f(x) + bu \tag{14}$$

To do so, we first introduce a new state $\tilde{q}$ as shown in 15 that will allow us to cancel out the control term.

$$\tilde{q} = q_0 - q_1 \Longrightarrow \ddot{\tilde{q}} = h(q_0 - \tilde{q}) \tag{15}$$

Then, we build a new state-space model such as 14 by performing a change of variables in terms of the original states q_i and $\tilde{q}$. Thus, we assign the new states to be:

$$x_1 = \tilde{q} \tag{16}$$
$$x_2 = q_0 \tag{17}$$
$$x_3 = \dot{\tilde{q}} \tag{18}$$
$$x_4 = \dot{q}_0 \tag{19}$$

expressing 16-19 as a state equation we have that:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ h(x_2 - x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{20}$$

where $f = [x_3 \ x_4 \ h(x_2 - x_1) \ 0]^T$, and $g = [0\ 0\ 0\ 1]^T$. In this way, our system is written in *regular form*, which means the control term appears only in the lower equation.

Next, we find the elements of $[g \ ad_f\ g \ ad_f^2\ g \ ad_f^3\ g]$ from 20 by using the Lie Bracket theorem [4]:

$$ad_f\ g = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

$$ad_f^2\ g = \frac{\partial ad_f\ g}{\partial x} f - \frac{\partial f}{\partial x} ad_f\ g = \begin{bmatrix} 0 \\ 0 \\ h' \\ 0 \end{bmatrix} \tag{22}$$

$$ad_f^3\ g = \frac{\partial ad_f^2\ g}{\partial x} f - \frac{\partial f}{\partial x} ad_f^2\ g = \begin{bmatrix} -h' \\ 0 \\ -x_3 h'' + x_4 h'' \\ 0 \end{bmatrix} \tag{23}$$

We arrange them into the corresponding matrix formed by g and each vector from 21-23 to check its linear independence.

$$\begin{bmatrix} g \\ ad_f\ g \\ ad_f^2\ g \\ ad_f^3\ g \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & -h' \\ 0 & -1 & 0 & 0 \\ 0 & 0 & h' & -x_3 h'' + x_4 h'' \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

As 24 is a full rank matrix, it is linearly independent, also meaning system 20 is globally controllable.

However, to be feedback linearizable, the system must also be *involutive* [5]. We verify system 20 is involutive if the Lie Bracket between any two vector combination from $[g, \ ad_f\ g, \ ad_f^2\ g]$ is zero. From g and 21-23 we obtain:

$$\begin{aligned} [g, \ ad_f\ g] &= \frac{\partial ad_f\ g}{\partial x} g - \frac{\partial g}{\partial x} ad_f\ g \\ &= 0\ g \ - \ 0 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 0 \end{aligned} \tag{25}$$

furthermore,

$$[g, \ ad_f^2\ g] = \frac{\partial ad_f^2\ g}{\partial x} g - \frac{\partial g}{\partial x} ad_f^2\ g \tag{26}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -h'' & h'' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Therefore, as either Lie Bracket between the column vectors of 24 is zero, 20 is *involutive*.

To find the system transformation matrix to the z coordinate system, we multiply each column vector in 24 by the partial derivative vector of $z(x)$ with respect to each state x_i.

$$\left[\frac{\partial z_1}{\partial x_1}\frac{\partial z_1}{\partial x_2}\frac{\partial z_1}{\partial x_3}\frac{\partial z_1}{\partial x_4}\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \frac{\partial z_1}{\partial x_4} = 0 \quad (27)$$

$$\left[\frac{\partial z_1}{\partial x_1}\frac{\partial z_1}{\partial x_2}\frac{\partial z_1}{\partial x_3}\frac{\partial z_1}{\partial x_4}\right] \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow \frac{\partial z_1}{\partial x_2} = 0 \quad (28)$$

$$\left[\frac{\partial z_1}{\partial x_1}\frac{\partial z_1}{\partial x_2}\frac{\partial z_1}{\partial x_3}\frac{\partial z_1}{\partial x_4}\right] \begin{bmatrix} 0 \\ 0 \\ h' \\ 0 \end{bmatrix} = 0 \Rightarrow \frac{\partial z_1}{\partial x_3} = 0 \quad (29)$$

$$\left[\frac{\partial z_1}{\partial x_1}\frac{\partial z_1}{\partial x_2}\frac{\partial z_1}{\partial x_3}\frac{\partial z_1}{\partial x_4}\right] \begin{bmatrix} -h' \\ 0 \\ -x_3h'' + x_4h'' \\ 0 \end{bmatrix} \neq 0 \quad (30)$$

$$\Rightarrow -\frac{\partial z_1}{\partial x_1}h' + \frac{\partial z_1}{\partial x_3}x_3h'' + \frac{\partial z_1}{\partial x_3}x_4h'' \neq 0 \quad (31)$$

This results in the conditions we must satisfy 27-31 to assign each feedback linearization state z_i. Therefore, the new state z_1 is not a function of x_4 or x_2, neither of x_3. However, the only unknown partial derivative in 31 is expressed as $\frac{\partial z_1}{\partial x_1} \neq 0$. Thus, we assume $z_1 = x_1$, that satisfies the required boundary conditions.

We obtain transformations for the other states by using the Lie Derivative theorem as follows [5]:

$$z_2 = L_f z_1 = \frac{\partial z_1}{\partial x} f \quad (32)$$

$$= [1 \ 0 \ 0 \ 0] \begin{bmatrix} x_3 \\ x_4 \\ h(x_2 - x_1) \\ 0 \end{bmatrix} = x_3 \quad (33)$$

$$z_3 = L_f^2 z_1 = \frac{\partial z_2}{\partial x} f \quad (34)$$

$$= [0 \ 0 \ 1 \ 0] \begin{bmatrix} x_3 \\ x_4 \\ h(x_2 - x_1) \\ 0 \end{bmatrix} = h(x_2 - x_1)$$

$$z_4 = L_f^3 z_1 = \frac{\partial z_3}{\partial x} f = \quad (35)$$

$$= -x_3h'(x_2 - x_1) + x_4h'(x_2 - x_1)$$

The transformation in the control is found from the following equation:

$$u = \alpha + \beta v \quad (36)$$

where,

$$\alpha = -\frac{L_f^4 z_1}{L_g L_f^3 z_1}, \quad \beta = \frac{1}{L_g L_f^3 z_1} \quad (37)$$

$$L_g L_f^3 z_1 = \frac{\partial L_f^3 z_1}{\partial x} g = L_g L_f^3 z_1 = b_1 h'(x_2 - x_1) \quad (38)$$

$$L_f^4 z_1 = \frac{\partial z_4}{\partial x} f \quad (39)$$

Simplifying, for z_4 in 35, and f in 20 we obtain:

$$\begin{aligned} L_f^4 z_1 &= x_3^2 h''(x_2 - x_1) - x_3x_4h''(x_2 - x_1) + \\ &- x_3x_4h''(x_2 - x_1) + x_4^2 h''(x_2 - x_1) + \\ &- h(x_2 - x_1)h'(x_2 - x_1) \end{aligned} \quad (40)$$

Thus,

$$\begin{aligned} L_f^4 z_1 &= x_3^2 h''(x_2 - x_1) - 2x_3x_4h''(x_2 - x_1) \quad (41) \\ &+ x_4^2 h''(x_2 - x_1) - h(x_2 - x_1)h'(x_2 - x_1) \end{aligned}$$

Finally, we obtain the state equations from the previous transformations to z_i given by the assumptions shown in 32-35 and the states in 20.

Letting $v = -k_1z_1 - k_2z_2 - k_3z_3 - k_4z_4$, we design our controller such that each k_n corresponds to the eigenvalues of the system matrix by the new system in Brunovsky form,

$$\dot{z}_1 = z_2 \quad (42)$$

$$\dot{z}_2 = z_3 \quad (43)$$

$$\dot{z}_3 = z_4 \quad (44)$$

$$\dot{z}_4 = -k_1z_1 - k_2z_2 - k_3z_3 - k_4z_4 \quad (45)$$

The system matrix for 42-45 is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 \end{bmatrix} \quad (46)$$

In this system, each k_n is selected such that the eigenvalues of matrix A lie on the left-hand side complex plane ($\lambda_{\mathbb{R}} < 0$).

3 Simulations

The closed loop control system is implemented in a SIMULINK block diagram as shown in Figure 3. Each gain k_n in 46 is found to place the eigenvalues of our new system matrix A in the negative plane.

In Figure 4 and 5, each mode q_i from the original system is shown to be stabilized in finite time. The nonlinear behavior of the beam quickly attenuates in each orginal system state q_i by using the feedback linearizing control shown in Figure 9. This controller supresses the vibration caused by the flexible movement of the beam with nonlinear characteristics $h(q_i)$.

It is clearly seen in the phase portraits of Figure 6 to 8 that the original system (q_0 vs. q_1) is asymptotically stable and converges to the origin as $t \to \infty$.

4 Conclusions

In this paper, we showed that a controller design based on feedback linearizarion provides quick stabilization in each mode of a flexible structure model with nonlinear properties. Specifically, an analysis for a two-mode flexible structure was made to demonstrate this technique quickly compensates the nonlinear motions in the original states by transforming the original system into a linear model and creating a feedback to eliminate these motions in the original system. Conditions such as using only the uncoupled flexible modes and assuming no damping were considered in order to avoid higher complexity in the solution. Simulations illustrating how this design method provides fast stabilization for the nonlinear oscillations generated in each mode summarize our results.

5 References

[1] R. Salinas Villarreal, “On a time-optimal controller for flexible structures,” Master’s thesis, Tulane University, 1997.

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[5] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive, and Robust*. Prentice-Hall, London, 1995.

[6] R. Salinas Villarreal, E. Barbieri, and S. V. Drakunov, “Time sub-optimal/sliding mode controller for a flexible structure,” in *Proceedings of the* 28^{th} *IEEE Southeastern Symposium on System Theory*, pp. 253–257, April 1996.

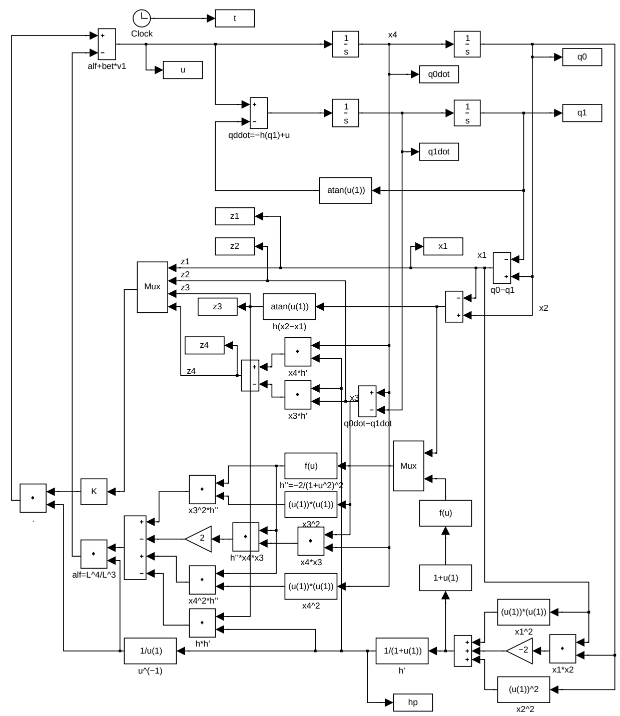

Figure 3: SIMULINK block diagram of the feedback linearization controller.

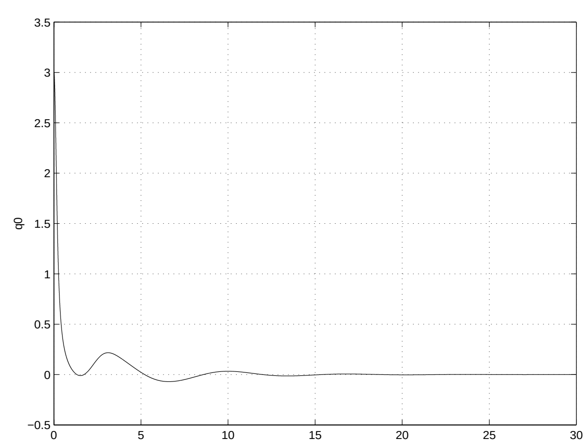

Figure 4: Plot showing the mode q_0

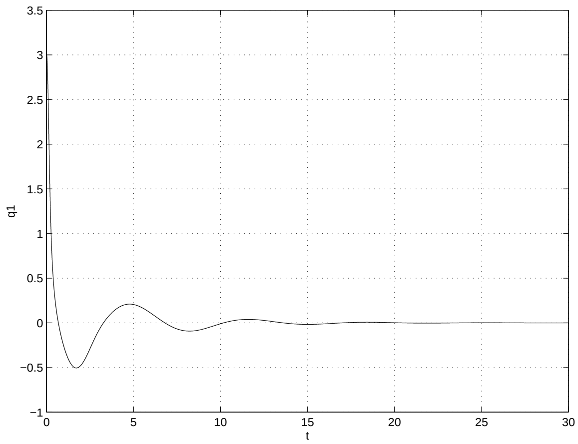

Figure 5: Plot showing the mode q_1

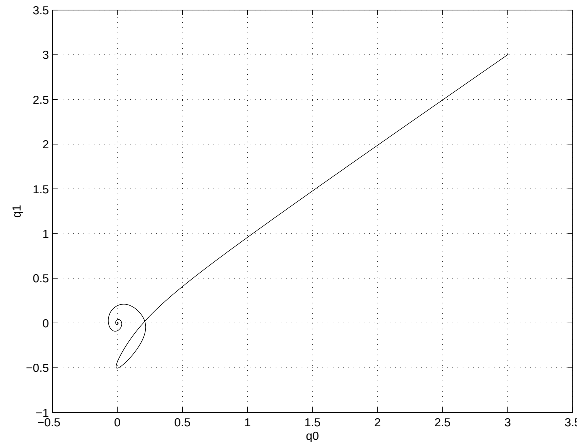

Figure 8: Plot showing q_0 vs q_1

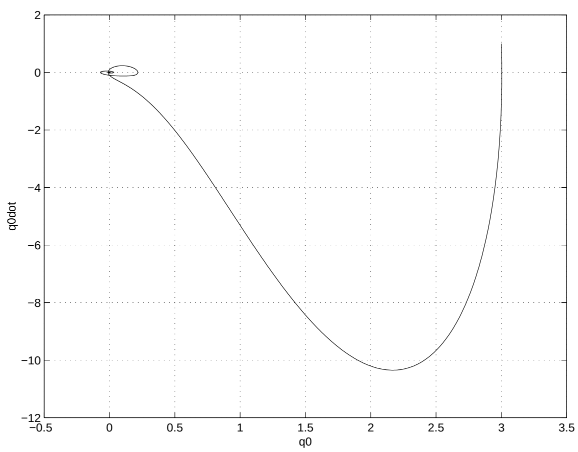

Figure 6: Phase portrait of mode q_0

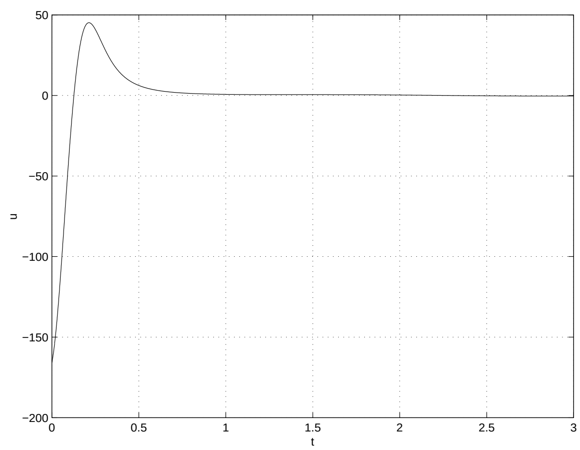

Figure 9: Plot of the feedback linearizing control

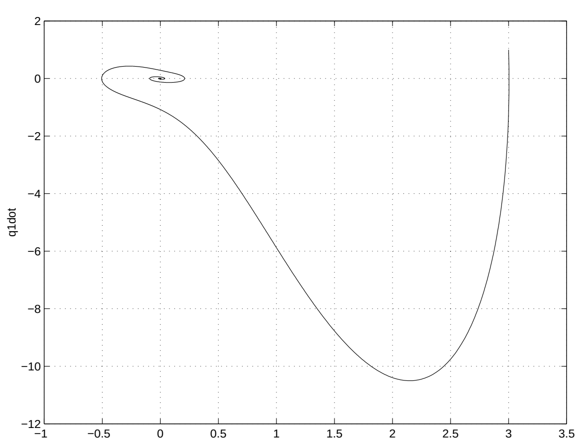

Figure 7: Phase portrait of mode q_1

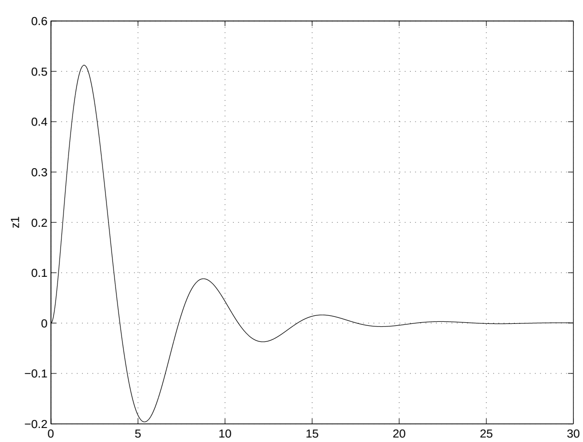

Figure 10: Plots of state z_1 vs. time

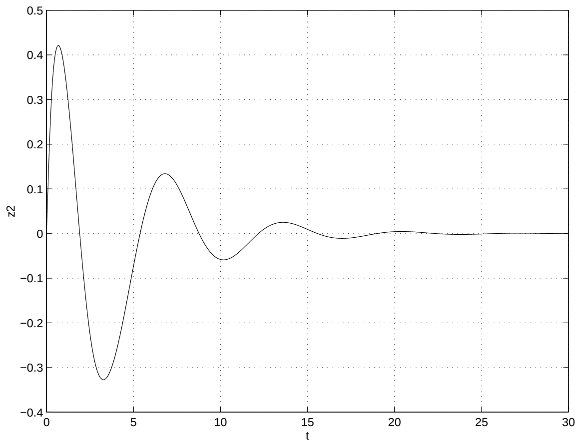

Figure 11: Plots of state z_2 vs. time

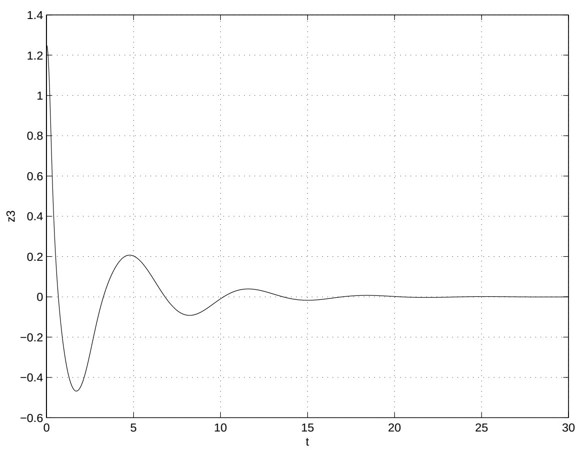

Figure 12: Plots of state z_3 vs. time

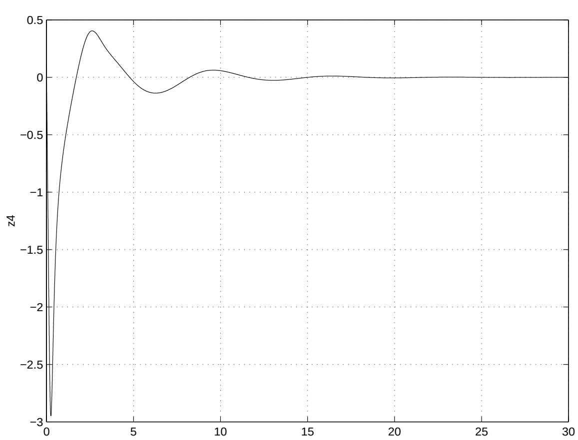

Figure 13: Plots of state z_4 vs. time